Notes on BFSS Matrix Theory

Xi Yin
Harvard University

D0-branes in M-theory

Type IIA string at strong coupling is dual to 11-dimensional M-theory compactified on a circle, whose radius R is related to the string coupling by

$$R = g^{2/3} l_P = g l_s$$

where $l_s = \alpha'^{1/2}$ is the string length scale, l_P is the Plank length.

D0-branes correspond to BPS momentum modes in X^{11}-direction, $p_{11} = 1/R$. In type IIA theory they have 2^8 massless excitations generated by the 8 broken supersymmetries. In M-theory this corresponds to an ultrashort massless graviton multiplet running in X^{11}-direction. BPS modes of momentum $p_{11} = N/R$ corresponds to bound states of N D0-branes. D-branes of higher dimensions and NS-5 brane are realized as M2- and M5-branes either transverse to X^{11} or wrapped on the S^1.

In weakly coupled string theory, the dynamics of N D0-branes can be described by the gauge-fixed light cone Lagrangian

$$L = \frac{1}{2g} \operatorname{Tr} \left(\dot{X}^i \dot{X}^i + 2\theta^T \dot{\theta} - \frac{1}{2}[X^i, X^j]^2 - 2\theta^T \gamma_i [\theta, X^i] \right)$$

where spatial coordinates X^i's are $N \times N$ matrices, θ is an $N \times N$ matrix of $SO(9)$ spinors. This can be thought as 10-dimensional SYM reduced to 0 spatial dimensions. In Plank units, we can write the Lagrangian in covariant form

$$L = \operatorname{Tr} \left(\frac{1}{2R} D_t Y^i D_t Y^i - \frac{1}{4} R [Y^i, Y^j]^2 - \theta^T D_t \theta - R \theta^T \gamma_i [\theta, Y^i] \right)$$

where $Y^i = g^{-1/3} X^i$, $D_t = \partial_t + iA$ (A can be gauged away). The supersymmetry transformation can be written as

$$\begin{aligned}
\delta X^i &= -2\epsilon^T \gamma^i \theta \\
\delta \theta &= \frac{1}{2} \left(D_t X^i \gamma_i + \gamma_- + \frac{1}{2}[X^i, X^j] \gamma_{ij} \right) \epsilon + \epsilon' \\
\delta A &= -2\epsilon^T \theta
\end{aligned}$$

The Hamiltonian is

$$H = R \operatorname{Tr} \left(\frac{1}{2} \Pi_i \Pi_i + \frac{1}{4}[Y^i, Y^j]^2 + \theta^T \gamma_i [\theta, Y^i] \right) \tag{1}$$

When Y^i's become large, corresponding to D0-branes separated at large distances, finite energy configuration requires the VEV of Y^i's to be mutual commuting.

They can be diagonized simultaneously and become coordinates for the N D0-branes. On the other hand, as we'll see later, in the short distance regime the dynamics of D0-branes are govern by noncommutative geometry on the membrane.

Discrete light cone quantization of M-theory

Label the spacetime coordinates $X^\mu = (t, z, X^i)$, and introduce light cone coordinate $X^\pm = \frac{1}{\sqrt{2}}(t \pm z)$. In the light cone frame, X^+ is regarded as time variable. For a system of mass M, the light cone Hamiltonian is

$$H = P_+ = \frac{P_i^2 + M^2}{2P_-}$$

This looks like a non-relativistic relation. The subgroup of the Poincare group including rotations and boosts in transverse directions is isomorphic to the Galilean group. For example, the Galilean boost acts as

$$P_i \to P_i + P_- V_i$$

The extended Supergalilean group has 32 supergenerators, splitting into 2 $SO(9)$ spinors under the transverse rotation group. They satisfy commutation relation

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= \delta_{\alpha\beta} H = \delta_{\alpha\beta} P_+, \\
\{q_\alpha, q_\beta\} &= \delta_{\alpha\beta} P_- \\
\{Q_\alpha, q_\beta\} &= \gamma^i_{\alpha\beta} P_i
\end{aligned}$$

The discrete light cone quantization (DLCQ) starts by compactifying X^- on a circle of radius R. The momentum is discretized as $P_- = N/R$. Since P_- is conserved, the system splits into superselection sectors characterized by N. The decompactification limit is obtained by taking $N, R \to \infty$ while holding P_- fixed.

