Physics 253b Section Notes # 7

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General remarks on the geometry of gauge fields

We know that gauge fields do not have to be globally well defined. For example, in the presence of a Dirac monopole, one cannot write down a globally well defined $U(1)$ gauge field. If one tries to do this, the gauge field will become singular along a line starting from the origin of the monopole and extends to infinity, known as the "Dirac string". This is related to the nontrivial topology of the spacetime, since the origin of the monopole is not included. What one can do is to take two coordinate patches U_+ and U_-, which do not contain the negative z-axis and positive z-axis, respectively. On U_+, the gauge field can be written in a particular gauge as

$$\vec{A} = Q\left(-\frac{y}{r(r+z)}, \frac{x}{r(r+z)}, 0\right)$$

On U_-, we can write the gauge field as

$$\vec{A}' = Q\left(\frac{y}{r(r-z)}, -\frac{x}{r(r-z)}, 0\right)$$

They do not agree on the overlap $U_+ \cap U_-$. This is not a problem, as long as they are related by a gauge transformation, in this case

$$\vec{A}' - \vec{A} = -2Q\,\vec{\nabla}\tan^{-1}(y/x)$$

So in spacetimes of nontrivial topology, we only expect the gauge field A and matter field ψ to be defined on patches of the spacetimes, and on the overlap of different patches they are related by gauge transformations.

In the non-Abelian case, let us consider a gauge field $A = A^a T^a$ with gauge group G, and matter field $\psi = (\psi_1, \cdots, \psi_n)$ in an n-dimensional representation R. R is a group homomorphism from G to $GL(n, \mathbf{R})$, namely it specifies the group action of G on an n-dimensional vector space V. Now at each point in the space, we can think of the field ψ as living in a copy of vector space V, on which the gauge transformations act. We can pick a coordinate patch U_α in spacetime, and the matter field on U_α is $\psi_{(\alpha)}$ (in a particular gauge). On a different patch U_β, we have matter field $\psi_{(\beta)}$. On the overlap $U_\alpha \cap U_\beta$, $\psi_{(\alpha)}$ and $\psi_{(\beta)}$ are in general not necessarily the same - we made different choices of gauge on different patches. So we can only demand them to be related by gauge transformation

$$\psi_{(\alpha)} = g_{(\alpha\beta)}\psi_{(\beta)} \qquad \text{on } U_\alpha \cap U_\beta.$$

We see that although at each point x, ψ takes value in a representation space V_x, there are no natural identification between V_x and V_y for different points x and y. Rather we should think of each V_x as a "fiber" growing upon the point x, and this structure of the spacetime manifold together with the fibers is known as a vector bundle E. Such ψ is called a section of the vector bundle E. For example, in GR vectors live in the tangent bundle.

Just like the Levi-Civita connection in GR, the gauge field A is a connection on the vector bundle E. On U_α we can write the gauge field as

$$A_{(\alpha)} = A_{\mu(\alpha)} dx^\mu$$

On the overlap $U_\alpha \cap U_\beta$, the gauge fields are related by gauge transformation

$$A_{\mu(\alpha)} = g_{(\alpha\beta)} A_{\mu(\beta)} g^{-1}_{(\alpha\beta)} + g_{(\alpha\beta)} \partial_\mu g^{-1}_{(\alpha\beta)}.$$

Then we have covariant derivative

$$D_\mu = \partial_\mu + A_\mu$$

and field strength (curvature)

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Written in forms, this is

$$F = dA + A \wedge A$$

where $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$.

Connection between gravity and gauge theory

Classical general relativity can be formulated in terms of vierbein (tetrad) e_i^a and spin connection $\omega_i{}^a{}_b$, where we use $i, j, \cdots$ for spacetime indices and $a, b, \cdots$ for local Lorentzian frame indices. The metric is determined from the vierbein by

$$g_{ij} = \eta_{ab} e_i^a e_j^b, \quad g^{ij} = \eta^{ab} e_a^i e_b^j.$$

The spin connection is defined as

$$\omega_i{}^a{}_b = e^a_j \nabla_i e^j_b = -e^j_b \nabla_i e^a_j.$$

It is determined from the Levi-Civita connection. We will think of $e^a_i, \omega_i{}^a{}_b$ as gauge fields, with $a, b, \cdots$ being internal indices, acted by the Lorentz group $SO(d-1,1)$. It will become clear why this is a sensible thing to do.

