Some vanishing theorems

Notes on Griffiths & Harris: section 1.2.

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Review of harmonic theory

Let us recall that a harmonic form ψ by definition satisfies

$$\Delta\psi = 0 \Leftrightarrow d\psi = d^*\psi = 0.$$

A closed form is harmonic if and only if it has minimal norm in its de Rham cohomology class. This is essentially the spirit of the Hodge theorem. In the context of complex manifolds we can also talk about $\overline{\partial}$-harmonic forms, defined in an analogous way. The space of harmonic forms of type (p,q) is called the harmonic space $\mathcal{H}^{p,q}(M)$. The complex version of the Hodge theorem states

Theorem 1 *Let M be a compact complex manifold, we have*
1. $\mathcal{H}^{p,q}(M)$ is finite dimensional, so the orthogonal projection $\mathcal{H} : A^{p,q} \to \mathcal{H}^{p,q}$ is well defined.
2. There is a unique Green's operator $G : A^{p,q} \to A^{p,q}$, satisfying $G(\mathcal{H}^{p,q}) = 0$, $[G,\bar{\partial}] = [G,\bar{\partial}^*] = 0$, and

$$\mathbf{1} = \mathcal{H} + \Delta G.$$

Consequently given a form ψ we have decomposition

$$\psi = \mathcal{H}(\psi) + \bar{\partial}(\bar{\partial}^* G\psi) + \bar{\partial}^*(\bar{\partial} G\psi).$$

If ψ is closed, this says ψ can be uniquely expressed as the sum of a harmonic form and a $\bar{\partial}$-exact form. So there is a 1-1 correspondence between the harmonic space and the Dolbeault cohomology group.

The idea of the proof is that, given a form η with $\mathcal{H}(\eta) = 0$, we can always solve

$$\Delta\psi = \eta$$

in the weak sense. Namely we can find a ψ in the completion of $A^{p,q}$ such that

$$(\psi, \Delta\varphi) = (\eta, \varphi)$$

for all $\varphi \in A^{p,q}$. Then by a regularity lemma, the map $\eta \mapsto \psi$ is a smoothing operator. More precisely if φ is in the Sobolev space H_s, ψ is in H_0, then one can deduce that ψ is in H_{s+2}. This will show that the weak solution one gets is actually smooth, and proves the decomposition identity.

The Hodge theorem generalizes to tensor bundles associated with vector bundle E over M in the straightforward way. Note that the Hodge dual generalizes to

$$*_E : A^{p,q}(E) \to A^{n-p,n-q}(E^*)$$

by

$$(\eta, \psi) = \int_M \eta \wedge *_E \psi$$

where $(\cdot, \cdot)$ is the natural inner product on sections of $A^{p,q}(E) = A^{p,q}(M) \otimes \Gamma(M, E)$. The Hodge dual in particular gives rise to Kodaira-Serre duality

$$H^q(M, \mathcal{O}(E)) \cong H^{0,q}_{\bar\partial}(E) \cong H^{n,n-q}_{\bar\partial}(E^*) \cong H^{n-q}(M, \mathcal{O}(E^* \otimes K_M)).$$

Let $L : A^{p,q}(E) \to A^{p+1,q+1}(E)$ be defined by $s \mapsto \omega \wedge s$, and $\Lambda = L^*$. Let the metric connection on E be $D = D' + \bar\partial$. The Hodge identity generalizes in the context of vector bundles to

$$[\Lambda, \bar\partial] = -\frac{i}{2} D'^*.$$

We will omit its proof, which is straightforward following the standard Hodge identity $[\Lambda, \bar\partial] = -\frac{i}{2}\partial^*$. The later will be sketched, among other things, in the next section.

Manipulating L and Λ

The identities involving L and Λ we'll use later are

$$[\Lambda, \bar\partial] = -i\partial^*, \quad [\Lambda, \partial] = i\bar\partial^*, \quad [\Lambda, L] = n - p - q.$$

One can choose a holomorphic coordinate system such that the Kähler metric has vanishing derivatives at a given point. So in the calculation involving only first derivatives, we can simply replace the Kähler form ω by the trivial one $\frac{i}{2}\sum dz^k \wedge d\bar z^k$.

