Black Holes, Anti de Sitter space, and Topological Strings

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Abstract

This thesis is devoted to the study of black holes in string theory, their connection to two and three dimensional anti de-Sitter space, and topological strings. We start by proposing a relation between supersymmetric black holes in four and five dimensions, as well as connections between multi-centered black holes in four dimensions and black rings in five dimensions. This connection is then applied to counting supersymmetric dyonic black holes in four dimensional string compactifications with 16 and 32 supersymmetries, respectively. We then turn to the near horizon attractor geometry $AdS_2 \times S^2 \times CY_3$, and study the classical supersymmetric D-branes in this background. We also find supersymmetric black hole solutions in supergravity in $AdS_2 \times S^2$, although the solutions have regions of closed timelike curves. Finally we consider the M-theory attractor geometry $AdS_3 \times S^2 \times CY_3$, and compute the elliptic genus of the dual $(0, 4)$ CFT by counting wrapped M2-brane states in the bulk in a dilute gas approximation. This leads to a derivation of the conjectured relation between black hole partition function and topological string amplitudes.

Contents

Citations to Previously Published Work

The content of chapter 2 has appeared in the following two papers:

> "New Connections Between 4D and 5D Black Holes", D. Gaiotto, A. Strominger and X. Yin, JHEP 0602 (2006) 024, arXiv: hep-th/0503217;
>
> "5D Black Rings and 4D Black Holes", D. Gaiotto, A. Strominger and X. Yin, JHEP 0602 (2006) 023, arXiv: hep-th/0504126.

Most of chapter 3 has appeared in the following two arXiv preprints:

> "Recounting Dyons in N=4 String Theory", D. Shih, A. Strominger and X. Yin, arXiv: hep-th/0505094;
>
> "Counting Dyons in N=8 String Theory", D. Shih, A. Strominger and X. Yin, arXiv: hep-th/0506151.

The first two sections of chapter 4 are contained in the paper

> "Supersymmetric Branes in $AdS_2 \times S^2 \times CY_3$", A. Simons, A. Strominger, D. Thompson and X. Yin, Phys.Rev. **D71** (2005) 066008, arXiv: hep-th/0406121.

The last section of chapter 4 is contained in the yet unpublished work of D. Gaiotto, A. Strominger and the author. Most of chapter 5 has appeared in the arXiv preprint:

> "From AdS3/CFT2 to Black Holes/Topological String", D. Gaiotto, A. Strominger and X. Yin, arXiv: hep-th/0602046.

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dedicated to my parents Chuying and Jizhou

Chapter 1

Introduction

The quest for a quantum theory of gravity has been one of the central problems in theoretical physics for decades, and is certainly continuing to be. Black holes play a key role as our conceptual bridge toward understanding quantum gravity. Two of the basic questions are: what are the microstates of a black hole? and what happens when a particle falls into a black hole?

A lot progress has been made in string theory toward answering the first problem [66], at least for black holes that preserve certain amount of supersymmetries. Charged black holes in string compactifications typically have dual description in terms of D-branes. States of a black hole have dual descriptions in terms of the world volume theory that describes excitations of D-branes. Extremal black holes correspond to ground states of D-branes with given charges, which are relatively simple to describe. Most results on counting supersymmetric black holes are obtained by reducing the world volume theory of the D-brane to a 1+1 dimensional conformal field theory (CFT) on a circle. A precise counting of such states is one of the main

topics of this thesis.

The second problem mentioned above is perhaps conceptually more important, and much less understood. What happens to a particle that falls through the horizon of a black hole and hits the geometric singularity? In string theory black holes can have holographic dual descriptions in terms of states in gauge theories. It is mostly believed now that the infalling particle will eventually return to outside the black hole through Hawking radiation, although how this happens exactly and how such a unitary evolution can be compatible with semiclassical analysis remains a mystery. In the cases where the black hole has a unitary holographic dual description, such as Schwarzschild black holes in asymptotically anti-de Sitter (AdS) space, it is often difficult to say what it means to be behind the horizon in the dual theory.

A possible alternative approach is to consider string theory in the near horizon geometry of a black hole. The latter takes the form $AdS_2 \times S^2$ for extremal black holes in four dimensions. It has been suggested [65] that quantum gravity on AdS_2 has dual description in terms of conformal quantum mechanics. There are many puzzles and little understanding of this duality. One basic problem is: what are the states in $AdS_2 \times S^2$? This problem is more subtle than its higher dimensional analog, due to the large back reaction in 1+1 dimensions. On the other hand, it could hold the key to many puzzles of black holes, since particles travelling in global $AdS_2 \times S^2$ crosses the horizon of the black hole and should be described as states in AdS_2. Some preliminary investigation of these questions are carried out in this thesis.

In this introductory chapter we will give an overview of the string theory setup of our problems, as well as the goals and results. Section 1 reviews the attractor

mechanism for supersymmetric black holes in $d=4, \mathcal{N}=2$ supergravity coupled to vector multiplets. We also briefly review the microscopic counting of the black hole entropy, obtained to leading order from a 1+1 dimensional CFT. Section 2 sketches the main idea of the connection between four and five dimensional black holes, as well as other supersymmetric objects. In section 3, we will describe the near horizon geometry of an extremal black hole. In section 4, we review the conjectured relation between black hole partition function and topological string amplitudes, which will be investigated in chapter 5.

1.1 Black holes in $d=4, \mathcal{N}=2$ supergravity

Let us start by considering type IIA string theory compactified on a Calabi-Yau threefold X. The resulting low energy effective theory in four dimensions is $\mathcal{N}=2$ supergravity coupled to h^{11} vector multiplets and $h^{21}+1$ hypermultiplets, where h^{11}, h^{21} are Hodge numbers of X.

There are in total $h^{11}+1$ gauge fields, contained in the vector multiplets and the graviton multiplet, which will be denoted by A^Λ, $\Lambda = 0, \cdots, h^{11}$. One can identify $F^0 = dA^0$ with the four dimensional part of the Ramond-Ramond 2-form field strength F_2^{RR}, and F^A $(A = 1, \cdots, h^{11})$ with the reduction of the RR 4-form F_4^{RR} on a basis of 2-cycles of X. The vector multiplets also contain a total number of h^{11} complex scalar fields. These fields parameterize the Kähler moduli of X. There are also $2(h^{2,1}+1)$ complex scalars coming from the hypermultiplets, corresponding to the complex structure moduli of X, the reduction of RR 3-form potential C_3^{RR} on the 3-cycles of X, as well as the string coupling.

It is convenient to introduce projective coordinates X^Λ on the Kähler moduli space of X, so that the h^{11} complex scalars in the vector multiplets can be written as $Z^A = X^A/X^0$. In $d = 4, \mathcal{N} = 2$ supergravity, the kinetic term of the vector multiplets are determined in terms of a single holomorphic function $F(X)$ homogeneous in X^Λ of degree 2. $F(X)$ is known as the prepotential. There is no coupling between vector multiplets and (neutral) hypermultiplets at the level of kinetic terms [25, 19, 24, 6]. In type IIA (or IIB) string theory the string coupling is part of the universal hypermultiplet, and does not coupling to the vector multiplet kinetic term. Hence the prepotential is protected from string loop corrections. It does however receive world sheet instanton corrections. In the large volume limit the world sheet instantons are exponentially suppressed, and we can write

$$F(X) = D_{ABC}\frac{X^A X^B X^C}{X^0} + \cdots \tag{1.1}$$

where D_{ABC} is 1/6 times the triple intersection number $\#(\Sigma_A \cap \Sigma_B \cap \Sigma_C)$, Σ_A being a basis of 4-cycles in X that is compatible with X^A.

