Marshallian Demand

Hicksian Demand

Slutsky Equation

1. What is Marshallian Demand?

- The Marshallian demand function is the solution to the following consumer utility maximization problem when he faces price vector $p$ and has income $Y$.

\[
\max_x u(x) \quad \text{s.t.} \quad p \cdot x \leq Y
\]

- We typically denote the Marshallian demand for good $j$ as $x_j(p, Y)$.

- The indirect utility function represents the level of utility attained given prices $p$, income $Y$.

\[
v(p, Y) = u(x_1(p, Y), x_2(p, Y), \ldots, x_n(p, Y))
\]

**How should we think about this?** Marshallian demand $x_j(p_j, p_{-j}, Y)$ is the optimal quantity (i.e. ‘solution’) of input $j$, chosen for a given parameter vector. Consider the parameter ‘own price’, i.e. $p_j$. When $p_j$ changes to $p_j'$, we expect this optimal solution to change to $x_j(p_j', p_{-j}, Y)$. This represents ‘demand’, which can be loosely thought about as the relationship between quantity chosen and parameters.

- Marshallian demand is also commonly known as ‘uncompensated demand’. A change to input demand $x_i(p_i', p_{-i}, Y)$ will lead to a new level of indirect utility, $v(p_i', p_{-i}, Y)$. We say that Marshallian demand is uncompensated because it solves the new optimal level of input without factoring in the consideration that the agent now achieves a different level of utility.

- Roy’s identity allows us to relate Marshallian demand of good $j$ to the derivatives of the indirect utility function.

Roy’s identity: $x_j(p, Y) = -\frac{v_{p_j}(p, Y)}{v_Y(p, Y)}$

- **Proof.** Recall that $v(p, Y)$ is the maximand of the constrained optimization problem

\[
\max_{x, \lambda} \mathcal{L}(x, \lambda; p, Y) = \max_{x, \lambda} u(x) + \lambda(Y - p \cdot x) = \mathcal{L}(x^*(p, Y), \lambda^*; p, Y)
\]

Applying the envelope theorem, we take derivatives with respect to the parameters for $j$

\[
v_{p_j}(p, Y) = \mathcal{L}_{p_j}(x^*(p, Y), \lambda^*; p, Y) = -\lambda x_j(p, Y)
\]

\[
v_Y(p, Y) = \mathcal{L}_Y(x^*(p, Y), \lambda^*; p, Y) = \lambda
\]

Dividing these two expressions yields us our result.

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1 These notes draw heavily on the notes from previous years, and in particularly from last year’s TF, Matt Fiedler.

2 Recall that $p$ is the price vector $(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_n)$. For notational convenience, I re-write this as $(p_i, p_{-i})$, isolating $p_i$ as the own price for good $i$, and grouping everything else as $p_{-i}$.
What does this mean? The intuition can be seen by re-arranging the identity as

\[ v_{p_j}(p, Y) = -x_j(p, Y) \cdot v_Y(p, Y) \]

When we increase price \( p_j \) by one dollar, the left hand side says that our indirect utility changes by \( v_{p_j} \). Given that we demand \( x_j \) of input \( j \), this is the same as saying that our wealth have changed by \( -x_j \) dollars which impacts our indirect utility by \( -x_j v_Y \). This is given by the right hand side. Roy’s identity balances this equivalence (using the envelope).

2. What is Hicksian Demand?

• The Hicksian demand function is the solution to the following consumer expenditure minimization problem when he faces price vector \( p \) and must achieve utility level \( u \).

\[
\min_x p \cdot x \quad \text{s.t.} \quad u(x) \geq u
\]

• We typically denote the Hicksian demand for good \( j \) as \( h_j(p, u) \).

• The expenditure function represents the level of expenditure required to attain utility \( u \), given prices \( p \).

\[ e(p, u) = p \cdot h(p, u) \]

How should we think about this? Hicksian demand \( h_j(p_j, p_{-j}, u) \) is the optimal quantity (i.e. ‘solution’) of input \( j \), chosen for a given parameter vector, constrained for a fixed utility level \( u \). Consider: There are two things that happen when price \( p_j \) changes to \( p'_j \).

- First, suppose utility was not constrained to be this fixed level \( u \). The optimal choice of \( x_j \) will change. This will lead to a new level of utility \( u' \). Demand (uncompensated) would be \( h_j(p'_j, p_{-j}, u') \).

- But, Hicksian demand fixes the utility level at \( u \). Intuitively, we now give the agent a compensation that brings him from \( u' \) back to the same utility level as before \( u \). This final demand is compensated, because it solves the new optimal level of input \( h_j(p'_j, p_{-j}, u) \), factoring in what it takes to preserve the agent at the original utility level.

