Lesson Presentation Guidelines

- Each lesson should last approximately 45 minutes. This will leave us with some time at the end for constructive feedback.
- You may do your lesson individually or with one other member of the class.
- Each person/pair will do one or two lessons; the actual number will depend on how many students enroll in the seminar.
- You may include some time for students (that is, your classmates) to participate by doing problems or the like.
- A list of suggested topics follows. If you would like to do a lesson on a topic about limits not listed below, please contact me to discuss this.
- Please feel free to consult me on any questions both during our workshop days (when we will have free time to prepare the lessons) and by email.
- The lessons do not need to be presented in the order below, although some do rely on others, as noted.

Suggested Lesson One: The Limit Definition of the Derivative

Recall that in lecture we defined $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$.

When working through examples using this definition, it is almost inevitable that students will ask good questions: “Aren’t we just making $h$ equal 0 at the end?” or “But doesn’t this mean we’re dividing by 0?” or “So the derivative is just approaching 6 but not really exactly equal to 6?” or various others that are hard to answer precisely without a rigorous definition of the limit.

Now that we have such a definition of the limit, we can answer these tough questions.

Idea: Use the limit definition of the derivative to compute $f'(3)$ if $f(x) = 2x^2$ or some similar derivative. But, in contrast to how we did such problems in lecture, now use the $\epsilon - \delta$ definition of the limit to prove formally what the limit is. You should also mention how the 0 in the $0 < |x - c| < \delta$ portion of our limit definition is crucial in every one of these derivative examples.

Suggested Lesson Two: Flawed Definitions of the Limit

In order to understand the limit definition well, it can be useful to consider some possible alternate definitions. Here we give several flawed definitions. In each case, the flawed definition will result in a different answer about the value (or existence) of the limit than what our intuition tells us. Explain in your lesson the flaws in some of these definitions and illustrate with example functions.

Flawed Definition 1. We write $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$ there exists an $x_0$ such that $|f(x_0) - L| < \epsilon$. 
For this example, use a function such as \( f(x) = \sin(1/x) \) at \( x = 0 \) to show why this definition will not work.

**Flawed Definition 2.** We write \( \lim_{x \to c} f(x) = L \) if \( f(c) = L \).

For this example, use a function such as \( f(x) = \frac{x^2 - 1}{x^2 + 1} \) at \( x = 1 \) or the function \( g(x) \) below at \( x = 2 \) to show why this definition will not work.

\[
g(x) = \begin{cases} 5, & \text{if } x < 2 \\ 8, & \text{if } x \geq 2 \end{cases}
\]

**Flawed Definition 3.** We write \( \lim_{x \to c} f(x) = L \) if for every \( \epsilon \geq 0 \) there exists a \( \delta > 0 \) such that if \( 0 < |x - c| < \delta \) then \( |f(x) - L| < \epsilon \).

For this example, consider what happens when we allow \( \epsilon \) to be 0.

**Flawed Definition 4.** We write \( \lim_{x \to c} f(x) = L \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( |x - c| < \delta \) then \( |f(x) - L| < \epsilon \).

For this example, use a function such as \( f(x) \) below at \( x = 3 \) to show why this definition will not work.

\[
f(x) = \begin{cases} x^2 - 9, & \text{if } x \neq 3 \\ 8, & \text{if } x = 3 \end{cases}
\]

### Suggested Lesson Three: Infinite Limits

What do we mean when we say that the output values of a function “go to infinity”? For example, we can write \( \lim_{x \to 3} \frac{1}{(x - 3)^2} = \infty \), but we need a formal definition of what this notation means.

**Definition.** We write \( \lim_{x \to c} f(x) = \infty \) if for every \( M > 0 \) there exists a \( \delta > 0 \) such that if \( 0 < |x - c| < \delta \) then \( f(x) > M \).

This definition means that no matter what \( M \) we pick, we can guarantee that the \( y \)-values of the function will stay above this \( M \) by choosing \( x \)-values sufficiently close to \( c \).

Idea: Consider a function such as the one given above and first look graphically and numerically at its behavior near \( x = 3 \) in order to see why this limit should be infinite. Then explain the definition and use it to prove formally that the limit is infinite.

You may also formulate a similar definition for the notation \( \lim_{x \to c} f(x) = -\infty \) and give an example of this sort of infinite limit.

### Suggested Lesson Four: Limits at Infinity
What do we mean when we say that a function has a horizontal asymptote? For example, we can write \( \lim_{x \to \infty} \left( 2 + \frac{1}{x} \right) = 2 \), but we need a formal definition of what this notation means.

**Definition.** We write \( \lim_{x \to \infty} f(x) = L \) if for every \( \epsilon > 0 \) there exists an \( x_0 \) such that if \( x > x_0 \) then \( |f(x) - L| < \epsilon \).