There are black holes that carry electric and magnetic charges (q_Λ, p^Λ) with respect to the gauge fields A^Λ. It is useful to define $F_\Lambda = \partial F/\partial X^\Lambda$, which are homogeneous in X^Λ's. (X^Λ, F_Λ) forms a symplectic vector, in the sense that the effective action is invariant under the symplectic rotation on (X^Λ, F_Λ). The central charge of the black hole of charge (q_Λ, p^Λ) is given by

$$Z = -q_\Lambda X^\Lambda + p^\Lambda F_\Lambda \tag{1.2}$$

The central charge is the charge with respect to the graviphoton field, which lies in

the graviton multiplet. Its field strength can be written

$$T^-_{\mu\nu} = F_\Lambda \hat{F}^{\Lambda -}_{\mu\nu} + X^\Lambda G^-_{\Lambda\mu\nu} \tag{1.3}$$

where the superscript $-$ denotes the anti-self-dual part. $\hat{F}^\Lambda_{\mu\nu}$ are a set of $h^{11}+1$ gauge field strengths, and $G_{\Lambda\mu\nu}$ are a set of dual fields. More precisely, one has

$$\begin{aligned}
G^+_\Lambda &= 2\overline{\mathcal{N}}_{\Lambda\Sigma}\hat{F}^{+\Sigma}, \\
\mathcal{N}_{\Lambda\Sigma} &= \frac{1}{4}\overline{F}_{\Lambda\Sigma} - \frac{N_{\Lambda\Delta}N_{\Sigma\Omega}X^\Delta X^\Omega}{N_{IJ}X^I X^J}, \\
\mathcal{N}_{\Lambda\Sigma} &= \frac{1}{4}(F_{\Lambda\Sigma} + \overline{F}_{\Lambda\Sigma}), \\
F_{\Lambda\Sigma} &= \partial_\Lambda F_\Sigma.
\end{aligned} \tag{1.4}$$

The electric and magnetic charges (q_Λ, p^Λ) are defined as the fluxes of G_Λ and $\hat{F}^\Lambda$ measured on an S^2 at infinity,

$$\begin{aligned}
q_\Lambda &= \frac{1}{2\pi}\mathrm{Re}\int_{S^2} iG^+_\Lambda, \\
p^\Lambda &= \frac{1}{2\pi}\mathrm{Re}\int_{S^2} \hat{F}^{+\Lambda}.
\end{aligned} \tag{1.5}$$

In type IIA string theory (q_0, q_A, p^A, p^0) are the RR charges associated to D0 and D2, D4, D6-branes wrapped on X, respectively. The BPS bound on the mass of the black hole has the form

$$M \geq e^{\mathcal{K}/2}|Z| \tag{1.6}$$

where $\mathcal{K} = -\ln i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)$ is the Kähler potential determined by the $\mathcal{N}=2$ prepotential. The supersymmetric black holes saturate this bound. An important fact about supersymmetric black holes in $\mathcal{N}=2$ supergravity is that the value of the moduli fields at the horizon of the black hole is independent of the moduli at

infinity, and are solely determined by the charges of the black hole.[1] If this were not the case, the horizon area of the black hole in Planck units might vary as one changes the asymptotic moduli, and it would be unlikely to have an interpretation of the Bekenstein-Hawking entropy as counting microscopic states which are intrinsic of the black hole.

The moduli at the horizon are determined by the attractor equation [64]

$$\begin{aligned} \mathrm{Re}(CX^\Lambda) &= p^\Lambda, \\ \mathrm{Re}(CF_\Lambda) &= q_\Lambda, \end{aligned} \tag{1.7}$$

where C is an arbitrary complex constant. Since (X^Λ, F_Λ) are only defined up to rescaling, we can aborb C into X^0 and set $C = 1$. Although the equations (1.7) look deceivingly simple, they are rather nontrivial to solve since F_Λ is in general a complicated function of X^Λ. Using the tree level prepotential, the attractor equations can be solved explicitly for generic charges in [62].

The solution for the metric of the black hole is

$$ds^2 = -\left(1 + \frac{R}{r}\right)^{-2} dt^2 + \left(1 + \frac{R}{r}\right)^2 (dr^2 + r^2 d\Omega_2^2) \tag{1.8}$$

with the radius $R = e^{\mathcal{K}/2}|Z|$.

As an example, let us consider the black hole corresponding to D4-D2-D0 bound states, with charge (p^A, q_A, q_0). The attractor moduli are

$$CX^0 = i\sqrt{\frac{D}{\hat{q}_0}}, \qquad CX^A = p^A + \frac{i}{6}\sqrt{\frac{D}{\hat{q}_0}} D^{AB} q_B \tag{1.9}$$

[1] There are exceptions to this statement, as the moduli at the horizon can jump when the moduli at infinity cross certain "walls" in the moduli space. This is related to the split attractor flow [27, 28] and will be discussed in chapter 2.

where

$$
\begin{aligned}
D &\equiv D_{ABC}p^Ap^Bp^C, \\
\hat{q}_0 &\equiv q_0 + \tfrac{1}{12}D^{AB}q_Aq_B, \\
D_{AB} &\equiv D_{ABC}p^C, \\
D^{AB}D_{BC} &= \delta^A_C.
\end{aligned}
\tag{1.10}
$$

The (macroscopic) entropy of the black hole can be calculated from Bekenstein-Hawking formula,

$$S_{BH} = \frac{\text{Horizon Area}}{4G_N} = 2\pi\sqrt{\hat{q}_0 D} \tag{1.11}$$

The entropy (1.11) can be understood microscopically by either counting the D4-D0 bound states directly [66, 67] or by lifting to M-theory and count the M5-brane states [56]. The latter gives a more precise description, which we shall briefly review here and will revisit in chapter 5. The D4-D0 system can be lifted in M-theory to an M5-brane wrapped on the M-theory circle times the 4-cycle in X where the D4-brane is wrapped. The world volume theory of the M5-brane reduces at low energies to a $(0,4)$ superconformal field theory (SCFT) in 1+1 dimensions. The D0-brane charge turns into the net left-moving momentum carried by massless modes on the M5-brane. This $(0,4)$ CFT has left and right central charges

$$c_L = 6D + c_2 \cdot P, \qquad c_R = 6D + \frac{1}{2}c_2 \cdot P \tag{1.12}$$

where c_2 is the second Chern class of X and $c_2 \cdot P = c_{2A}p^A$. In particular the right central charge c_R is always a multiple of 6, coming from supermultiplets each consisting of 4 bosons and 4 fermions. A supersymmetric D4-D0 black hole is described by states in this $(0,4)$ with Ramond ground state in the right-moving sector, and

arbitrary left excitations, with the left momentum L_0 equal to the D0-brane charge. The number of states at high momenta is given by Cardy's formula,

$$\# \text{ states} \sim \exp\left(2\pi\sqrt{\frac{c_L L_0}{6}}\right) \tag{1.13}$$

which indeed agrees with (1.11) to leading order. D2-brane charges can be added by turning on 3-form fluxes on the M5-brane world volume, which gives rise to right moving momenta and shifts the D0-brane charge for a state of given L_0.

Although this very simple counting gives the leading macroscopic entropy, it is rather difficult to extend it beyond the leading order, since not much is known about the $(0,4)$ CFT living on the M5-brane other than the central charge. We will come back to this CFT in chapter 5, and study its chiral primary states via the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality.

1.2 The 4D/5D connection

A simple and surprising relation between four and five dimensional black holes was found in [37]. The idea is to consider a four dimensional black hole in a Kluza-Klein theory that carries one unit of KK monopole charge. The KK monopole lifts to the Taub-NUT geometry in five dimensions, and correspondingly the 5D lift of the 4D black hole with KK monopole charge describes a 5D spinning black hole located at the center of the Taub-NUT space. The existence of such 5D solution is independent of the radius of the Taub-NUT space, and one can consider the limit where the radius goes to infinity, in which case the Taub-NUT space opens up into the flat $\mathbf{R}^4$. In the case of a supersymmetric black hole, the ground states of the black hole, or rather,

Figure 1.1: A black hole spinning at the tip of the Taub-NUT space becomes one in asymptotically 5D flat space as the radius of Taub-NUT space is taken to infinity.

a supersymmetric index that counts the number of ground states with appropriate signs, should be invariant as one varies the asymptotic radius of the Kluza-Klein compactification. In particular, such a supersymmetric index that counts 4D black holes of a given set of charges should be equal to that of the corresponding 5D black hole.

Indeed this relation can be demonstrated at the level of supergravity by writing down explicitly the solutions of black holes spinning at the tip of Taub-NUT space. It follows that the classical Bekenstein-Hawking entropy of the corresponding 4D and 5D black holes are the same, since the horizon of the latter is the KK lift of the former. More surprisingly, there are evidences that this correspondence is an exact statement on the degeneracy of black hole states string theory. Assuming this correspondence we are able to compute the indices that count certain 4D black holes from the 5D

indices, which turns out to be better understood. Such examples involve the 1/4 BPS black holes in $\mathcal{N} = 4$ string compactifications, which leads to a derivation of a conjecture of Dijkgraaf, Verlinde and Verlinde [31], and the 1/8 BPS black holes in $\mathcal{N} = 8$ string theory, which agrees with direct countings in four dimensions and reveals interesting subtleties in the D-brane bound states.

In $d = 4, \mathcal{N} = 2$ supergravity coupled to vector multiplets, there are also supersymmetric multi-centered black holes that carry intrinsic angular momenta and have moduli fields varying over the space. These solutions were found in a series of papers by Denef and collaborators [27, 28, 7]. One can choose one of the centers to be a KK monopole, another center to be a 4D black hole with no KK monopole charge. This two-centered configuration can be lifted to 5D, corresponding to a supersymmetric black ring near the center of Taub-NUT space. As one takes the asymptotic radius of the Taub-NUT space to infinity, one recovers precisely the 5D black ring in asymptotic flat spacetime. This connection may help understanding the transition among various black objects in five dimensions.

In chapter 2 we will describe the connection between 4D and 5D black holes, as well as between 4D multi-center black holes and 5D black rings. In chapter 3 the 4D/5D connection is applied to the counting of BPS dyons in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ four dimensional string compactifications.