It is useful to first work in the frame

$$ds^2 = d\tilde{X}^+ d\tilde{X}^- - \varepsilon^2 d\tilde{X}^- d\tilde{X}^- - dX^i dX^i$$

and then take the limit $\varepsilon \to 0$. To reproduce light like coordinate X^- of period R, the theory is compactified on a spacelike circle of radius

$$R_c = \varepsilon R$$

The limit $\varepsilon \to 0$ is taken while holding N fixed, whereas $\tilde{P}_-$ goes to infinity. Then we can boost the system back to the original light cone frame where the compactification radius is R.

Suppose we compactify M-theory on a spacelike circle of radius R_c, and focus on the sector of momentum N/R_c. This corresponds to type IIA theory of coupling

$$g_s = \left(\frac{R_c}{l_P}\right)^{3/2}$$

and the sector with N units of D0-brane charge. In the limit $R_c \to 0$, the string coupling $g_s \to 0$, the mass of D0-branes goes to infinity. Therefore one would expect that the theory is equivalent to the Galilean quantum mechanics of D0-branes in 9 spatial dimensions.

In sum, the BFSS conjecture is that, M-theory can be formulated in a world with a compact lightlike direction and quantized by DLCQ (then take the large N limit); the sector of momentum $P_- = N/R$ is exactly described by the Hamiltonian (1) with 16 supersymmetries and $U(N)$ symmetry. For example, the Galilean boost acts as

$$\Pi \to \Pi + \frac{v}{R} \otimes \mathbf{1}$$

Scattering of supergravitons

Define the center of mass and total momentum

$$X_{c.m.} = \frac{1}{N}\mathrm{Tr}X, \quad P_{c.m.} = \mathrm{Tr}P$$

The Hamiltonian has the form

$$H = \frac{P_{c.m.}^2}{2P_-} + H_{rel}$$

As type IIA/M-theory duality requires normalizable threshold BPS bound states of D0-branes. For these states $H_{rel} = 0$, the energy is given by $\frac{P_{c.m.}^2}{2P_-}$. They are identified with massless 11-dimensional supergravitons.

We are in particular interested in the long range scattering of two supergravitons. The configuration

$$X^i = \begin{pmatrix} X^i_{N_1 \times N_1} & 0 \\ 0 & Y^i_{N_2 \times N_2} \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_{N_1 \times N_1} & 0 \\ 0 & \phi_{N_2 \times N_2} \end{pmatrix}$$

corresponds to two supergravitons of momentum N_1/R and N_2/R. Assuming they are far apart,

$$r = \left| \frac{1}{N_1}\mathrm{Tr}X - \frac{1}{N_2}\mathrm{Tr}Y \right| \gg l_P$$

In the background of X and Y, the Lagrangian for an off diagonal element w takes the form

$$L = \frac{1}{2R}\dot{w}^2 - R\,|X - Y|^2\, w^2$$

The off diagonal components behave like harmonic oscillators of frequency

$$\omega = R\,|X - Y|$$

At large distance, $\omega \to \infty$, one would expect w to be frozen to ground states. However the zero point energy is of order

$$N_1 N_2 \omega = N_1 N_2 R\,|X - Y|$$

This would imply confinement of supergravitons, which is certainly wrong. In fact the bosonic zero point energy is exactly cancelled by supersymmetry.

One can also expect to integrate out the massive W-bosons to obtain the effetive action for the slowly varying "Higgs" supergravitons. The interaction terms quadratic in $\dot{X}_{c.m.}$ and $\dot{Y}_{c.m.}$ vanish as a consequence of non-renormalization theorem. The lowest order interaction is quartic in velocities, which can be computed from Feynman diagrams as in SYM. The propagator for the W-boson w is

$$\Delta_F = \frac{R}{\omega^2 + m^2}$$

where the mass $m \simeq Rr$. The quartic interaction vertex is $\sim R$. The 1-loop contribution to the X^4 term from the bosonic part takes the form

$$\int \frac{d\omega}{(\omega^2 + m^2)^2} R^4 N_1 N_2$$

This is, however, cancelled by fermionic contribution by supersymmetry, since the BPS threshold requires the flat directions to be exact. The contribution to $\dot{X}^4$ interaction takes the form

$$L_{eff} \sim |\dot{r}|^4 \int \frac{d\omega}{(\omega^2 + m^2)^4} R^4 N_1 N_2 \sim \frac{R^4 N_1 N_2}{m^7} \dot{r}^4 = \frac{G_N N_1 N_2}{r^7 R^3} \left| \dot{X} - \dot{Y} \right|^4$$