Define $e_k : \varphi \mapsto dz^k \wedge \varphi$, $\bar e_k : \varphi \to d\bar z^k \wedge \varphi$, and $i_k, \bar i_k$ their adjoints. For example, the action of i_k on a form is to take away dz^k and multiply by 2 (since $||dz^k|| = 2$), or zero if the form doesn't contain dz^k. $e_k, \bar{e}_k, i_k, \bar{i}_k$ satisfy commutators

$$\{i_k, e_l\} = 2\delta_{kl}, \quad \{\bar{i}_k, \bar{e}_l\} = 2\delta_{kl}$$

One can conveniently write

$$L = \frac{i}{2} \sum e_k \bar{e}_k, \quad \Lambda = -\frac{i}{2} \sum \bar{i}_k i_k, \quad \partial = \partial_k e_k, \quad \text{etc.}$$

Now it is trivial to check the identities. For example,

$$[\Lambda, L] = \frac{1}{4} \sum_{k,l} [\bar{i}_k i_k, e_l \bar{e}_l] = \frac{1}{2} (\bar{i}_k \bar{e}_k - e_k i_k) = (n-q) - p$$

when acting on a (p, q)-form. Using these identities one can easily derive the the important fact that $\frac{1}{2}\Delta_d$, Δ_∂, $\Delta_{\bar{\partial}}$ coincide on a Kähler manifold.

Basic setup

We will always take M to be a compact Kähler manifold. A line bundle L over M is positive if there is a metric on L such that the curvature 2-form is positive definite. It follows that $c_1(L)$ can be represented by a positive form in $H^2_{DR}(M)$. More nontrivially, the converse is also true. To show this, we first need the following

Lemma 1 *If η is an exact (p, q)-form on compact Kähler manifold M, then*

$$\eta = \partial \bar{\partial} \gamma$$

for some $(p-1, q-1)$-form γ.

Proof: $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$ on M. The associated Green's operators are related by $G_\partial = G_{\bar{\partial}} = 2G_d$. So $d, \partial, \bar{\partial}$ and their adjoints all commute with $G_{\bar{\partial}}$. Now since η is exact, its harmonic projection is zero. By Hodge decomposition,

$$\eta = \bar{\partial} \bar{\partial}^* G_{\bar{\partial}} \eta$$

Now $\partial(\bar{\partial}^* G_{\bar{\partial}} \eta) = \pm \bar{\partial}^* G_{\bar{\partial}}(\partial \eta) = 0$ since η is of pure type (p, q). The harmonic projection of $\bar{\partial}^* G_{\bar{\partial}} \eta$ is certainly zero, so using Hodge decomposition once again,

$$\bar{\partial}^* G_{\bar{\partial}} \eta = \partial \partial^* G_\partial (\bar{\partial}^* G_{\bar{\partial}} \eta) = \partial(\partial^* \bar{\partial}^* G^2_{\bar{\partial}} \eta) = \partial \gamma.$$

This finishes the proof. **QED.**

Back to the positivity of line bundles. Suppose $c_1(L)$ is represented by a real form $\frac{i}{2\pi}\varphi$, then the curvature

$$\Theta = \partial\bar{\partial}\rho + \varphi$$

for some real smooth function ρ. If we rescale the metric on L (corresponding to Θ) by e^{ρ}, the curvature form becomes φ. This shows that if $c_1(L)$ can be represented by a positive form then L is positive.

Kodaira vanishing theorem and consequences

The Kodaira-Nakano vanishing theorem is a statement that certain cohomology groups of positive line bundles vanish.

Theorem 2 *If L is a positive line bundle over M, then*

$$H^q(M, \Omega^p(L)) = 0 \quad \text{for } p+q > n.$$

Dualizing this statement, if L is negative, then

$$H^q(M, \Omega^p(L)) = 0 \quad \text{for } p+q < n.$$

Proof: We know that $H^q(M, \Omega^p(L)) \cong \mathcal{H}^{p,q}(L)$, so it suffices to look at harmonic forms.

The key point is to find a metric on L such that the curvature form $\frac{i}{2\pi}\Theta$ is a Kähler metric on M. This is always possible since L is positive. Equivalently, the curvature operator $\Theta = D^2$ is the same as $-2\pi i L$, where L is the operator appearing in the Hodge identity.