1.3 The near horizon attractor geometry

It will be useful to describe the near horizon geometry of a supersymmetric black hole in type IIA string theory compactified on X. The near horizon geometry of the

4D supersymmetric black hole is $AdS_2 \times S^2$ with the moduli fields at their attractor values. The corresponding 10-dimensional geometry is $AdS_2 \times S^2 \times X$. For simplicity we will restrict ourselves to the attractor geometry of a D4-D2-D0 black hole. The radius R of AdS_2 and S^2, which are the same as the radius of the black hole, is determined in terms of the charges (p^A, q_A, q_0) via

$$R = \sqrt{2}\,(D\hat{q}_0)^{\frac{1}{4}} \tag{1.14}$$

in four-dimensional Planck units.

The metric on the Poincaré patch of $AdS_2 \times S^2$ is

$$ds^2 = R^2\left(\frac{-dt^2 + d\sigma^2}{\sigma^2} + d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1.15}$$

while the metric is

$$ds^2 = R^2(-\cosh^2\chi d\tau^2 + d\chi^2 + d\theta^2 + \sin^2\theta d\phi^2) \tag{1.16}$$

in global coordinates. In the case $q_A = 0$, the RR field strengths are

$$F_{(2)} = \frac{1}{R}\omega_{AdS_2}, \qquad F_{(4)} = \frac{1}{R}\omega_{S^2} \wedge J, \tag{1.17}$$

where $\omega_{AdS_2} = R^2 \cosh\chi d\tau \wedge d\chi$ is the volume form on AdS_2, $\omega_{S^2} = R^2 \sin\theta d\theta \wedge d\phi$ is the volume form on the S^2, and J is the Kähler form on the Calabi-Yau. In particular, the Kähler volume of the 2-cycles α^A are determined by the charges as

$$\frac{1}{2\pi\alpha'}\int_{\alpha^A} J = 2\pi p^A \sqrt{\frac{q_0}{D}} \tag{1.18}$$

One can also consider M-theory on an $AdS_3 \times S^2 \times X$ attractor geometry, where X is a Calabi-Yau threefold, with 4-form flux

$$G_4 = \omega_{S^2} \wedge p^A \omega_A \tag{1.19}$$

Here ω_A is a basis of harmonic 2-forms dual to 2-cycles α^A in X with intersection numbers $\int_X \omega_A \wedge \omega_B \wedge \omega_C = 6D_{ABC}$. The metric is :[2]

$$\int_{\alpha^A} J = (2\pi)^2 \frac{p^A}{\ell} \tag{1.20}$$

$$ds_{11}^2 = \ell^2(-\cosh^2\chi d\tau^2 + d\chi^2 + \sinh^2\chi d\psi^2) + \frac{\ell^2}{4}(d\theta^2 + \sin^2\theta d\phi^2) + ds_X^2, \tag{1.21}$$

where ℓ is the radius of AdS_3 and J is the Kähler form on X . This is the near horizon geometry of an M5-brane wrapped on the 4-cycles $p^A\Sigma_A$ (where Σ_A are a basis of 4-cycles dual to ω_A) in X and forming an extended string in the noncompact five dimensions. One expects M-theory on $AdS_3 \times S^2 \times X$ to be dual to the $(0,4)$ CFT on the M5-brane world volume (dimensionally reduced on P). This CFT has leading order left central charge $c_L = \frac{3\ell}{2G_3} = 6D_{ABC}p^Ap^Bp^C$, where the 3D Newton constant is $G_3 = \frac{16\pi^7}{Vol_X Vol_{S^2}}$ [56].

The M-theory attractor geometry with flux (1.19) can also be obtained by lifting the IIA geometry $AdS_2 \times S^2 \times X$ with D0 and D4 fluxes, and then decompactifying the M-theory circle.

It has long been suspected [54, 65] that quantum gravity on AdS_2 has a dual description in terms of a superconformal quantum mechanics. This correspondence is more subtle than the higher dimensional examples of AdS/CFT correspondence, for two main reasons. Firstly, it is less clear how to define states in AdS_2, since the space is 1+1 dimensional, and any charged particle will cause a large back reaction which could alter (at least part of) the asymptotic boundary. Secondly, the dual 0+1 dimensional quantum mechanics is expected to have superconformal symmetry, or even super-Virasoro symmetry. This is unusual for theories of quantum mechanics,

[2]We adopt 11D Planck units in which, as in [60], the action is $(2\pi)^{-8}\int d^{11}x\sqrt{-g}R + \cdots$.

Figure 1.2: The near horizon geometry of an extremal Reisner-Nordstrom black hole. The red line represents the trajectory of a particle travelling in the global AdS_2.

which often have mass gaps. Attempts to construct the dual superconformal quantum mechanics directly include [37]. In this thesis we will investigate the bulk problem, e.g. what are the states of quantum gravity on AdS_2.

The most naive approach is to consider particles in $AdS_2 \times S^2$ as probes, ignoring back reactions completely. This will be investigated in chapter 4 for various wrapped D-branes in type IIA string theory compactified on $AdS_2 \times S^2 \times X$, where X is a Calabi-Yau threefold. Indeed, by analyzing the kappa-symmetry of the classical world volume action of the D-branes, one finds that the holomorphically wrapped D-branes with any given set of charges can be supersymmetric in $AdS_2 \times S^2$. The quantum states of such supersymmetric particles/D-branes form short representations of the superconformal algebra. Their analogs in $AdS_3 \times S^2$ will be studied in more detail in chapter 5.

It is in fact possible to include backreaction in the supergravity approximation, by writing down explicitly the gravity solutions for supersymmetric black holes in $AdS_2 \times S^2$. These solutions will be presented in the second part of chapter 4. They however suffers from regions of closed timelike curves (CTCs), which indicates some kind of classical instability of these solutions in string theory. It is not clear how or if one can remove CTCs by modifying these solutions. The answer might give hints on how to think about quantum states in AdS_2.

1.4 Black holes and topological strings

As review in section 1, the leading macroscopic entropy of a supersymmetric black hole in $d = 4, \mathcal{N} = 2$ supergravity can be computed from the Bekenstein-Hawking formula. This is made possible by the attractor mechanism, which determines the moduli fields at the black hole horizon in terms of the charges of the black holes, regardless of the asymptotic moduli. On the other hand, the leading macroscopic entropy is also reproduced from the microscopic description of the black hole states in terms of 1+1 dimensional CFT.

One would like to go beyond the leading order and compute all the subleading corrections to the entropy in the large charge expansion. To do so one must extend the notion of Bekenstein-Hawking entropy to theories of gravity with higher order curvature corrections beyond the Einstein-Hilbert action. A surprisingly simple candidate for the generalized BH entropy is given by Wald's formula,

$$S = \frac{\pi}{2} \int_{S^2} \epsilon_{ab} \epsilon_{cd} \frac{\delta \mathcal{L}_{eff}}{\delta R_{abcd}} \qquad (1.22)$$

where $\mathcal{L}_{eff}$ is the full covariant effective action expressed in terms of the Riemann tensor R_{abcd} and the matter fields. For the Einstein-Hilbert action (1.22) reproduces the standard Bekenstein-Hawking area formula. In $\mathcal{N} = 2$ supergravity arising in type II string compactified on a Calabi-Yau threefold, the relevant terms in the effective action are the $\mathcal{N} = 2$ F-terms [13, 4, 5, 52, 58],

$$\int d^4x d^4\theta F_g(X^\Lambda)(W^{ab}W_{ab})^g = \int d^4x F_g(X^\Lambda) R_-^2 T_-^{2g-2} + \cdots \tag{1.23}$$

where $F_g(X^\Lambda)$ is homogeneous in X^Λ of degree $2 - 2g$, here X^Λ stands for the vector multiplet written as a superfield in $\mathcal{N} = 2$ (chiral) superspace and W_{ab} is the Weyl superfield. R_- and T_- denote the anti-self-dual part of the Riemann tensor and graviphoton field strength, and we omitted the contraction of indices, which involve terms like $R_-^2(T_-^2)^{g-1}$ and $(R_-T_-)^2(T_-^2)^{g-2}$. F_0 is the prepotential, and $F_g(X^\Lambda)$ are computed by the free energy of topological strings on X at genus g [13], where X^A/X^0 are the Kähler moduli and $1/X^0$ is identified with the topological string coupling constant.