This gives rise to an effective interaction Hamiltonian of the form

$$H_{eff} = R \left[\frac{P^2}{2P_-} + \frac{P'^2}{2P'_-} + \frac{a}{R^2} \frac{G_N P_- P'_-}{r^7} \left(\frac{P}{P_-} - \frac{P'}{P'_-} \right)^4 \right]$$

where a is a numerical constant that can be computed exactly. It has been shown that this result agrees with the tree level scattering amplitude in supergravity with no P_- exchange. The latter would correspond to processes with exchange of N. Higher matrix theory loop corrections would contains additional powers of N/r^3, the amplitude will be corrected by a function

$$F\left(\frac{N}{r^3}\right) = 1 + C_2 \frac{N}{r^3} + C_3 \left(\frac{N}{r^3}\right)^2 + \cdots$$

The physically relevant limit is $N \to \infty$ while holding r fixed (in Plank units). Hence the agreement of 1-loop calculation suggests a non-renormalization theorem to protect the amplitude from higher loop corrections. The proof is still unknown.

Background field calculation

Even though the diagramatic computation sketched above is instructive, the easiest way to obtain the graviton scattering amplitude is via background field method. Still consider the Lagrangian

$$L = \frac{1}{2R} \mathrm{Tr} \left(D_t X^i D_t X^i + \frac{1}{2}[X^i, X^j]^2 + i\theta^T \dot{\theta} - \theta^T \gamma_i [X^i, \theta] \right)$$

And expand the bosonic field around background B^i,

$$X^i = B^i + Y^i$$

Let's consider two single D0-branes moving toward each other along X^1-direction with relative velocity v and impact parameter b. The background is given by

$$B^1 = \frac{1}{2}\begin{pmatrix} vt & 0 \\ 0 & -vt \end{pmatrix}, \quad B^2 = \frac{1}{2}\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$$

where we neglected polarization issues. The system consists of 10 bosonic fields, Y^i and gauge field A, 16 fermions θ, together with 2 ghosts. We want to integrate out the off diagonal components, which are complex fields. The quadratic part of the Lagrangian of the off diagonal components has mass matrix for bosonic fields, ghosts, and fermions:

$$\Omega_b^2 = \mathrm{diag}\left\{\begin{pmatrix} r^2 & -2iv \\ 2iv & r^2 \end{pmatrix}, r^2, \cdots, r^2\right\}, \quad \Omega_g^2 = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \Omega_f^2 = r^2\mathbf{1}_{16\times16} + v\gamma_1$$

where $r^2 = b^2 + v^2\tau^2$ in the Euclidean theory ($t = -i\tau$). One can write down the exact propagator for the kinetic operator $-\partial_\tau^2 + b^2 + v^2\tau^2$, but this will not be necessarily when we are only computing the leading order contribution. At large distance (quasi-static), the effective potential can be approximated by

$$\begin{aligned}
V_{qs} &= \sum_b \omega_b - \sum_g \omega_g - \frac{1}{2}\sum_f \omega_f \\
&= \sqrt{r^2 + 2v} + \sqrt{r^2 - 2v} + 8r - 2r - \frac{1}{2}\left(8\sqrt{r^2 + v} + 8\sqrt{r^2 - v}\right) \\
&= -\frac{15}{16}\frac{v^4}{r^7} + \mathcal{O}\left(\frac{v^6}{r^{11}}\right)
\end{aligned}$$

