Physics 253b Section Notes # 8

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Topological Yang-Mills Theory

The simplest topological action for non-Abelian gauge field A is

$$S = \int_M \mathrm{Tr} F \wedge F = \frac{1}{4} \int_M d^4x \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} F_{\mu\nu} F_{\rho\sigma}$$

where M is an oriented 4-manifold. This action is manifestly independent of the metric $g_{\mu\nu}$. If we can make sense of it as a full quantum theory, the observables - correlation functions of gauge invariant operators, will be independent of the metric and give topological invariants of the "spacetime" manifold M.

Naively the path integral with this action is ill defined, due to the enormous number of gauge symmetries. In addition to the "ordinary" gauge symmetry, the action is invariant under general coordinate transformation. A general infinitesimal gauge transformation is parameterized by Lie algebra valued functions $\varepsilon_\mu(x)$ and $\varepsilon(x)$,

$$\begin{aligned} \delta A_\mu(x) &= D_\mu \varepsilon(x) + \varepsilon_\mu(x) \\ &= \partial_\mu \varepsilon + [A_\mu, \varepsilon] + \varepsilon_\mu. \end{aligned}$$

Let's first analyze the ghost fields we need for gauge fixing. To fix the gauge parameter ε_μ and ε we need fermionic ghosts ψ_μ, c. We will often use the 1-form $\psi = \psi_\mu dx^\mu$.

One further observes that there is a redundancy in the parameterization of the gauge transformation above,

$$\varepsilon_\mu \to \varepsilon_\mu + D_\mu \lambda, \qquad \varepsilon \to \varepsilon - \lambda.$$

This redundancy gives a gauge symmetry on the ghosts ψ_μ, c. To fix this gauge symmetry we need to introduce ghost of the ghost, a bosonic field ϕ. The numbers of the degrees of freedom for the fields are listed below.

A_μ	ψ_μ	c	ϕ
4	-4	-1	$+1$

They are assembled conveniently in the following table

$$\begin{array}{ccccc} & A & & & F = dA + A^2 \\ c & & & \psi & \\ & & \phi & & \end{array}$$

where the ghost number increases towards left, and the degree of forms increases as one moves upward.

The BRST operator s should act on the gauge field A by the gauge transformation with gauge parameters $\varepsilon_\mu, \varepsilon$ replaced by ghosts ψ_μ, c. For example,

$$\delta A_\mu = D_\mu \varepsilon + \varepsilon_\mu \ \longrightarrow \ sA = -Dc + \psi$$

It must also be nilpotent

$$s^2 = 0.$$

We will use [,] to denote the graded (anti)commutator. An operator $\mathcal{O}$ is even (odd) if its degree as a form plus ghost number is even (odd). For example, (A, c) are odd, and (F, ψ, ϕ) are even. Equivalently, we can think of [,] as commutator only on the Lie algebra. As an example,

$$c = c^a T^a, \quad \frac{1}{2}[c, c]^a = (cc)^a = \frac{i}{2} f^{abc} c^b c^c.$$

The BRST transformations are determined by the conditions stated above,

$$\begin{aligned} sA &= -Dc + \psi, \\ sc &= -\frac{1}{2}[c, c] + \phi, \\ s\psi &= -D\phi - [c, \psi], \\ s\phi &= -[c, \phi], \\ (sF &= D\phi - [c, F]) \end{aligned}$$

Note that s anticommutes with d. Let us check the nilpotency for s acting on A.

$$\begin{aligned} s^2 A &= -s(Dc) + s\psi \\ &= d(sc) - [sA, c] + [A, sc] + s\psi \\ &= -\frac{1}{2} d[c, c] + d\phi - [-Dc + \psi, c] + [A, -\frac{1}{2}[c, c] + \phi] - D\phi - [c, \psi] \\ &= -\frac{1}{2} d[c, c] - \frac{1}{2}[A, [c, c]] + [Dc, c] \\ &\quad + d\phi + [A, \phi] - D\phi - [\psi, c] - [c, \psi] \\ &= 0. \end{aligned}$$

Now if we assemble the fields into

$$\mathcal{A} = A + c, \quad \mathcal{F} = F + \psi + \phi,$$

The BRST equations can be rewritten in the compact form

$$
\begin{aligned}
(d+s)\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}] &= \mathcal{F}, \\
(d+s)\mathcal{F} + [\mathcal{A},\mathcal{F}] &= 0.
\end{aligned}
$$

We see that $\mathcal{A}$ is some sort of generalized gauge field multiplet, and $\mathcal{F}$ is the corresponding field strength. The nilpotency condition $(d+s)^2 = 0$ is simply the Bianchi identity for $\mathcal{F}$.