Wald's formula (1.22) is applied to compute the $1/Q$ corrections to the black hole entropy in [52, 51, 58]. It was observed by [59] that the black hole entropy is essentially the Legendre transform of the topological string amplitude with respect to the electric potentials, where the moduli fields are identified with the magnetic charge and electric potential via the attractor formula. [59] then conjectured an exact relation between a suitably defined index that counts the entropy of black holes in a mixed ensemble, Z_{BH}, with the square of the topological string partition function. Essentially,

$$Z_{BH}(p^\Lambda, \phi^\Lambda) = \sum \Omega_{BH}(p^\Lambda, q_\Lambda) e^{-q_\Lambda \phi^\Lambda} = |Z_{top}(g_{top} = \frac{4\pi i}{X^0}, t^A = \frac{X^A}{X^0})|^2 \tag{1.24}$$

where $X^\Lambda = p^\Lambda + \frac{i}{\pi}\phi^\Lambda$. There are many subtleties to (1.24). Firstly, it is unclear a priori how the degeneracy Ω_{BH} is defined, although in the case of $\mathcal{N} = 2$ black holes it appears to be the Witten index. The black hole partition Z_{BH} in the mixed ensemble involves a summation over all possible electric charges, and naively this sum is divergent and needs to be regularized. Further and most importantly, the relation (1.24) is not exactly true, and there are corrections to this equality, as shown in many examples where the black hole degeneracy can be counted exactly [68, 22].

Extending the observation of [59], Sen showed that the entropy of an extremal black hole as defined by (1.22) is given by the extremizing an entropy function with respect to the moduli fields as well as the radius at the horizon. The entropy function is given by the Legendre transform of the integral of the Lagrangian density over the horizon. This explains for example that Wald's entropy is independent of hypermultiplet scalars: if this weren't the case, the extremization of the entropy function at the horizon would fix the hypermultiplet scalars in terms of the charges as well. This argument, however, does not forbid the contribution from $\mathcal{N} = 2$ D-terms that involve only vector and graviton multiplets to Wald's entropy formula. It is plausible that one needs to modify (1.22) to get the macroscopic entropy corresponding to a supersymmetric index, which only receives contribution from the $\mathcal{N} = 2$ F-terms.

In chapter 5 we will identified the OSV partition function of an $\mathcal{N} = 2$ black hole with the elliptic genus of the $(0, 4)$ CFT of [56]. This $(0, 4)$ CFT (in its NS sector) is holographically dual to M-theory on $AdS_3 \times S^2 \times X$. We can compute the elliptic genus by counting states in the bulk AdS_3 from massless modes in supergravity, and more importantly, wrapped M2-branes, in a dilute gas approximation. Surprisingly, via a

relation between BPS M2-branes and the topological string amplitude of Gopakumar and Vafa [41, 42], the elliptic genus computed from the bulk is nothing but $|Z_{top}|^2$, thus proving the OSV relation in this approximation. We regard this as a framework to systematically think about corrections to the OSV conjecture, although much work is to be done to fully understand them.

Chapter 2

Black Holes in 4D and 5D

In this chapter we will propose a simple and direct connection between a certain BPS partition function Z_{5D} of the general 5D spinning BPS black hole in a Calabi-Yau compactification of M-theory and Z_{4D} of the general 4D BPS black hole in a Calabi-Yau compactification of the IIA theory. Generalizing this, we will also find the connection between 5D BPS black rings and multi-centered 4D BPS black holes of [27, 28, 7].

We begin in section 1 by deriving the basic 4D-5D connection. Exact 5D supersymmetric solutions were found in [39] which can be described as a 5D black hole with $SU(2)_L$ spin J_L^3 and M2 charges q_A^{5D} sitting at the center of a charge p^0 Taub-NUT space. The explicit solution will be constructed in section 2. Since Taub-NUT space is locally asymptotic to flat $R^3 \times S^1$ this implements a $5 \to 4$ compactification. When the compactification radius R, a modulus of the Taub-NUT solution, becomes small the 4D picture becomes appropriate. We show that in the 4D picture we have a black hole with D6-D2-D0 charges $(p^0, \frac{q_A^{5D}}{p^0}, \frac{2J_L^3}{(p^0)^2})$, and vanishing D4 charge $p^A = 0$.

In section 3 we argue that an appropriate BPS partition function (i.e. index) Z should not depend on the radius R, yielding an equality of the form $Z_{4D} = Z_{5D}$ with a certain relation between the arguments. The microscopic description for many (but not all) 5D spinning black holes is known [66, 17]. Hence this 5D-4D relation gives a microscopic description of 4D black holes for many cases in which it had previously been unknown. As a check these relations are found to correctly, and in a rather intricate manner, reproduce the entropy formula at leading order.

In section 4 the result is generalized to include general D4 charge p^A. From the 5D M-theory perspective this involves turning on a four form $F^{(4)} \sim \omega_{NUT} \wedge p^A \alpha_A$, where ω_{NUT} is a normalizable harmonic 2-form on Taub-NUT space and α_A is an integral basis of harmonic Calabi-Yau two forms. The 4D partition function for *any* set of D-brane charges may then be identified with that of a spinning 5D black hole in this Taub-NUT-flux background. This identification is again shown to intricately yield the correct leading-order entropy.

Section 5 reviews a general ansatz for supersymmetric solutions in 4D $\mathcal{N} = 2$ supergravity coupled to vector multiplets, which is used to construct the multi-centered black hole solutions in [27, 28, 7]. This ansatz will be applied again in chapter 4 to construct black holes in $AdS_2 \times S^2$.

In section 6, we construct a general solution of 5D supergravity describing black rings and black holes in a multi-Taub-NUT-flux geometry, based on the ansatz of [39, 33]. We then show that the solutions of section 6 are nothing but the M-theory lift of the multi-centered solutions of section 5. A basic example is the lift of the bound state of a D6-brane with a D4-D2-D0 black hole to a 5D black ring.

2.1 Lifting 4D black holes to 5D

Consider p^0 D6 branes wrapping a Calabi-Yau space X in a IIA string compactification. In the M-theory picture this is described as the product of a Taub-NUT space with X:

$$ds_M^2 = \left(1 + \frac{p^0 R}{r}\right) d\vec{r}^2 + R^2 \left(1 + \frac{p^0 R}{r}\right)^{-1} (dx^{11} + p_0 \cos\theta d\phi)^2 + ds_X^2 - dt^2 \qquad (2.1)$$

where $x^{11} \sim x^{11} + 4\pi$. The Taub-NUT geometry has a $U(1)_L \times SU(2)_R$ isometry, where the $U(1)_L$ generates x^{11} translations. The radius R here is related to the ten-dimensional IIA coupling via

$$R = g_{10}^{2/3}. \qquad (2.2)$$

At strong coupling, or large R, there is a large region with $r \ll R$ in the core of the Taub-NUT geometry in which the 5D metric reduces to

$$ds_5^2 = \frac{p^0 R}{r}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 + r^2(dx^{11}/p_0 + \cos\theta d\phi)^2) - dt^2. \qquad (2.3)$$

This is the flat metric on R^4/Z_{p^0} tensored with the time direction. For $p^0 = 1$ we simply have 5D Minkowski space.

Calabi-Yau compactifications of M theory to 5D admit a second set of supersymmetric solutions with $U(1)_L \times SU(2)_R$ isometries. These are the 5D spinning black holes [17], characterized by membrane charges q_A and angular momentum J_L associated to the $U(1)_L$ isometry. Their characteristic size r_{BH} grows as the square root of the graviphoton charge $\sqrt{Q}$ which in turn is proportional to the membrane charge q_A.

Let us now suppose that $\sqrt{Q} \ll R$ and $p^0 = 1$. Then we can make an approximate BPS solution by inserting the spinning black hole at the center of the $p^0 = 1$ Taub-

NUT, symmetries aligned, well inside the region where the R^4 is flat. Aligning the symmetries requires the black hole to be exactly at the center of the Taub-NUT. In fact an exact solution of this form exists for all Q, R [39] and is reproduced in the next section. Of course for large $\sqrt{Q} \gg R$ it can no longer be described as a black hole in the center of Taub-NUT, but this is irrelevant for our purposes since the BPS quantities we consider should be independent of R.