• Shephard’s Lemma\(^3\) allows us to find the Hicksian demand function for good \( j \) directly from the expenditure function by simple partial differentiation.

\[ \text{Shephard’s Lemma:} \quad e_{p_j}(p, u) = h_j(p, u) \]

• Proof. Recall that \( e(p, u) \) is the minimand of the constrained optimization problem

\[
\min_{x, \lambda} \mathcal{L}(x, \lambda; p, u) = \min_{x, \lambda} p \cdot x + \lambda (u - u(x)) = \mathcal{L}(h(p, u), \lambda^*; p, u)
\]

By an application of the envelope theorem, we differentiate with respect to \( p_j \) an obtain the result

\[ e_{p_j}(p, u) = \mathcal{L}_{p_j}(h(p, u), \lambda^*; p, u) = h_j(p, u) \]

• What does this mean? Intuitively, the first-order effect of a price increase on expenditure is just that we pay more for each unit of the good that we are currently consuming.

\(^3\)You should note the spelling of “Shephard”. It can be non-intuitive.
In words, Shephard’s lemma states that the cost minimizing point of any given good \( j \) with price \( p_j \) is unique\(^4\). The idea is that a consumer will buy a unique ideal amount of each item to minimize the cost of obtaining a certain level of utility given the market price of goods.

2\(\frac{1}{2}\). **Translating mathematical insights into non-mathematical intuition**

- In this version of the course, it is best to think about everything as a mathematical function.
- That said, it may be helpful to view some diagrams. First, note that a demand curve is different from a demand function.
- The Marshallian demand curve (fig. on right) plots out the relationship between the price of a good \( p_j \) and the quantity of that good \( x_j (p_j, p_{-j}, Y) \) optimally chosen by an agent, assuming that all other demand parameters are held constant (i.e. income \( Y \), price of other goods \( p_{-j} \), agent’s preferences \( u(\cdot) \)).

**Graphical interpretation of Marshallian demand.** When a parameter \( p_j' \) changes to \( p_j'' \), the tradeoff (slope) or MRS changes. To optimize, the agent will shift his indifference curve \( I \) out as much as possible to obtain the new solution bundle for Marshallian demand.

**Graphical interpretation of Hicksian demand.** When a parameter \( p_j' \) changes to \( p_j'' \), the tradeoff (slope) or MRS changes but we hold utility constant. In optimizing, the agent stays on the same indifference curve in obtaining the new solution bundle for Hicksian demand.

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\(^4\) Given that the indifference curves of the expenditure function are convex, i.e. a convex set of preferences.
In other words, we can decompose the consumer’s reaction into substitution and income effects when $p_j$ changes.

**Substitution effect.** When $p_j$ changes, even if the individual remains on same indifference curve, his optimal choice $x_j$ will change because the MRS must equal the new price ratio.

Intuitively, the change in $p_j$ makes a good relatively less attractive at the margin and induces substitution away from that good.

**Income effect.** When $p_j$ changes, even if MRS remains the same, his optimal choice $x_j$ will change because real income changes and he must move to a new indifference curve.

Intuitively, the change in $p_j$ changes the cost of inframarginal units of good $j$, changing the budget set and thus utility.

Marshallian demand reflects both substitution and income effects.

- **Substitution effect:** Hold utility $u$ constant, but allow relative price of good $x$ to change.
- **Income effect:** Hold tradeoff between goods $x$ and $y$ constant, shift out ‘real income’.

Hicksian (Compensated) demand reflects only the substitution effect.

### 3. Slutsky Equation

Utility maximization and expenditure minimization are dual problems. Formally,

$$
x (p, Y) = h (p, v (p, Y))
$$

The bundle of goods that solves the utility maximization problem (Marshallian) with prices $p$ and income $Y$ also solves the expenditure minimization problem (Hicksian) with prices $p$ and utility target $v (p, Y)$.

$$
h (p, u) = x (p, e (p, u))
$$

The bundle of goods that solve the expenditure minimization problem (Hicksian) with prices $p$ and utility target $u$ also solve the utility maximization problem (Marshallian) with prices $p$ and income $e (p, u)$.

This duality allows us to derive the Slutsky equation, which relates changes in Marshallian demand to changes in Hicksian demand.

**Slutsky Decomposition Equation:** The change in demand due to price can be decomposed into a substitution effect and an income effect.

$$
\frac{\partial x_j}{\partial p_j} = \frac{\partial h_j}{\partial p_j} - x_j \frac{\partial x_j}{\partial Y}
$$

Demand response to price changes Substitution effect Income effect
• Proof.