Using the Bianchi identity, it is easy to see that

$$
(d+s)\mathrm{Tr}\mathcal{F}\mathcal{F} = 0.
$$

where, as always, we have omitted the wedge product $\wedge$. We can derive from above a string of relations

$$
s \cdot \mathcal{O}_i = d\mathcal{O}_{i-1}, \quad i = 0, \cdots, 4.
$$

where each $\mathcal{O}_i$ is a degree i form defined as follows

$$
\begin{aligned}
\mathcal{O}_0 &= -\frac{1}{2}\mathrm{Tr}\phi\phi, \\
\mathcal{O}_1 &= \mathrm{Tr}\psi\phi, \\
\mathcal{O}_2 &= \mathrm{Tr}(-\frac{1}{2}\psi\psi - \phi F), \\
\mathcal{O}_3 &= -2\mathrm{Tr}\psi F, \\
\mathcal{O}_4 &= \mathrm{Tr}FF.
\end{aligned}
$$

Now we can introduce the gauge fixing terms. We will fix the following functions

$$
\partial_\mu A^\mu, \quad F^{\mu\nu} \pm \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \quad D_\mu\psi^\mu.
$$

by introducing multiplers $b, b_{\mu\nu}, \bar{\eta}$. For example,

$$
S_{GF} \supset \int d^4x\, b\,(\partial_\mu A^\mu)
$$

corresponds to the Lorentz gauge. To make the gauge-fixing part of the action BRST invariant, we will take $b = s\bar{c}$ for some "anti-ghost" $\bar{c}$, and similarly for $b_{\mu\nu}$ and $\bar{\eta}$. The name "anti-ghost" comes from the fact that they have negative ghost number.

These are summarized in the following table

$$\begin{tikzcd}
\partial_\mu A^\mu \arrow[d, leftrightarrow] & F^{\mu\nu} \pm \tilde{F}^{\mu\nu} \arrow[d, leftrightarrow] & D_\mu\psi^\mu \arrow[d, leftrightarrow] \\
b\ \ (0) \arrow[d] & b_{\mu\nu}\ \ (0) \arrow[d] & \bar{\eta}\ \ (-1) \arrow[d] \\
\bar{c}\ \ (-1) & \bar{\chi}_{\mu\nu}\ \ (-1) & \bar{\phi}\ \ (-2)
\end{tikzcd}$$

where the numbers in the parenthesis denotes the ghost numbers. The BRST transformations on the anti-ghosts are

$$\begin{array}{ll} s\bar{c} = b, & sb = 0, \\ s\bar{\chi}_{\mu\nu} = b_{\mu\nu}, & sb_{\mu\nu} = 0, \\ s\bar{\phi} = \bar{\eta}, & s\bar{\eta} = 0. \end{array}$$

Our field content can be summarized as

$$\begin{array}{cccccccc} & A & & & & F & & \\ c & & \bar{c} & & \psi & & \bar{\chi}_{\mu\nu} & \\ & b & & \phi & & b_{\mu\nu} & & \bar{\phi} \\ & & & & & & \bar{\eta} & \end{array}$$

where the ghost number increases toward left.

We can now write down the path integral

$$Z = \int dA\, dc\, d\bar{c}\, db\, d\psi_\mu\, d\bar{\chi}_{\mu\nu}\, db_{\mu\nu}\, d\phi\, d\bar{\phi}\, d\bar{\eta}\, e^{-S}$$

where

$$S = \int_M \mathrm{Tr} F \wedge F + S_{GF}$$

A convenient choice of the gauge fixing term is

$$\begin{aligned} \mathcal{L}_{GF} &= s \cdot \mathcal{V} \\ &= s \cdot \mathrm{Tr}\left[\bar{\chi}_{\mu\nu}(F^{\mu\nu} \pm \tilde{F}^{\mu\nu}) \pm \frac{1}{2}\bar{\chi}_{\mu\nu} b^{\mu\nu} + \bar{\phi} D_\mu \psi^\mu + \bar{c}\partial_\mu A^\mu + \frac{1}{2}\bar{c} b\right] \\ &= \mathrm{Tr}\Big\{ b_{\mu\nu}(F^{\mu\nu} \pm \tilde{F}^{\mu\nu}) \pm \frac{1}{2} b_{\mu\nu} b^{\mu\nu} - \bar{\chi}_{\mu\nu}(D^{[\mu}\psi^{\nu]} \pm \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} D_{[\rho}\psi_{\sigma]}) \\ &\quad -\bar{\chi}_{\mu\nu}[c, F^{\mu\nu} \pm \tilde{F}^{\mu\nu}] + \bar{\eta} D_\mu \psi^\mu + \bar{\phi}[\psi_\mu, \psi^\mu] + \bar{\phi} D_\mu D^\mu \phi \\ &\quad + b\partial_\mu A^\mu + \frac{1}{2} b^2 - \bar{c}\partial_\mu D^\mu c - \bar{c}\partial_\mu \psi^\mu \Big\} \end{aligned}$$

This is a rather complicated expression. Nevertheless we can see the "ordinary" gauge fixing term $b\partial_\mu A^\mu + \frac{1}{2}b^2$ (integrating out b), and the ghost term $-\bar{c}\partial_\mu D^\mu c$ as in ordinary non-Abelian gauge theories.