At distances large compared to R, this solution is effectively a spherically symmetric black hole in a four dimensional IIA compactification carrying D6 charge $p^0 = 1$, and D2 charge q_A. In addition J_L, which is the eigenvalue of $U(1)_L$ rotations, becomes proportional to D0 charge q_0, since $U(1)_L$ generates x^{11} translations. To get the proportionality factor, consider an orbit of the asymptotic $U(1)_L$ in the S^3 near the tip $\mathbf{R}^4$. The angular momentum in the 1-2 plane J_1 and that in the 3-4 plane J_2 are related to J_L, J_R by $J_1 = J_L + J_R, J_2 = J_L - J_R$. An orbit of the $U(1)_L$ is a helix going along a circle in the 1-2 plane and a circle in the 3-4 plane at the same time. The wave function of angular momentum $(J_L, J_R = 0)$ picks up a factor $e^{2\pi i(J_1+J_2)} = e^{4\pi i J_L}$ as one goes around the S^1 orbit. Therefore we conclude

$$q_0 = 2J_L. \tag{2.4}$$

A similar construction works for integral $p^0 > 1$. We simply take the exact 5D solution and quotient it by the Z_{p^0} subgroup of the $U(1)_L$ isometry, which acts freely outside the horizon. At infinity, this quotients the Kaluza-Klein circle and changes its radius from R to $\frac{R}{p^0}$, while the topology of the 5D horizon becomes S^3/Z_{p^0}. The

corresponding 4D black hole then has zero-brane charge[1]

$$q_0 = \frac{2J_L}{(p^0)^2}. \tag{2.5}$$

Moreover, since the $S^2 \times S^1$ at infinity over which the 4D charges are given as field strength integrals is divided by p^0, we have

$$q_A = \frac{q_A^{5D}}{p^0}. \tag{2.6}$$

2.2 Spinning black hole in Taub-NUT space

In this section we write down explicitly the solution of a 5D spinning black hole at the center of Taub-NUT space. The Killing spinor equation of $\mathcal{N} = 2$ 5D supergravity is

$$\left[d + \frac{1}{4}\omega_{ab}\Gamma^{ab} + \frac{i}{4\sqrt{3}} e^a (\Gamma^{bc}{}_a F_{bc} - 4\Gamma^b F_{ab}) \right] \epsilon = 0 \tag{2.7}$$

where e^a are the frame 1-forms and ω_{ab} is the spin connection. The metric for the supersymmetric spinning black hole in Taub-NUT space is [39]

$$ds^2 = -(1 + \frac{\tilde{Q}}{Rr})^{-2} \left(dt + \frac{\tilde{J}a}{p^0 R^2} \right)^2 + (1 + \frac{\tilde{Q}}{Rr}) ds_{TN}^2 \tag{2.8}$$

where

$$a = \left(1 + \frac{p^0 R}{r} \right) (dx^{11} + p^0 \cos\theta d\phi) - dx^{11} \tag{2.9}$$

and $x^{11} \sim x^{11} + 4\pi$. R is the asymptotic radius of the Taub-NUT space and the graviphoton field

$$F = \frac{\sqrt{3}}{2} d \left[(1 + \frac{\tilde{Q}}{Rr})^{-1} \left(dt + \frac{\tilde{J}a}{p^0 R^2} \right) \right] \tag{2.10}$$

[1] Writing the D0 charge schematically as a 4D spatial integral $q_0 \sim \int d^4\Sigma^b K^a T_{ab}$ of the $U(1)_L$ Killing field K contracted with the stress tensor, one factor of p^0 comes from the division of the domain of the integrand, while the second comes from demanding that K be normalized so as to generate unit translations of the Kaluza-Klein circle at infinity.

Similarly to the calculation of [40], the Killing spinor equations are solved by

$$i\Gamma^0\epsilon = \epsilon \tag{2.11}$$

and the self-duality of da and of the spin connection of Taub-NUT space.

With a redefinition of variable $r = \rho^2/R$, in the limit $R \to \infty$, the solution (2.8) becomes

$$ds^2 = -(1+\frac{\tilde{Q}}{\rho^2})^{-2}\left[dt + \frac{\tilde{J}}{\rho^2}(dx^{11} + p^0\cos\theta d\phi)\right]^2 + 4p^0(1+\frac{\tilde{Q}}{\rho^2})(d\rho^2 + \rho^2 d\tilde{\Omega}_3^2) \tag{2.12}$$

where

$$d\tilde{\Omega}_3^2 = \frac{1}{4}\left[d\theta^2 + \sin^2\theta d\phi^2 + \frac{1}{(p^0)^2}(dx^{11} + \cos\theta d\phi)^2\right] \tag{2.13}$$

is the metric on the unit $S^3/\mathbf{Z}_{p^0}$. (2.12) is nothing but a spinning black hole at the center of the orbifold space $\mathbf{C}^2/\mathbf{Z}_{p^0}$. Note that the area of the black hole horizon is independent of R, and is given by

$$A = 16\pi^2\sqrt{p^0\tilde{Q}^3 - (p^0\tilde{J})^2} \tag{2.14}$$

$\tilde{Q}$ and $\tilde{J}$ are related to the standard normalized 5D charges Q, J [17] by a rescaling,

$$Q = 2\pi^{2/3}\tilde{Q}, \quad J = 2\sqrt{2}\pi\tilde{J}. \tag{2.15}$$

2.3 The entropy of D6-D2-D0 system

The preceding classical construction suggests the quantum conjecture that the supersymmetric partition function of a 4D black hole with D-brane charges $(p^0, 0, q_A, q_0)$ is directly related to that of a Z_{p^0} orbifold (which is trivial for $p^0 = 1$) of a 5D black hole with membrane charges q_A and spin $q_0/2$. A precise conjecture relating certain

4D and 5D supersymmetric indices will be made in the next section. In this section we will check the conjecture at the level of the leading semiclassical entropy.

The macroscopic entropy of a 5D spinning black hole is [47]

$$S_{5DBH} = 2\pi\sqrt{Q^3 - J_L^2}, \tag{2.16}$$

where $Q^3 = D_{ABC}Y^AY^BY^C$ with Y^A's satisfying $3D_{ABC}Y^BY^C = q_A^{5D}$. A 4D black hole is obtained by inserting this 5D black hole in the center of Taub-NUT. For the special case $p^0 = 1$, we identify $J_L = q_0/2$, and (2.16) becomes

$$S_{4DBH}(p^0 = 1) = 2\pi\sqrt{Q^3 - \frac{1}{4}(q_0)^2}. \tag{2.17}$$

This agrees precisely with the known 4D result for no D4 charges and $p^0 = 1$ [62].

This is to be expected: in the reduction from 5D supergravity to 4D supergravity the radius of the fifth dimension is identified with an appropriate combination of the 4D scalar moduli, and the Taub-NUT radius is the asymptotic value of that scalar modulus at infinity. The entropy of a 4D BPS black hole does not depend of the asymptotic values of the scalar moduli at infinity.

Therefore, any microscopic accounting of a 5D black hole with charges q_A directly descends to a microscopic accounting of a 4D black hole with D6 charge $p^0 = 1$, D4 charge $p^A = 0$, and arbitrary D2-D0 charges $q_A,\ q_0$.

Now consider $p^0 > 1$. Dividing by p^0 divides the area and hence the entropy by p^0. Therefore, in terms of the parameters J_L and Q_{5D} of the unquotiented 5D black hole the 4D entropy is

$$S_{4DBH} = \frac{2\pi}{p^0}\sqrt{Q_{5D}^3 - J_L^2}. \tag{2.18}$$

Using (2.5) and (2.6) then gives

$$S_{4DBH} = 2\pi\sqrt{p^0 Q^3 - \frac{1}{4}(p^0 q_0)^2} \tag{2.19}$$

in precise agreement with the 4D entropy formula for general nonzero D0, D2 and D6 charges [62].

For $p^0 > 1$ a microscopic accounting of 5D entropy does not descend so directly to an accounting of 4D entropy, because we still have to understand the effect of the Z_{p^0} orbifold action on the dual quantum microsystem describing the black hole. The dual quantum microsystem is not known in general so we can't describe the orbifold action. In order to proceed we assume a microscopic picture of the kind discovered in [66, 17], in which the $U(1)_L$ corresponds to a conserved left-moving current of a 2D CFT. Z_{p^0} is then an orbifold action, and the entropy is dominated by the "long string" of the maximally twisted sector. This effectively increases the 2D central charge by a factor of p^0 so that $Q^3 \to p^0 Q^3$. At the same time the relation between worldvolume momentum and target space one is rescaled as well $q_0 \to p^0 q_0$, and we recover (2.19). With this assumption, any microscopic accounting of a 5D black hole with charges q_A directly descends to a microscopic accounting of a 4D black hole with D4 charge $p^A = 0$, and arbitrary D6-D2-D0 charges $p^0,\ q_A,\ q_0$. In section 5 we will relax the restriction $p^A = 0$.

We can now conjecture the exact relation between the partition function of 4D extremal black holes and 5D spinning black holes, as follows

$$Z_{4D}(\phi^A, \phi^0) = Z_{5D}(\phi^A, 2\phi^0 + 2\pi i) \tag{2.20}$$

where these partition functions are Witten indices of the form

$$Z_{4D}(\phi^A, \phi^0) = \mathrm{Tr}'_{p^0=1, p^A=0}(-1)^{2J^3} e^{-\phi^A q_A - \phi^0 q_0 - \beta H} \tag{2.21}$$

and

$$Z_{5D}(\phi^A, \mu) = \mathrm{Tr}(-1)^{2J^3_L + 2J^3_R} e^{-\phi^A q_A - \mu J^3_L - \beta H}. \tag{2.22}$$

Tr' here denotes the trace over all 4D states with the overall center-of-mass multiplet factored out[2] and J^3 generates a 4D spatial rotation. The 4D trace is restricted to the sector with $p^0 = 1$ and $p^A = 0$. Z_{5D} has IR divergences from black holes which fragment and separate: we regulate these by putting them in Taub-NUT space of radius R which forces all black holes to sit at the center (where they do not break supersymmetry),[3]and then taking $R \to \infty$.