1. Start from duality equation (2) for good \( j \)

\[
x_j(p, e(p, u)) = h_j(p, u)
\]

2. Differentiate with respect to \( p_j \)

\[
\frac{\partial x_j(p, e(p, u))}{\partial p_j} + \frac{\partial x_j(p, e(p, u))}{\partial Y} \cdot \frac{\partial e(p, u)}{\partial p_j} = \frac{\partial h_j(p, u)}{\partial p_j}
\]

3. Substitute in the following identities

\[
\frac{\partial e(p, u)}{\partial p_j} = h_j(p, u) \quad \text{(from Shephard’s Lemma)}
\]

\[
Y = e(p, u) \quad \text{(Budget Constraint: income = expenditure)}
\]

\[
h_j(p, u) = x_j(p, Y) \quad \text{(from duality)}
\]

leading to

\[
\frac{\partial x_j(p, Y)}{\partial p_j} + \frac{\partial x_j(p, Y)}{\partial Y} \cdot x_j(p, Y) = \frac{\partial h_j(p, u)}{\partial p_j}
\]

4. Rearrange to obtain the result.

• Consider the substitution effect. This is exactly the definition of the Hicksian demand curve, which tell us the effect on demand of price changes, after we have negated any effects on overall utility. The negative slope of the Hicksian demand curve tells us that this term is always negative.

• Consider the income effect. Intuitively, the first-order effect on our budget when \( p_j \) rises by a dollar is that we are \( x_j \) dollars poorer. We scale this response by \( \frac{\partial x_j}{\partial Y} \) which tells us how sensitive demand for good \( j \) is to changes in wealth.

• A normal good is one where \( \frac{\partial x_j}{\partial Y} > 0 \). This effect reinforces the substitution effect.

• On the other hand, an inferior good is one where \( \frac{\partial x_j}{\partial Y} < 0 \). The income effect would then counteract the substitution effect.

• The following is a useful schematic that shows how the utility maximization problem (UMP) and expenditure minimization problem (EMP) are connected.
4. Identities from differentiating the budget constraint

- Define

| \( \eta_i = \frac{p_i x_i}{Y} \) | expenditure on \( i \) total budget | Budget share of good \( i \) |
| \( \varepsilon_{pi}^i = \frac{\partial x_i}{\partial p_j} p_j \) | % change in qty demanded of \( i \) % change in price of good \( j \) | Elasticity of good \( i \) with respect to price of good \( j \) |
| \( \varepsilon_{Y}^i = \frac{\partial x_i}{\partial Y} x_i \) | % change in qty demanded of \( i \) % change in income | Elasticity of good \( i \) with respect to income \( Y \) |

- We start from the satisfied budget equality

\[ Y = \sum_{i=1}^{n} p_i x_i(p, Y). \]

- To obtain the first identity, differentiate both sides with respect to income \( Y \)

\[
1 = \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial Y} = \sum_{i=1}^{n} p_i x_i \frac{Y}{Y} \frac{\partial x_i}{\partial Y} x_i = \sum_{i=1}^{n} \eta_i \varepsilon_{Y}^i.
\]

  Intuitively, the average of the income elasticities \( \varepsilon_{Y}^i \), weighted by the share of the total budget spent on each good \( \eta_i \), must be one. This is a consequence of the fact that the budget constraint must always hold, so when income increases by one percent, spending on the average good must increase by one percent. Note that this identity implies, among other things, that at least one good must be a normal good at any given \( (p, Y) \) bundle, for if \( \varepsilon_{Y}^i < 0 \) for all \( i \), then this equation would not hold.

- We could also differentiate the budget constraint with respect to the price \( p_j \)

\[ 0 = x_j + \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial p_j}. \]

- Rearranging, multiplying both sides by \( p_j \), and dividing both sides by \( Y \), we obtain

\[
\frac{p_j x_j}{Y} = -\sum_{i=1}^{n} p_i \frac{p_j}{Y} \frac{\partial x_i}{\partial p_j} \quad \eta_j = -\sum_{i=1}^{n} \frac{p_i x_i}{Y} \frac{p_j}{\partial p_j} x_i = -\sum_{i=1}^{n} \eta_i \varepsilon_{p_j}^i.
\]

  Intuitively, the average of the price elasticities \( \varepsilon_{p_j}^i \), weighted by the share of the budget spent on each good \( \eta_i \), must be equal to the expenditure share of the good whose price changed. This is also a consequence of the fact that the budget constraint must always hold. When the price \( p_j \) increases by one percentage point, this leads to an \( \eta_j \) percentage point increase in total spending (at the old quantities), so average spending across all goods must fall by \( \eta_j \) percentage points to ensure that the budget constraint continues to hold.