Let $\eta \in \mathcal{H}^{p,q}(L)$, then

$$\Theta\eta = D^2\eta = (\bar{\partial}D' + D'\bar{\partial})\eta = \bar{\partial}D'\eta.$$

Now

$$\begin{aligned} 0 &\le (D'\eta, D'\eta) = (D'^*D'\eta, \eta) \\ &= 2i\left((\bar{\partial}\Lambda - \frac{i}{2}D'^*)D'\eta, \eta\right) \\ &= 2i(\Lambda\bar{\partial}D'\eta, \eta) = 2i(\Lambda\Theta\eta, \eta), \end{aligned}$$

and

$$\begin{aligned} 0 &\leq (D'^*\eta, D'^*\eta) = (D'D'^*\eta, \eta) \\ &= -2i\left(D'(\Lambda\bar{\partial} + \frac{i}{2}D'^*)\eta, \eta\right) \\ &= -2i(D'\bar{\partial}\Lambda D'\eta, \eta) = -2i(\Theta\Lambda\eta, \eta). \end{aligned}$$

Putting these together, we get

$$2i([\Lambda, \Theta]\eta, \eta) = 4\pi([\Lambda, L]\eta, \eta) \geq 0.$$

On the other hand, we have algebraic identity

$$[\Lambda, L] = n - p - q.$$

So for $p + q > n$, $||\eta||^2 = 0$, which implies $\eta = 0$. **QED.**

Using Kodaira vanishing theorem, we can prove the Lefschetz hyperplane theorem

Theorem 3 *Let M be an n-dimensional compact complex manifold, $V \subset M$ a smooth hypersurface with $L = [V]$ positive. Then the map induced by inclusion $V \hookrightarrow M$,*

$$H^q(M, \mathbf{Q}) \to H^q(V, \mathbf{Q})$$

is an isomorphism for $q \leq n - 2$, and injective for $q = n - 1$.

Comments: The name "hyperplane theorem" comes from the fact that if M is a submanifold of $\mathbf{P}^N$, $V = M \cap H$ for a hyperplane H satifies the hypothesis of the theorem. In fact, the theorem holds for $\mathbf{Z}$-coefficient cohomology groups, and even for homotopy groups. The proof of the more general cases will be sketched in the next section using Morse functions.

Proof: It suffices to prove for $\mathbf{C}$-coefficients. This will be implicit in the rest of the proof.

By Hodge decomposition and Dolbeault theorem, it suffices to look at

$$H^q(M, \Omega^p_M) \to H^q(V, \Omega^p_V)$$

We will prove that this map is an isomorphism for $p+q \leq n-2$, and injective for $p + q = n - 1$. Since $[-V]$ is negative, from Kodaira vanishing theorem we know that

$$\begin{aligned} H^q(M, \Omega^p_M([-V])) = 0, &\quad p + q < n; \\ H^q(V, \Omega^{p-1}_V([-V])) = 0, &\quad p + q < n. \end{aligned}$$

To make connection with our case, observe the exact sequence of sheaves on M,

$$0 \to \Omega^p_M([-V]) \to \Omega^p_M \to \Omega^p_M|_V \to 0$$

This follows from the correspondence between forms vanishing along V and sections of $\Omega^p_M([-V])$. There is an analogous sequence of sheaves on V,

$$0 \to \Omega^{p-1}_V([-V]) \to \Omega^p_M|_V \to \Omega^p_V \to 0$$

To see this requires some thought. First by adjunction formula, $[-V] = N^*_V$ - df_α's give a global nonzero section of $N^*_V \otimes [V]$. The conormal bundle N^*_V by definition fits into the sequence

$$0 \to N^*_{V,p} \to T^{*\prime}_p(M) \to T^{*\prime}_p(V) \to 0$$

Since $N^*_{V,p} \hookrightarrow T^*_p(M)$ has dimension 1, there is

$$0 \to N^*_{V,p} \otimes \bigwedge^{p-1} T^{*\prime}_p(V) \to \bigwedge^p T^{*\prime}_p(M) \to \bigwedge^p T^{*\prime}_p(V) \to 0$$