2.4 The D6-D4-D2-D0 system

In this section we generalize our construction to 4D extremal black hole of generic charges (p^0, p^A, q_A, q_0).

2.4.1 $p^0 = 1$

In this subsection we take $p^0 = 1$ and then generalize to $p^0 > 1$ in the next subsection. Consider turning on a constant worldvolume $U(1)$ gauge field $F_{world} = p^A \alpha_A$ on a IIA $D6$ brane wrapping the Calabi-Yau X. The coupling of F_{world} to RR

[2]In 5D, this degree of freedom is part of the background Taub-NUT geometry which is frozen.

[3]More precisely, the quantum wave function of a hypermultiplet has one supersymmetric ground state corresponding to the unique normalizable self-dual harmonic two form ω_{NUT}. An interesting generalization, on which we hope to report, involves the supersymmetric black ring.

potential gives an object in 4D with charges

$$(1, p^A, 3p^Ap^BD_{ABC}, -p^Ap^Bp^CD_{ABC}). \tag{2.23}$$

Solving the attractor equations for such charges, we find simply $CX^A = p^A, CX^0 = 1$ (see [64] for notation). The leading order macroscopic entropy formula [64] then gives vanishing entropy. This is consistent with the microscopic picture in which there is a unique F_{world}.

Now let us try to understand the 11-dimensional description of this configuration. The M-theory lift is again a Taub-NUT geometry, with nonzero four form flux turned on:

$$F^{(4)} = \omega_{NUT} \wedge \sum p^A \alpha_A. \tag{2.24}$$

ω_{NUT} here is the unique self-dual harmonic two form on Taub-NUT space [43], and α_A is a basis for harmonic Calabi-Yau two-forms. This flux sources D2 charge via the coupling $\int C^{(3)} \wedge F^{(4)} \wedge F^{(4)}$, yielding $q_A = 3p^Bp^CD_{ABC}$ as in (2.23). There is a nonzero Poynting vector corresponding to the momentum along the M-theory circle. From the 4D point of view this is interpreted as D0 charge $q_0 = -p^Ap^Bp^CD_{ABC}$ as in (2.23). So, by turning on $F^{(4)}$ as in (2.24), we produce a configuration with $p^0 = 1$, arbitrary D4 charges, but predetermined D2-D0 charges and no entropy.

To get a configuration with arbitrary D2-D0 charges, we now insert a 5D spinning black hole with charges q_A^{5D} and angular momentum J_L^3 in the middle of this Taub-NUT-flux configuration. The exact solution can be found in [39]. This yields a configuration with asymptotic 4D charges

$$(1, p^A, 3p^Ap^BD_{ABC} + q_A^{5D}, -p^Ap^Bp^CD_{ABC} - p^Aq_A^{5D} + 2J_L^3) \tag{2.25}$$

Notice the extra shift in $D0$ brane charge coming from placing the charged 5D black hole in the nontrivial magnetic four form field. This is a higher dimensional generalization of Dirac's observation that a static electric charge in a magnetic field carries angular momentum.

Now we wish to identify the partition function of the 4D black hole with that of the spinning 5D black hole. 5D black holes doesn't carry p^A charge, so in order for this to be correct it must be the case that, for the special values of charges given in (2.25), the index Z_{4D} is independent of p^A. This can be seen as a consequence of symplectic invariance, as follows.

The index Z_{4D} is naturally a function of $CX^\Sigma = p^\Sigma + i\frac{\phi^\Sigma}{\pi}$ ([59]). For a cubic prepotential $p^0 = 1$ and $p^A = 0$, the electric potentials ϕ^Σ are determined from the charges by

$$q_0 = -Im\frac{CD_{ABC}X^AX^BX^C}{(X^0)^2} = Re\frac{D_{ABC}\phi^A\phi^B\phi^C}{\pi(\pi + i\phi^0)^2}, \tag{2.26}$$

$$q_A = 3Im\frac{CD_{ABC}X^BX^C}{X^0} = -3Im\frac{D_{ABC}\phi^B\phi^C}{\pi(\pi + i\phi^0)}. \tag{2.27}$$

Now consider the symplectic transformation

$$CX'^0 = CX^0, \qquad\qquad CX'^A = CX^A + p^ACX^0, \tag{2.28}$$

under which Z_{4D} is presumed invariant.[4] For the values of the moduli under consideration this results in

$$X'^0 = 1 + i\frac{\phi^0}{\pi}, \quad CX'^A = p^A + i(\frac{\phi^A}{\pi} + p^A\frac{\phi^0}{\pi}). \tag{2.29}$$

[4] In principle it might transform as a modular form, but this would not affect the leading order computation given here.

Comparing with (2.26) we see that the new charges are related to the old ones by

$$q_0' = -Im\frac{CD_{ABC}X'^AX'^BX'^C}{(X'^0)^2} = q_0 - p^Aq_A - D_{ABC}p^Ap^Bp^C, \tag{2.30}$$

and

$$q_A' = q_A + 3D_{ABC}p^Bp^C. \tag{2.31}$$

Taking $(q_A, q_0) = (q_A^{5D}, 2J_L^3)$, this shift agrees exactly with that encountered in (2.25). Therefore we can use a symplectic transformation to shift from $p^A = 0$ to arbitrary nonzero p_A and Z_{4D} remains unchanged. Physically this corresponds to the fact that putting a 5D spinning black hole in a background $F^{(4)}$ shifts some charges but does not change the number of microstates.

2.4.2 $p^0 > 1$

A similar analysis holds for $p^0 > 1$. The asymptotic charges (2.25) for a spinning black hole become

$$(p^0, p^A, \frac{3}{p^0}p^Ap^BD_{ABC} + q_A^{5D}, -\frac{1}{(p^0)^2}p^Ap^Bp^CD_{ABC} - \frac{p^Aq_A^{5D}}{p^0} + 2J_L^3). \tag{2.32}$$

p^A can then be shifted away as before via the symplectic transformation

$$CX'^0 = CX^0, \qquad CX'^A = CX^A + \frac{p^A}{p^0}CX^0. \tag{2.33}$$

2.5 Multicenter BPS black holes in 4D

In a series of beautiful papers [27, 28, 7] Denef and collaborators have explicitly constructed multi-center 4D BPS black hole solutions which in general carry angular

momenta. The black holes in these solutions can have different sets of charges and they are bound to one another in the sense that the black holes separations are fixed in terms of their charges and the asymptotic values of the moduli.

In the following sections we will construct exact "flux-Taub-NUT-black-ring" solutions describing a black ring in Taub-NUT with four-form flux turned on. We further show that these solutions are precisely the lift to 5D of the Denef multi-center solutions, and the 4D black hole separations become the radii of the 5D rings.

Let us now review, and slightly reformulate, the general multi-center BPS solution of 4D $\mathcal{N} = 2$ supergravity[5] found in [7]. The solution is characterized by electromagnetic charges and asymptotic moduli. It may be expressed in terms of $2h_{11} + 2$ real harmonic functions on R^3

$$\begin{pmatrix} H^\Lambda(\vec{x}) \\ H_\Lambda(\vec{x}) \end{pmatrix} = \begin{pmatrix} h^\Lambda \\ h_\Lambda \end{pmatrix} + \sum_s \begin{pmatrix} p_i^\Lambda \\ q_{\Lambda,i} \end{pmatrix} \frac{1}{|\vec{x} - \vec{x}_i|} \tag{2.34}$$

where $(p_i^\Lambda, q_{\Lambda i})$ is the electromagnetic charge located at the spatial position $\vec{x}_i$, and (h^Λ, h_Λ) are constants which will shortly be related to the asymptotic moduli. The projective scalar moduli X^Λ as a function of spatial positions are then given by

$$CX^\Lambda(\vec{x}) = H^\Lambda(\vec{x}) + \frac{i}{\pi} \left. \frac{\partial S_{bh}(p^\Lambda, q_\Lambda)}{\partial q_\Lambda} \right|_{p^\Lambda = H^\Lambda(\vec{x}),\ q_\Lambda = H_\Lambda(\vec{x})} . \tag{2.35}$$

The complex function C depends on the choice of projective gauge (and may be set to one by an appropriate choice). As a function of the moduli and prepotential (or periods F_Λ) S_{bh} here is given by

$$S_{bh}(CX^\Lambda(\vec{x})) = \frac{\pi}{2} \mathrm{Im}[CX^\Lambda \bar{C} \bar{F}_\Lambda]. \tag{2.36}$$

[5]The solutions reviewed here solve the leading order equations,and do not incorporate R^2 corrections.