The action is quadratic in $b_{\mu\nu}$ and b. We can integrate them out and write

$$Z = \int dA\, dc\, d\bar{c}\, d\psi_\mu\, d\bar{\chi}_{\mu\nu}\, d\phi\, d\bar{\phi}\, d\bar{\eta}\, e^{-S'}$$

where

$$\begin{aligned} S' &= \int_M d^4x \mathrm{Tr}\Big\{ F^{\mu\nu} F_{\mu\nu} - \bar{\chi}_{\mu\nu}(D^{[\mu}\psi^{\nu]} \pm \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} D_{[\rho}\psi_{\sigma]}) \\ &\quad -\bar{\chi}_{\mu\nu}[c, F^{\mu\nu} \pm \tilde{F}^{\mu\nu}] + \bar{\eta} D_\mu \psi^\mu + \bar{\phi}[\psi_\mu, \psi^\mu] \\ &\quad + \bar{\phi} D_\mu D^\mu \phi - \frac{1}{2}(\partial_\mu A^\mu)^2 - \bar{c}\partial_\mu D^\mu c - \bar{c}\partial_\mu \psi^\mu \Big\} \end{aligned}$$

The original "gauge invariant" part of the action $\int \mathrm{Tr} F \wedge F$ is cancelled by terms that appears when we integrate out $b_{\mu\nu}$. S' is our final gauge fixed action of topological Yang-Mills theory! Started with the topological action $\int \mathrm{Tr} F \wedge F$, after a series of gauge fixing we obtain the ordinary Yang-Mills plus ghosts interactions.

Why is this theory "topological"? Recall that the action takes the form

$$S = \int \mathrm{Tr} F \wedge F + s \cdot \mathcal{V}$$

The only dependence on the metric $g_{\mu\nu}$ comes from $\mathcal{V}$, where we had to invoke the metric to fix the gauge. We are interested in correlators of BRST invariant operators, namely those satisfying

$$s \cdot \mathcal{O} = 0 \qquad (\text{or } [Q, \mathcal{O}] = 0)$$

These operators can be constructed from the local operator $\mathcal{O}_i$, which satisfy

$$s \cdot \mathcal{O}_i = d\mathcal{O}_{i-1}$$

as we have already seen. $\mathcal{O}_i$ is an i-form, we can integrate it on an i-dimensional closed submanifold Σ_i of M. Then

$$s \cdot \int_{\Sigma_i} \mathcal{O}_i = \int_{\Sigma} d\mathcal{O}_{i-1} = 0$$

We have correlation functions

$$\langle \int_{\Sigma} \mathcal{O} \cdots \rangle = \int [d\varphi]\, e^{-(\int \mathrm{Tr} F \wedge F + s \cdot \mathcal{V})} \int_{\Sigma} \mathcal{O} \cdots$$

The variation with respect to the metric is

$$\begin{aligned} \frac{\delta}{\delta g_{\mu\nu}} \langle \int_{\Sigma} \mathcal{O} \cdots \rangle &= \int [d\varphi]\, e^{-S} (s \cdot \frac{\delta \mathcal{V}}{\delta g_{\mu\nu}}) \int_{\Sigma} \mathcal{O} \cdots \\ &= \int [d\varphi]\, e^{-S} \frac{\delta \mathcal{V}}{\delta g_{\mu\nu}} (s \cdot \int_{\Sigma} \mathcal{O}) \cdots + \cdots \\ &= 0, \end{aligned}$$

where we used the fact

$$\int [d\varphi]\, e^{-S} s \cdot \mathcal{O} = 0, \qquad \forall \mathcal{O},$$

which follows from the BRST invariance of the action S and the path integral measure $[d\varphi]$. In fact one can show that there is no perturbative anomaly in the measure. For further details the reader is referred to L. Baulieu and I.M. Singer, *Topological Yang-Mills Symmetry*, Nucl.Phys.Proc.Suppl.**5B**, 12 (1988). Also check out Witten's lecture on cohomological field theories.