This gives our desired exact sequence of sheaves on V. They give rise to LES of cohomology groups,

$$\begin{aligned} &H^q(M, \Omega^p_M([-V])) \to H^q(M, \Omega^p_M) \to H^q(M, \Omega^p_M|_V) \\ &\cong H^q(V, \Omega^p_M|_V) \to H^{q+1}(M, \Omega^p_M([-V])), \end{aligned}$$

and

$$H^q(V, \Omega^{p-1}_V([-V])) \to H^q(V, \Omega^p_M|_V) \to H^q(V, \Omega^p_V) \to H^{q+1}(V, \Omega^{p-1}_V([-V])).$$

The map

$$H^q(M, \Omega^p_M) \to H^q(M, \Omega^p_M|_V) \cong H^q(V, \Omega^p_M|_V) \to H^q(V, \Omega^p_V)$$

is therefore isomorphism when $p+q \le n-2$ and injective when $p+q = n-1$. **QED.**

Suppose V is a hypersurface in $\mathbf{P}^n$. By intersecting it with hyperplanes, the theorem tells us that $H^{2k-1}(V) = 0$ for $k \neq n/2$, and $H^{2k}(V)$ is generated by the intersection of V with k hyperplanes for $k < n/2$. For example, take the quintic 3-fold X in $\mathbf{P}^4$. The Lefschetz hyperplane theorem says that $h^{1,0}(X) = h^{2,0}(X) = 0$, and $H^{1,1}(X)$ is generated by the Poincaré dual of the intersection of X with a hyperplane, so $h^{1,1} = 1$. The theorem doesn't say that b^3 is zero. In fact we know that $h^{2,1}$ is the number of complex structure deformations, which equals 101 for the quintic X. Also, we see that any hypersurface in $\mathbf{P}^n$ for $n \ge 3$ is simply connected.

A little Morse theory

In this section we sketch an interesting proof of Lefschetz hyperplane theorem using Morse theory.

Let A be a compact manifold, and $B \subset A$ a submanifold. Let $\varphi : A \to \mathbf{R}^+$ be a smooth function with $B = \varphi^{-1}(0)$. We call $x \in A$ a critical point if $d\varphi(x) = 0$, and $\varphi(x)$ a critical value. The Hessian of φ, $H(\varphi) = \partial^2\varphi/\partial u_i \partial u_j$ is a quadratic form on $T_x(A)$. φ is a Morse function if $H(\varphi)$ at all the critical points are nondegenerate. The index of a critical point is defined to be the number of negative eigenvalues of the Hessian at that point.

We'll make use of the following

Lemma 2 *Suppose φ is a Morse function, and $H(\varphi)$ is nonsingular on the normal bundle N_B. When we vary $t \in \mathbf{R}^+$, the homotopy type of the space $A_t = \varphi^{-1}([0,t])$ does not change if t does not cross a critical value; it changes by attaching a cell of dimension k when t crosses a critical value of index k.*

We will not attempt to prove it, but motivate it in a simple example. Consider a torus embedded in $\mathbf{R}^3$ in the standard way, placed vertically with its bottom touching the xy-plane, so that z-coordinate is a Morse function. When we increase z from 0 we meet critical points of index 1, 1, and 2. A_ϵ is a disk for small ϵ. When we cross the first critical value, A_t becomes a tube, or effectively a 1-cell is attached to the boundary of the disk. When we cross the second critical value, another 1-cell is attached to the two ends of the tube, so the homotopy type is the same as a torus with one hole. Finally when we cross the third critical value and get the full torus, a 2-cell is attached to fill in the hole.

Now we are ready to give another proof of the Lefschetz hyperplane theorem. Let M be a compact complex manifold, $L = [V]$ a positive line bundle over M. There is a holomorphic section s whose zero locus is $V = (s)$. Pick a metric on L such that $\frac{i}{2\pi}\Theta$ is positive. Using the formula

$$\Theta = \partial\bar{\partial} \ln |s|^{-2}$$

we choose $\varphi = \ln |s|^2$ to be a Morse function. Note that although the complex Hessian matrix for φ is hermitian and negative definite, the real Hessian might be singular. If that's the case, we can always choose a Morse function sufficiently close to φ in C^2 topology. It is slightly different from our above convention in that $V = \varphi^{-1}(-\infty)$, but this doesn't matter as long as $d|s| \neq 0$ along V.