This definition means that by picking a big enough \( x_0 \), we can guarantee that at all \( x \)-values to the right of this \( x_0 \), \( f(x) \) will be within \( \epsilon \) of the limit \( L \).

Idea: Consider a function such as the one given above and first look graphically and numerically at its behavior for large \( x \) in order to see what its limit should be. Then explain the definition and use it to prove formally that the limit is \( L \).

You may also formulate a similar definition for the notation \( \lim_{x \to -\infty} f(x) = L \) and give an example of this sort of infinite limit.

**Suggested Lesson Five: One-Sided Limits**

We have used notation such as \( \lim_{x \to 3^+} f(x) = 2 \) in class, but we have not defined precisely what such a one-sided limit means.

**Definition.** We write \( \lim_{x \to c^+} f(x) = L \) if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( 0 < x - c < \delta \) then \( |f(x) - L| < \epsilon \).

Idea: Explain this definition and formulate a similar one for the left-hand limit, then use the definitions to prove what each one-sided limit is for a function such as the following.

\[
f(x) = \begin{cases} 
2x, & \text{if } x < 1 \\
3x + 1, & \text{if } x \geq 1 
\end{cases}
\]

You may also like to explain that if \( \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L \), then \( \lim_{x \to c} f(x) = L \).

Finally, you might discuss why the limit must fail to exist at \( x = c \) (see Lesson on Limits that Fail to Exist) if the one-sided limits are not equal.

**Suggested Lesson Six: Limits that Fail to Exist**

Consider the function

\[
f(x) = \begin{cases} 
5, & \text{if } x < 2 \\
8, & \text{if } x \geq 2 
\end{cases}
\]

What can we say about \( \lim_{x \to 2} f(x) \)? If we try to use our limit definition to show that this limit is 5, we will fail; the same will happen if we try to show this limit is 8. In fact, this limit does not exist.
Idea: Prove that this or a similar limit does not exist. The usual method to prove that such a limit does not exist is proof by contradiction (which you may have already seen in the standard proofs that there are infinitely many primes or that $\sqrt{2}$ is irrational). These proofs start by assuming that the statement to be proved is false. They then show that such an assumption leads to a contradiction; therefore, the assumption must have been incorrect. In this case, you can begin by assuming the limit is some number $L$ and showing that there is an $\epsilon > 0$ for which every possible value of $\delta$ leads to a contradiction.

Limits can also fail to exist because the function oscillates wildly. You might also like to look at a function such as $f(x) = \sin(1/x)$ as $x \to 0$ and show why this limit fails to exist too.

**Suggested Lesson Seven: Limits are Unique**

When trying to understand our $\epsilon - \delta$ definition of the limit, it is instructive to see why, if a limit exists, it must be unique.

Idea: Illustrate what goes wrong when an incorrect value for $L$ is chosen. For example, take an example we have already done of a limit that exists and show how the algebra fails if we try a different value of $L$.

You may also like to show a proof that limits are unique. Such a proof is usually done using the method of proof by contradiction (which you may have already seen in the standard proofs that there are infinitely many primes or that $\sqrt{2}$ is irrational). These proofs start by assuming that the statement to be proved is false. They then show that such an assumption leads to a contradiction; therefore, the assumption must have been incorrect. In this case, you can begin by assuming $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = M$ where $L \neq M$. From there, you will arrive at a contradiction.

**Suggested Lesson Eight: Geometric Series**

**Prerequisite: Lesson on Limits at Infinity**

One amazing consequence of our limit definition is that we can use it to prove that in some cases adding up infinitely many positive numbers will result in a finite sum.

**Definition.** An **infinite geometric series** is an expression of the form

$$a + ar + ar^2 + ar^3 + \ldots$$

where $a$ and $r$ are real numbers.

A concrete example is the infinite geometric series $1 + 1/3 + 1/9 + 1/27 + \ldots$.

We would like to be able to determine whether a given infinite geometric series results in a finite sum (and if so, what that sum is). To do this, we define a finite geometric series (below) with $n$ terms and take the limit as $n$ goes to infinity.

**Definition.** A **finite geometric series** is an expression of the form

$$a + ar + ar^2 + \ldots + ar^{n-1}$$
where \( a \) and \( r \) are real numbers and \( n \) is a positive integer. For shorthand, we can write 
\[ S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \]

To get an infinite geometric series \( S \), we take the limit of \( S_n \) as \( n \) goes to infinity; that is, we compute 
\[ \lim_{n \to \infty} S_n. \]

In our example here, \( S_n = 1 + 1/3 + 1/9 + \cdots + (1/3)^{n-1} \).