The Riemann tensor can be written as

$$\begin{aligned} R_{ij}{}^a{}_b &= R_{ij}{}^k{}_l e^a_k e^b_l \\ &= \partial_i \omega_j{}^a{}_b - \partial_j \omega_i{}^a{}_b + \omega_i{}^a{}_c \omega_j{}^c{}_b - \omega_j{}^a{}_c \omega_i{}^c{}_b \\ &= \partial_i \omega_j{}^a{}_b - \partial_j \omega_i{}^a{}_b + [\omega_i, \omega_j]^a{}_b \end{aligned}$$

We see that this is nothing but the field strength (curvature) associated to the Lie algebra $so(d-1,1)$-valued gauge field ω_i. We can assemble them into forms

$$\omega^a{}_b = \omega_i{}^a{}_b dx^i, \quad R^a{}_b = \frac{1}{2} R_{ij}{}^a{}_b dx^i \wedge dx^j.$$

Then

$$R = d\omega + \omega \wedge \omega, \qquad \text{where } (\omega \wedge \omega)^a{}_b = \omega^a{}_c \wedge \omega^c{}_b.$$

The Einstein-Hilbert action is 3+1 dimensions can be written as

$$S = \frac{1}{4} \int_M \epsilon^{ijkl} \epsilon_{abcd} (e^a_i e^b_j R_{kl}{}^{cd})$$

Let's check that it is equivalent to the ordinary form

$$\begin{aligned} S &= \frac{1}{4} \int \epsilon^{ijkl} \epsilon_{abcd} e^a_i e^b_j e^c_m e^d_n R_{kl}{}^{mn} \\ &= \frac{1}{4} \int \epsilon^{ijkl} \epsilon_{ijmn} (\det e) R_{kl}{}^{mn} \\ &= \frac{1}{4} \int 2(\delta^k_m \delta^l_n - \delta^k_n \delta^l_m)(\det e) R_{kl}{}^{mn} \\ &= \int \sqrt{-g} R. \end{aligned}$$

We can also rewrite the action in terms of forms

$$S = \frac{1}{2} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}$$

It is natural to expect e_i^a and ${\omega_i}^a{}_b$ to be gauge fields associated to local translation (diffeomorphism) and local Lorentz transformation, respectively. They combine to give a gauge field associated to local Poincare invariance, so the gauge group is the Poincare group $ISO(d-1,1)$.

If this is the case, under infinitesimal translation ξ^a and local Lorentz rotation $v^a{}_b$, the vierbein and spin connection should transform as

$$\begin{aligned} \delta e_i^a &= D_i\xi^a + v^a{}_b e_i^b = \partial_i \xi^a + {\omega_i}^a{}_b \xi^b + v^a{}_b e_i^b, \\ \delta {\omega_i}^a{}_b &= D_i v^a{}_b = \partial_i v_b^a + [\omega_i, v]^a{}_b. \end{aligned}$$

as $ISO(d-1,1)$ gauge fields. Actually this is not quite the same as general coordinate transformation, but they are equivalent when the fields are on-shell. Namely they are the same up to terms that vanish when the equations of motion are imposed.

The equations of motion are obtained by

$$\text{varying } \omega \;\Rightarrow\; D_i e_j^a - D_j e_i^a = 0, \qquad \text{torsion free;}$$

and

$$\text{varying } e \;\Rightarrow\; e_a^j {R_{ij}}^a{}_b = 0, \qquad \text{vacuum Einstein equation.}$$

The trouble of actually treating 4D gravity as a gauge theory is that, the Einstein-Hilbert action is not the usual action of gauge theories.

Surprises in 2+1 dimensions

In 2+1 dimensions, the Einstein-Hilbert action can be written as

$$\begin{aligned} S &= \frac{1}{2}\int \epsilon^{ijk}\epsilon_{abc} e_i^a {R_{jk}}^{bc} \\ &= \frac{1}{2}\int \epsilon^{ijk}\epsilon_{abc} e_i^a (\partial_j {\omega_k}^{bc} - \partial_k {\omega_j}^{bc} + [\omega_i, \omega_j]^{bc}) \end{aligned}$$

This action has the form $\int AdA + A^3$. There is a well known gauge theory in 2+1 dimensions of this type, the Chern-Simons theory.

Consider a gauge field A locally as a Lie algebra valued 1-form on 3-manifold M. The Chern-Simons action is

$$S = \frac{1}{2}\int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

There can be an overall factor k in front of the action, playing the role of a coupling constant, which we have omitted for notational simplicity. The "Tr" can be any invariant nondegenrate bilinear form on the Lie algebra. For example, if we expand the gauge field on generators as $A = A^a T^a$, then the first term in the action is

$$\int \mathrm{Tr} A \wedge dA = \mathrm{Tr}(T^a T^b) \int A^a \wedge dA^b = d^{ab} \int A^a \wedge dA^b$$

where d^{ab} is a nondegenerate bilinear form.

In general the gauge field is only locally defined. So how do we know that the Chern-Simons action is well-defined? In fact it isn't, namely the action is not invariant under "large" (topologically nontrivial) gauge transformations. Nevertheless the action is well defined up to some fixed period, and if we choose the coupling constant appropriately so that this period is 2π, then the phase factor e^{iS} in the path integral is well defined. This also gives a quantization of the coupling constant. We will not explore the full quantum theory at this moment.