In order to find S_{bh} as a function of charges, as needed in (2.35), one must solve the algebraic attractor equations [35, 64]. This may or may not be analytically possible, depending on the form of the prepotential and the charge vector. Note that the S_{bh} used here is a function of position and is equal to the black hole entropy only at the horizon. The constants h encode the values of the moduli at spatial infinity, i.e. $Re[CX^\Lambda(\infty)] = h^\Lambda, Re[CF^\Lambda(\infty)] = h_\Lambda$.

Given the moduli fields $X^\Lambda(\vec{x})$ the four dimensional metric is then simply

$$ds_4^2 = -\frac{\pi}{S_{bh}}\left(dt + \omega^{(4)}\right)^2 + \frac{S_{bh}}{\pi}d\vec{x}^2, \tag{2.37}$$

where it is implicit that $S_{bh} = S_{bh}(X^\Lambda(\vec{x}))$ and $\omega^{(4)}$ is the solution of

$$d\omega^{(4)} = H_\Lambda *^3 dH^\Lambda - H^\Lambda *^3 dH_\Lambda. \tag{2.38}$$

The gauge fields strengths are

$$dA^\Lambda = d\left[S_{bh}^{-1}\frac{\partial S_{bh}}{\partial q_\Lambda}\left(dt + \omega^{(4)}\right)\right]_{p^\Lambda = H^\Lambda,\ q_\Lambda = H_\Lambda} + *^3 dH^\Lambda. \tag{2.39}$$

Finally the equilibrium positions $\vec{x}_j$ of the black hole centers are determined by the asymptotic moduli and the charges via the integrability condition following from (2.38), which may be written

$$\left[p_j^\Lambda H_\Lambda(\vec{x}) - q_{j\Lambda} H^\Lambda(\vec{x})\right]\Big|_{\vec{x}=\vec{x}_j} = 0. \tag{2.40}$$

The basic example of this section, which illustrates the connection to the black ring, is a bound state of a single $D6$ brane at $r = 0$ and a $D4 - D2 - D0$ black hole with charges (p^A, q_A, q_0) at $r = L,\ \theta = 0$. The harmonic functions are:

$$H^0 = \tfrac{4}{R_{TN}^2} + \tfrac{1}{r}$$

$$
\begin{aligned}
H^A &= \tfrac{p^A}{(r^2+L^2-2rL\cos\theta)^{\frac{1}{2}}} \\
H_A &= h_A + \tfrac{q_A}{(r^2+L^2-2rL\cos\theta)^{\frac{1}{2}}} \\
H_0 &= -\tfrac{q_0}{L} + \tfrac{q_0}{(r^2+L^2-2rL\cos\theta)^{\frac{1}{2}}}
\end{aligned}
\tag{2.41}
$$

The integrability condition (2.40) is then

$$
\frac{1}{L} + \frac{4}{R_{TN}^2} = \frac{h_A p^A}{q_0}. \tag{2.42}
$$

As the parameter R_{TN} goes to infinity the distance between the centers reaches a minimum value, while for R_{TN} small enough the distance between the centers will diverge, and the bound state disappears.

2.6 Supersymmetric solutions in 5D

This section will describe some new supersymmetric 5D black ring-Taub-NUT-flux solutions which generalize previous solutions of minimal supergravity [33, 50]. In the next section we will see they are simply the lift to 5D of the 4D multicenter solutions reviewed in the previous section.

$\mathcal{N}=2$ supergravity fields in 5D are organized by the so-called very special geometry [44], parameterized by h_{11} real scalar fields Y^A, subject to the constraint

$$
D_{ABC}Y^AY^BY^C = 1, \tag{2.43}
$$

for constant couplings D_{ABC}. It is useful to further define

$$
Y_A \equiv 3D_{ABC}Y^BY^C. \tag{2.44}
$$

BPS solutions in 5D $\mathcal{N}=2$ supergravity may be written following [39, 45, 38, 50]

$$
ds_5^2 = -2^{-4/3}f^{-2}\left(dt+\omega\right)^2 + 2^{2/3}f\,ds_X^2
$$

$$F^A = d\left[f^{-1}Y^A(dt+\omega)\right] + \Theta^A \tag{2.45}$$

where ds_X^2 is hyperkäler metric on a 4D hyperkäler space X, Θ^A are closed self-dual 2-forms on X, the self-dual part of $d\omega$ is $-fY_A\Theta^A$ and f is a function on X obeying

$$\nabla^2(fY_A) = 3D_{ABC}\Theta^B\cdot\Theta^C. \tag{2.46}$$

When the space X is Taub-NUT,[6] one has

$$ds_X^2 = H^0(\vec{x})d\vec{x}^2 + H^0(\vec{x})^{-1}(dx^5+\omega^0)^2, \qquad d\omega^0 = *^3 dH^0, \tag{2.47}$$

with H^0 a harmonic function as in (2.41), and the coordinate x^5 has periodicity 4π. Closed self-dual 2-forms are then given by

$$\Theta^A = d\left[\frac{H^A}{H^0}(dx^5+\omega^0)\right] + *^3 dH^A, \tag{2.48}$$

with harmonic H^A as in (2.41). Inserting (2.48) the equation (2.46) for f becomes

$$\nabla^2(fY_A) = 6D_{ABC}\nabla\left(\frac{H^B}{H^0}\right)\cdot\nabla\left(\frac{H^C}{H^0}\right). \tag{2.49}$$

This is magically solved by

$$fY_A = H_A + \frac{3D_{ABC}H^BH^C}{H^0}. \tag{2.50}$$

f and Y_A are then determined by (2.43) and (2.50). This immediately gives the self-dual part of $d\omega$. It is straightforward to show that ω can be written

$$\omega = -\left(H_0 + 2\frac{D_{ABC}H^AH^BH^C}{(H^0)^2} + \frac{H_AH^A}{H^0}\right)(dx^5+\omega^0) + \omega^{(4)}, \tag{2.51}$$

where $\omega^{(4)}$ satisfies the integrability condition (2.40).

As we will see more clearly in the next section, the solution (2.45) together with (2.48)(2.50)(2.51) describe black rings in the Taub-NUT space.

[6] The more general solution with X being a Gibbons-Hawking space was presented in [38].

2.7 4D black holes → 5D black rings

$\mathcal{N}=2$ 5D supergravity may be viewed as the circle decompactification of $\mathcal{N}=2$ 4D supergravity. In this section we see that the 5D black ring solution of the previous section is the lift from 4D of the general multi-center black hole solutions. A discussion of the relevant geometry is in [43].

Given a solution of $ds^2_{4D}, A^A_{4D}, A^0_{4D}, z^A = \frac{X^A}{X^0}$ of 4D supergravity, a solution of 5D supergravity is quite generally given by

$$\begin{aligned} ds^2_{5D} &= 2^{2/3}\mathcal{V}^2(dx^5 + A^0_{4D})^2 + 2^{-1/3}\mathcal{V}^{-1}ds^2_{4D}, \\ A^A_{5D} &= A^A_{4D} + \mathrm{Re}z^A(dx^5 + A^0_{4D}), \\ Y^A &= \mathcal{V}^{-1}\mathrm{Im}z^A, \qquad\qquad \mathcal{V} \equiv \left(D_{ABC}\mathrm{Im}z^A\mathrm{Im}z^B\mathrm{Im}z^C\right)^{\frac{1}{3}}. \end{aligned} \tag{2.52}$$

Inserting the multi-center 4D solution of section 2 then gives the general black ring solutions of section 3.

To be more explicit, let us consider the 4D prepotential

$$F(X) = \frac{D_{ABC}X^AX^BX^C}{X^0}. \tag{2.53}$$

The expression for the entropy as a function of the charges is known, although complicated [62]. It is

$$\begin{aligned} &S(p,q) = 2\pi\sqrt{Q^3p^0 - J^2(p^0)^2}, \\ &Q^{\frac{3}{2}} = D_{ABC}y^Ay^By^C, \\ &3D_{ABC}y^Ay^B = q_C + \frac{3D_{ABC}p^Ap^B}{p^0}, \\ &J = \frac{q_0}{2} + \frac{D_{ABC}p^Ap^Bp^C}{(p^0)^2} + \frac{p^Aq_A}{2p^0}. \end{aligned} \tag{2.54}$$

Correspondingly there will be certain functions $Q(\vec{x})$ and $J(\vec{x})$ built out of the harmonic functions $(H^\Lambda(\vec{x}), H_\Lambda(\vec{x}))$. The volume of the Calabi-Yau at $\vec{x}$ is $\mathcal{V}(\vec{x})^3$, with

$$\mathcal{V}(\vec{x}) = \frac{S(H^\Lambda, H_\Lambda)}{2\pi H^0 Q} \tag{2.55}$$

With a bit algebra, (2.52) then yields for the metric

$$\begin{aligned} ds^2_{5D} &= -2^{-4/3} Q(\vec{x})^{-2} \left(dt + \omega^{(4)} - 2J(\vec{x})(dx^5 + \omega^0) \right)^2 + 2^{2/3} Q(\vec{x}) ds^2_{TN} \\ ds^2_{TN} &= H^0(\vec{x}) d\vec{x}^2 + H^0(\vec{x})^{-1} (dx^5 + \omega^0)^2 \end{aligned} \tag{2.56}$$

where

$$d\omega^0 = *^3 dH^0 \tag{2.57}$$

and $\omega^{(4)}$ is given by (2.38). This is precisely the solution we found in section 3. The 4D multi-centered black hole solution in general has several horizons, which lifts to horizons in 5D. These horizons are circle bundles over S^2. Depending on whether each center has nonzero magnetic KK charge, the 5D horizons will be either quotients of S^3, corresponding to a 5D spinning black hole, or $S^2 \times S^1$, corresponding to a black ring.