The trick is to find a simpler way of expressing \( S_n \), as follows.

\[
S_n = 1 + 1/3 + 1/9 + \cdots + (1/3)^{n-1} \\
(1/3)S_n = 1/3 + 1/9 + 1/27 + \cdots + (1/3)^n \\
(2/3)S_n = 1 - (1/3)^n
\]

We've multiplied each side by \( 1/3 \).

Now we just divide each side of the third equation by \( 2/3 \) to get our “closed form” expression \( S_n = \frac{1-(1/3)^n}{2/3} \). This now allows use to easily evaluate, say, \( S_{100} \) without needing to add up 100 terms.

We can then use our definition of a limit at infinity (see Lesson on Limits at Infinity) to evaluate this limit by treating \( n \to \infty \) as \( x \to \infty \) in that definition.

Idea: Show an example similar to the one above and/or give an application from real life. For instance, if a patient takes 100 mg of a drug each morning and in each 24-hour period 20% of the drug is eliminated from her body, how much will she have in her body far in the future?

You may also wish to explain which values of \( r \) (only \( |r| < 1 \), in fact) make the limit of the infinite geometric series exist.

**Suggested Lesson Nine: Continuity**

Our intuitive notion is that a function is continuous at \( x = c \) if we can draw its graph there without lifting pencil from paper. But how can we make this idea mathematically rigorous?

**Definition.** A function \( f(x) \) is **continuous** at \( x = c \) if 
\[ \lim_{x \to c} f(x) = f(c). \]

Idea: Explain this definition and then use it to show that functions similar to the following are or are not continuous.

\[
f(x) = \begin{cases} 
\frac{x^2-5x+6}{x-3}, & \text{if } x \neq 3 \\
1, & \text{if } x = 3
\end{cases} \\
g(x) = \begin{cases} 
\frac{x^2-1}{x-1}, & \text{if } x \neq 1 \\
3, & \text{if } x = 1
\end{cases}
\]

You should also mention how the 0 in the \( 0 < |x - c| < \delta \) portion of our limit definition is crucial in an example such as \( g(x) \) above.

**Suggested Lesson Ten: Differentiability**

**Prerequisite: Lesson on Limits that Fail to Exist**

In class we have seen various examples of functions whose derivatives fail to exist at one or more points; we say that the function is not differentiable at those points. Using our definition of limits, we can formulate the meaning of “differentiable”
more precisely.

**Definition.** A function $f(x)$ is **differentiable** at $x = c$ if $f'(c)$ exists, i.e., if
\[
\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}
\]
exists.

Idea: After explaining this definition, use it to show whether functions such as the following are differentiable at the specified $x$-values.

$f(x) = |x - 3|$ at $x = 3$  \hspace{1cm} g(x) = x^2$ at $x = 3$

**Suggested Lesson Eleven: Infinite Limits at Infinity**

*Prerequisites: Lesson on Limits at Infinity, Lesson on Infinite Limits*

What should the notation $\lim_{x \to \infty} f(x) = \infty$ mean?

Idea: By looking at how we defined $\lim_{x \to \infty} f(x) = L$ and $\lim_{x \to \infty} f(x) = \infty$, formulate a definition of $\lim_{x \to \infty} f(x) = \infty$. Then use your definition to prove a limit such as $\lim_{x \to \infty} x^2 = \infty$.

You might also like to consider defining one or more of the following and giving an appropriate example or examples.

$\lim_{x \to \infty} f(x) = -\infty$  \hspace{1cm} $\lim_{x \to \infty} f(x) = \infty$  \hspace{1cm} $\lim_{x \to \infty} f(x) = -\infty$

**Suggested Lesson Twelve: Limits that Fail to Exist at Infinity**

*Prerequisites: Lesson on Limits at Infinity, Lesson on Limits that Fail to Exist*

What should it mean to say $\lim_{x \to \infty} f(x)$ is not infinite but yet fails to exist?

Idea: Consider a limit such as $\lim_{x \to \infty} \sin x$ and show that this limit is not infinite but also does not equal any real number $L$. A proof by contradiction (see above) will likely be useful here.

**Suggested Lesson Thirteen: Helpful Limit Theorems**

In practice, we usually don’t use the official $\epsilon - \delta$ definition to compute limits. For example, if we already know that $\lim_{x \to 3} f(x) = 5$ and $\lim_{x \to 3} g(x) = 7$, then we jump immediately to $\lim_{x \to 3} [f(x) + g(x)] = 12$. But we don’t really need to make such leaps of faith: we can prove theorems that allow us to manipulate limits in this way.

Two very useful such theorems to prove are the following:

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then $\lim_{x \to c} [f(x) + g(x)] = L + M$.

If $\lim_{x \to c} f(x) = L$ and $k$ is any real number, then $\lim_{x \to c} [kf(x)] = kL$.

Idea: Prove one or both of these theorems and then illustrate their use with some concrete examples.