M2- and M5-branes in matrix theory

Membranes are relatively easy to see in matrix theory. The world volume of the membrane is described by noncommutative geometry. Consider a noncommutative torus parameterized by angular coordinates p, q satisfying the relation

$$[p, q] = \frac{2\pi i}{N}$$

Define

$$U = e^{ip}, \quad V = e^{iq}$$

We have

$$UV = e^{\frac{2\pi i}{N}} VU$$

U, V can be represented as $N \times N$ matrices. Any $N \times N$ matrix Z has expansion

$$Z = \sum_{n,m=1}^{N} Z_{nm} U^n V^m$$

They can be regarded as functions on the noncommutative membrane. The classical limit corresponds to $N \to \infty$, when p, q become ordinary coordinates. In this limit we should replace

$$\begin{aligned} \frac{1}{N}\mathrm{Tr}Z &\to \int dpdqZ(p,q) \\ [Z, W] &\to \frac{i}{N}\{Z, W\}_{P.B.} \end{aligned}$$

In the semi-classical limit the Hamiltonian can be written as

$$H = \frac{R}{N}\int dp\, dq \left(\frac{1}{2}\Pi_i(p,q)^2 - \frac{1}{4}\{X^i, X^j\}^2_{P.B.} + \text{fermions}\right)$$

which is precisely the Hamiltonian for a classical membrane in the light cone frame. The energy is proportional to $R/N = 1/P_-$, which is the behavior of a localized object in the light cone frame.

To study infinite M2-brane, we can introduce IR cutoff L,

$$X^1 = \frac{pL}{2\pi}, \quad X^2 = \frac{qL}{2\pi}$$

The classical area is obtained by integrating $dX^1 \wedge dX^2$. The matrix generalization is

$$\mathrm{Tr}[X^1, X^2] \sim L^2$$

Now let $N = n^2$, and p, q be $n \times n$ matrices satisfying

$$[p, q] = \frac{2\pi i}{n}$$

Consider the configuration

$$\begin{aligned} X^1 = p \otimes \mathbf{1}_n L, &\quad X^2 = q \otimes \mathbf{1}_n L \\ X^3 = \mathbf{1}_n \otimes p\, L, &\quad X^4 = \mathbf{1}_n \otimes q\, L \end{aligned}$$

The energy grows as

$$E = \frac{R}{4}\mathrm{Tr}[X^i, X^j]^2 \sim RL^4$$

This is a 5-brane wrapped on the lightlike direction. It has "dissolved" 2-branes oriented in the 1-2 and 3-4 planes.

Compactification

Compactify X^9 on a circle of radius R_9. Consider a winding membrane satisfying

$$X^9(q, p + 2\pi) = X^9(q, p) + 2\pi R_9$$

In matrix representation we have

$$e^{-iNq} X^9 e^{iNq} = X^9 + 2\pi R_9$$

Obviously this can not be true for finite N. In the large N limit, $q \to \sigma/N$ for an angular variable σ. We have

$$X^9 = i\frac{\partial}{\partial\sigma} + \sum X_n^9 e^{in\sigma} = i\frac{\partial}{\partial\sigma} + A(\sigma)$$

In the rest of this section we write X^i for $i = 1, \cdots, 8$. The Hamiltonian becomes

$$H = R \int_0^{1/R_9} d\sigma \, \mathrm{Tr} \left\{ f^2 + (D\mathbf{X})^2 + [X^i, X^j]^2 + \theta^T[\gamma D + \mathbf{\Gamma X}, \theta] \right\}$$

where f is the field strength of A, γ's are $1+1$ dimensional Dirac matrices, θ transforms as $(L, 8_c) \oplus (R, 8_s)$ under $SO(1,1) \times SO(8)$, and some $SO(8)$ indices are suppressed in the expression. This is a $d = 2$ SYM with 16 supersymmetries.

In the limit $R_9 \to 0$, A can be gauged away, all X^i's commute with each other. The Hamiltonian reduces to

$$H = \int d\sigma \left[\dot{X}^2 + \left(\frac{\partial X}{\partial\sigma}\right)^2 + \theta^T \gamma_9 \frac{\partial\theta}{\partial\sigma} \right]$$

which is precisely the Green-Schwarz type IIA string!

References

1. T. Banks, W. Fischler, S. Shenker and L. Susskind, M Theory as a Matrix Model: a Conjecture, hep-th/9610043.

2. D. Bigatti and L. Susskind, Review of Matrix Theory, hep-th/9712072.

3. W. Taylor, M(atrix) Theory: Matrix Quantum Mechanics as a Fundamental Theory, hep-th/0101126.