Since $\partial^2\varphi/\partial z_i\bar{\partial} z_j$ is hermitian and negative definite, it is not hard to see that the real Hessian

$$H(\varphi) = \begin{pmatrix} \frac{\partial^2}{\partial x_i \partial x_j} & \frac{\partial^2}{\partial x_i \partial y_j} \\ \frac{\partial^2}{\partial y_i \partial x_j} & \frac{\partial^2}{\partial y_i \partial y_j} \end{pmatrix} \ln |s|^2$$

has at least n negative eigenvalues. This would also be true if we approximate φ in C^2 by a Morse function. Therefore when we vary t from $-\infty$ to $+\infty$, $M_t = \varphi^{-1}(-\infty, t]$ varies from V to M, only cells of dimension$\geq n$ are attached. This shows that $H_q(V) \to H_q(M)$ is isomorphism for $q \leq n-2$, and surjective for $q = n-1$. Passing to cohomology groups yields the Lefschetz hyperplane theorem.

Lefschetz theorem on $(1,1)$-classes

First we'll sketch a different vanishing theorem.

Theorem 4 *Let M be a compact complex manifold, $L \to M$ a positive line bundle, and $E \to M$ a holomorphic vector bundle. Then for $q > 0$ and sufficiently large μ,*

$$H^q(M, \mathcal{O}(L^\mu \otimes E)) = 0.$$

Comment: The case when E is a line bundle follows trivially from Kodaira vanishing theorem, since we can choose μ large enough to make $L^\mu \otimes E \otimes K_M^*$ positive.

Proof: By Kodaira-Serre duality and Dolbeault theorem, we can translate the theorem into

$$\mathcal{H}^{0,p}(L^{-\mu} \otimes E') = 0$$

for $p < n$ and sufficiently large μ, where $E' = E^* \otimes K_M$. Since E was arbitrary, we'll replace E' by E. Pick a metric on L such that $\omega = \frac{i}{2\pi}\Theta_L$ is a positive form. Then the curvature form of $L^{-\mu} \otimes E$ is

$$\Theta_{L^{-\mu}\otimes E} = 2\pi i \mu \ \omega \otimes 1_E + \Theta_E$$

Take a harmonic form $\eta \in \mathcal{H}^{0,p}(L^{-\mu} \otimes E)$. The same calculation as in the proof of Kodaira vanishing theorem gives

$$2i([\Lambda, \Theta]\eta, \eta) \geq 0$$

It follows that

$$4\pi\mu(n-p)||\eta||^2 \leq 2i([\Lambda, \Theta_E]\eta, \eta) \leq C||\eta||^2$$

So for $p < n$, we can choose μ large enough so that $4\pi\mu(n-p) > C$. Then inequality then implies $\eta = 0$. **QED.**

We now use the above vanishing theorem to prove the following important lemma.

Lemma 3 *Every line bundle L over a submanifold $M \subset \mathbf{P}^N$ is associated to a divisor D, i.e. $L = [D]$.*

Proof: It suffices to show that every line bundle $L \to M$ admits a nontrivial global meromorphic section. Let H be the restriction of the hyperplane bundle over $\mathbf{P}^N$ to M. Then it suffices to prove that for sufficiently large μ, $L + \mu H$ has a nontrivial global holomorphic section, or $H^0(M, \mathcal{O}(L + \mu H)) \neq 0$.

Let's prove it by induction on $\dim M = n$. Suppose the assertion is true for submanifolds of $\mathbf{P}^N$ of dimension$< n$. In particular it holds for the intersection $V = M \cap \mathbf{P}^{N-1}$. By Bertini's theorem, V is smooth for generic choice of hyperplane $\mathbf{P}^{N-1}$, then we have $H = [V]$. There is an exact sequence of sheaves

$$0 \to \mathcal{O}_M(L + (\mu - 1)H) \xrightarrow{\otimes s} \mathcal{O}_M(L + \mu H) \to \mathcal{O}_V(L + \mu H) \to 0$$

where the first map is given by tensoring with the section s of H which vanishes along V, and the second map is the restriction. Part of the associated LES is

$$H^0(M, \mathcal{O}(L + \mu H)) \to H^0(V, \mathcal{O}(L + \mu H)) \to H^1(M, \mathcal{O}(L + (\mu - 1)H))$$

For sufficiently large μ, $H^1(M, \mathcal{O}(L + (\mu - 1)H)) = 0$ by the vanishing theorem (actually Kodaira vanishing theorem already suffices), and $H^0(V, \mathcal{O}(L + \mu H)) \neq 0$ by assumption of induction. It follows that $H^0(M, \mathcal{O}(L+\mu H)) \neq 0$. **QED.**

Finally we prove the Lefschetz theorem on $(1,1)$-classes.