One way to define the action properly is the following. We can imagine M being the boundary of a 4-manifold Y, $M = \partial Y$, and extend the gauge field onto Y (this is always possible for oriented 3-manifolds). Then using

$$d\mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = \mathrm{Tr} F \wedge F, \qquad F = dA + A \wedge A,$$

and Stokes theorem, we can rewrite the action as an integral on the 4-manifold Y,

$$S = \frac{1}{2} \int_Y \mathrm{Tr} F \wedge F$$

Now the trace over $F \wedge F$ is clearly gauge invariant and well-defined. However we can have different choices of N. One can show that the difference of the action for different choices of N differ by integer multiplets of $4\pi^2$ (with proper normalization of Tr).

What about gauge symmetry? The transformation associated to an infinitesimal gauge parameter ξ (a Lie algebra valued function) is

$$A \to A + D\xi = A + d\xi + [A, \xi]$$

The variation of the Lagrangian is

$$\delta \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = \mathrm{Tr}(\delta A \wedge dA + A \wedge d\delta A + 2\delta A \wedge A \wedge A)$$

where we have used the cylic property of the trace. Now by integration by part, this becomes simply

$$\begin{aligned} 2\mathrm{Tr}(\delta A \wedge F) &= 2\mathrm{Tr}(d\xi + A\xi - \xi A) \wedge F \\ &\to -2\mathrm{Tr}\xi(dF - F \wedge A + A \wedge F) \\ &= 0 \end{aligned}$$

where we used the Bianchi identity in the last step. This shows that the Chern-Simons action is indeed invariant under infinitesimal gauge transformations.

We have seen that the natural gauge group that arises in the vierbein formulation of 2+1 dimensional gravity is the Poincare group $ISO(2,1)$. The Poincare algebra has generators translation P^a and rotation $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$. They satisfy

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc}J^c, \\ [J_a, P_b] &= \epsilon_{abc}P^c, \\ [P_a, P_b] &= 0. \end{aligned}$$

The “trace” is defined by

$$\mathrm{Tr}J_aP_b = \delta_{ab}, \quad \mathrm{Tr}J_aJ_b = \mathrm{Tr}P_aP_b = 0.$$

Note that such invariant nondegenerate bilinear form on $ISO(d-1,1)$ exists for $d = 3$ but not for $d = 4$. Now we expand the gauge field A on the $ISO(2,1)$ generators as

$$A_i = e_i^a P_a + \omega_i^a J_a, \qquad \omega_i^a = \frac{1}{2}\epsilon^{abc}\omega_{ibc}.$$

The punch line is that, as one would guess, the Chern-Simons action reproduces the Einstein-Hilbert action in 2+1 dimensions. e_i^a and ω_{iab} appearing in the gauge field are nothing but the vierbein and the spin connection, as our notation has already indicated. The gauge transformations in Chern-Simons theory are the same as the general coordinate transformations on the vierbein and spin connection *up to terms that vanish on shell.*

Now let's check this by explicitly evaluating the action. The field strength is

$$F_{ij} = [D_i, D_j]$$

$$
\begin{aligned}
&= \partial_i A_j - \partial_j A_i + [A_i, A_j] \\
&= P_a \left(\partial_i e^a_j - \partial_j e^a_i + \epsilon^{abc}(\omega_{ib} e_{jc} - \omega_{jb} e_{ic}) \right) \\
&\quad + J_a \left(\partial_i \omega^a_j - \partial_j \omega^a_i + \epsilon^{abc} \omega_{ib} \omega_{jc} \right).
\end{aligned}
$$

We will still use the trick that the Chern-Simons action is the same as the integral of $\frac{1}{2}\mathrm{Tr} F \wedge F$ on a 4-manifold Y with $\partial Y = M$. Now

$$
\begin{aligned}
\int_Y \mathrm{Tr} F \wedge F &= \frac{1}{4} \int_Y \epsilon^{ijkl} \mathrm{Tr} F_{ij} F_{kl} \\
&= \frac{1}{2} \int_Y \epsilon^{ijkl} \left(\partial_i e^a_j - \partial_j e^a_i + \epsilon^{abc}(\omega_{ib} e_{jc} - \omega_{jb} e_{ic}) \right) \left(\partial_k \omega_{la} - \partial_l \omega_{ka} + \epsilon_{ade} \omega^d_k \omega^e_l \right) \\
&= \int_Y \epsilon^{ijkl} \partial_i \left[e^a_j (\partial_k \omega_{la} - \partial_l \omega_{ka} + \epsilon_{ade} \omega^d_k \omega^e_l) \right] \\
&= \int_M \epsilon^{ijk} e_{ia} (\partial_j \omega^a_k - \partial_k \omega^a_j + \epsilon^{abc} \omega_{jb} \omega_{kc}).
\end{aligned}
$$

It is clear that the $ISO(2,1)$ Chern-Simons action is the same as the Einstein-Hilbert action in vierbein formalism.