Theorem 5 *For a submanifold $M \subset \mathbf{P}^N$, every cohomology class*

$$\gamma \in H^{1,1}(M) \cap H^2(M, \mathbf{Z})$$

is Poincaré dual to the fundamental class of a divisor D, namely $\gamma = \eta_D$.

Comments: The fundamental class of every $(n-p)$-dimensional subvariety V of an n-manifold is dual to a (p,p)-form, for the simple reason that only the integral of an $(n-p,n-p)$-form on V can be nonzero. Thus the converse of the theorem holds rather trivially.

Proof: Our goal is to find a line bundle L with $c_1(L) = \gamma$, then by lemma 3 $L = [D]$ for some divisor D, and then by the theorem proved in G&H section 1.1, $\gamma = c_1(L) = \eta_D$. The obstruction to finding such an L is clear from the exponential sheaf sequence. There is a diagram

$$\begin{tikzcd}
\mathrm{Pic}(M) = H^1(M,\mathcal{O}^*) \arrow[r] & H^2(M,\mathbf{Z}) \arrow[r] \arrow[d] & H^2(M,\mathcal{O}) \arrow[d, "\cong"] \\
 & H^2(M,\mathbf{C}) \arrow[r] & H^{0,2}_{\bar\partial}(M)
\end{tikzcd}$$

where the upper row is exact, and the lower row is given by the projection on forms. If we can show that this diagram commutes, we are done, since the image of $H^{1,1}(M) \cap H^2(M,\mathbf{Z}) \subset H^2(M,\mathbf{C})$ under the projection into $H^{0,2}(M) \cong H^2(M,\mathcal{O})$ is zero.

Let's show the commutativity by brutal force. Let $z = (z_{\alpha\beta\gamma})$ by a Čech cochain in $Z^2(M,\mathbf{Z}) \subset Z^2(M,\mathbf{C})$. Recall that the de Rham isomorphism is given by

$$H^2(\mathbf{C}) \xrightarrow{\delta^*} H^1(\mathcal{Z}^1_d) \xrightarrow{\delta^*} H^0(\mathcal{Z}^2_d)/dH^0(\mathcal{Z}^1_d) = H^2_{DR}(\mathbf{C})$$

Locally write $z_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha}$, then $(df_{\alpha\beta})$ is a closed cochain in $Z^1(\mathcal{Z}^1_d)$. It represents the image of z in $H^1(\mathcal{Z}^1_d)$. Now locally we can write $df_{\alpha\beta} = \omega_\beta - \omega_\alpha$, and $(d\omega_\alpha)$ represents the image of z in H^2_{DR}.

The Dolbeault isomorphism is given by

$$H^2(\mathcal{O}) \xrightarrow{\delta^*} H^1(\mathcal{Z}^{0,1}_{\bar\partial}) \xrightarrow{\delta^*} H^0(\mathcal{Z}^{0,2}_{\bar\partial})/dH^0(\mathcal{Z}^{0,1}_{\bar\partial}) = H^{0,2}_{\bar\partial}(\mathbf{C})$$

Consider the image of z under $H^2(\mathbf{C}) \to H^2(\mathcal{O})$. It is represented by $(\bar\partial f_{\alpha\beta}) \in Z^1(\mathcal{Z}^{0,1}_{\bar\partial})$. Now we can write locally $\bar\partial f_{\alpha\beta} = \omega^{0,1}_\beta - \omega^{0,1}_\alpha$, where $\omega^{0,1}_\alpha = \pi^{(0,1)}\omega_\alpha$. $(\bar\partial\omega^{0,1}) = (d\omega_\alpha)^{0,2}$ then represents of the image of z in $H^{0,2}_{\bar\partial}$. So the de Rham isomorphism followed by projection $\pi^{(0,2)}$ is the same as the Dolbeault isomorphism. This finishes the proof. **QED.**