Now let us further specialize to the bound state of a single D6-brane with a D4-D2-D0 black hole with charges (p^A, q_A, q_0). The relevant harmonic functions are given in (2.41). In the limit $R_{TN} \to \infty$, $H^0 \to \frac{1}{r}$, (2.56) becomes precisely the black ring solution in flat 5D spacetime [33]. The radius of the ring is $R_{ring} = L$. It is constructed from wrapped M5 branes with charges p^A, carries M2 charges $\tilde{q}_A = q_A + 3D_{ABC}p^B p^C$ and $SU(2)_L$ spin $J_L = q_0/2$.

Note that the entropy of the two-centered black hole comes from only the D4-D2-D0 system. It is amusing to verify directly that the tree level entropy of the

D4-D2-D0 system of charge (p^A, q_A, q_0) indeed agrees with that of the black ring [9, 21] with M5-M2 charge $(p^A, \tilde{q}_A)$ and angular momentum $J_L = q_0/2$.

More generally, when there are mulitple 4D black holes carrying D6 charge, the background geometry of the 5D lift will be a resolved multi-Taub-Nut-flux geometry. The black holes of the 4D solution that carry $D6$ charges will lift to 5D spinning black holes at the fixed points of the $U(1)_L$ isometry of the multi-Taub-NUT background. Those that do not will lift to 5D black rings tracing orbits of the isometry.

Chapter 3

Counting Supersymmetric Black Holes

Some years ago an explicit formula for the elliptic genus which counts 1/4 BPS dyons in four dimensional $\mathcal{N} = 4$ theories was presciently conjectured [31]. Using the 4D-5D connection described in chapter 1, we will derive this conjecture from the 5D elliptic genus, which is the elliptic genus of symmetric products of $K3$.

We will then extend this approach to the counting of 1/8 BPS dyons in $\mathcal{N} = 8$ string theory in four dimensions, using the 5D modified elliptic genus. In addition we describe a direct microscopic counting of D0-D2-D4 bound states which gives the same result.

Section 1 is a review of some basic facts about D1-D5 system on $K3 \times S^1$ and $T^4 \times S^1$, dual to five dimensional BPS black holes. In section 2 we will compute the elliptic genus of $\mathcal{N} = 4$ string theory based on the exact relation between 4D and 5D BPS states. In section 3 we derive the 4D index for the $\mathcal{N} = 8$ black hole from the

5D modified elliptic genus. We also sketch how this expression should follow (for one element of the U-duality class of black holes) from a microscopic analysis in 4D.

3.1 D1-D5 CFT on $K3$ and T^4

3.1.1 $K3$

We shall review some well-known facts about the D1-D5 system on $K3 \times S^1$ and $T^4 \times S^1$ in type IIB string theory. In this subsection we consider $K3$. One starts with $\tilde{Q}_1$ D1-branes wrapped on the S^1, bound to Q_5 D5-branes wrapped on $K3 \times S^1$. When the $K3$ is small compared to the size of the S^1, the low energy effective theory of this system can be described by a 1+1 dimensional CFT living on the circle. This is a $(4, 4)$ superconformal field theory, whose nontrivial is given by the sigma model on the moduli space of $\tilde{Q}_1$ $U(Q_5)$ instantons on $K3$, which is a smooth resolution of the symmetric product space $(K3)^N/S_N$, where $N = (\tilde{Q}_1 - Q_5)Q_5 + 1$. The shift of $\tilde{Q}_1$ is due to the induced D1-brane charge on a D5-brane wrapped on $K3$, $-c_2(K3)/24 = -1$. The number N is U-duality invariant, and in the case $Q_5 = 1$ $N = \tilde{Q}_1$, which agrees with the intuition that the target space of the sigma model is the moduli space of $\tilde{Q}_1$ points moving on the $K3$ world volume. For the rest of this chapter we will write $Q_1 = \tilde{Q}_1 - Q_5$ for the physical D1-brane charge.

We will be interested in the elliptic genus of the D1-D5 CFT, given by

$$Z_{D1-D5}(q, y) = \mathrm{Tr}_{RR}(-)^F q^{L_0} \overline{q}^{\overline{L}_0} y^{2J_L} \tag{3.1}$$

where J_L is a generator of the $SU(2)_L$ R-symmetry for the left movers. The elliptic genus only receives contribution from states that involve Ramond ground state on the

Figure 3.1: The D1-D5 system on $K3 \times S^1$

right, and is invariant under deformations of the CFT. In particular, one can compute (3.1) using the orbifold CFT $(K3)^N/S_N$. Let us write $\chi_N(\rho,\nu)$ for the elliptic genus of the orbifold CFT with $q = e^{2\pi i\rho}, y = e^{2\pi i\nu}$. It is shown in [30] that the weighted sum of the elliptic genera has a product representation:

$$\sum_{N\geq 0} \chi_N(\rho,\nu) e^{2\pi i N\sigma} = \frac{1}{\Phi'(\rho,\sigma,\nu)} \tag{3.2}$$

where Φ' is given by

$$\Phi'(\rho,\sigma,\nu) = \prod_{k\geq 0, l>0, m\in\mathbf{Z}} (1 - e^{2\pi i(k\rho + l\sigma + m\nu)})^{c(4kl-m^2)}, \tag{3.3}$$

with $c(4k - m^2) = d_5'(k,1,m)$ being the elliptic genus coefficients for a single $K3$ [48]. Namely the elliptic genus for the CFT on $K3$ has the expansion [32, 49]

$$Z_{K3}(q|y) = \sum_{k,m} c(4k-m^2) q^k y^m = 24 \left(\frac{\vartheta_3(q|y)}{\vartheta_3(q|1)}\right)^2 - 2\frac{\vartheta_4(q|1)^4 - \vartheta_2(q|1)^4}{\eta(q)^4} \left(\frac{\vartheta_1(q|y)}{\eta(q)}\right)^2 \tag{3.4}$$

In particular, $c(-1) = 2$, $c(0) = 20$, and $c(n) = 0$ for $n \leq -2$.

3.1.2 T^4

In this subsection, we want to summarize the work of reference [55] on counting the microstates of 1/8 BPS black holes in five dimensions. These can be realized in string theory as the usual D1-D5-momentum system of type IIB on $T^4 \times S^1$, with Q_1 D1-branes, Q_5 D5-branes and integral S^1 momentum n. The reason that microstate counting of this system is more difficult than for $K3$ compactification is because the usual supersymmetric index that counts these microstates, the orbifold elliptic genus of $Hilb^k(K3)$ with $k = Q_1Q_5$, vanishes when $K3$ is replaced with T^4. In [55], this difficulty was overcome by defining (and then computing) a new supersymmetric index $\mathcal{E}_2$, closely related with the elliptic genus, which is nonvanishing for T^4. We will refer to this new supersymmetric index as the modified elliptic genus of $Hilb^k(T^4)$. It is defined to be

$$\mathcal{E}_2^{(k)} = Tr\left[(-1)^{2J_L^3-2J_R^3}2(J_R^3)^2 q^{L_0}\overline{q}^{\overline{L}_0}y^{2J_L^3}\right] \tag{3.5}$$

where the trace is over states of the sigma model with target space $Hilb^k(T^4)$.[1] Here J_L^3 and J_R^3 are the left and right half-integral $U(1)$ charges of the CFT, and they are identified with generators of SO(4) rotations of the transverse R^4. The S^1 momentum is $n = L_0 - \overline{L}_0$. The usual elliptic genus is given by the same formula but without the $2(J_R^3)^2$ factor; it is these two insertions of J_R^3 that make $\mathcal{E}_2$ nonvanishing for T^4.

As for $K3$, here it is convenient to define a generating function for the modified elliptic genus:

$$\mathcal{E}_2 = \sum_{k\geq 1} p^k \mathcal{E}_2^{(k)} \tag{3.6}$$

[1] A free sigma model on $R^4 \times T^4$ is factored out here, and our definition differs by a factor of 2 from [55].

In [55], this was shown to be given by the following sum

$$\mathcal{E}_2(p,q,y) = \sum_{s,k,n,l} s(p^k q^n y^l)^s \widehat{c}(nk,l) \tag{3.7}$$

with the sum running over $s, k \geq 1$, $n \geq 0$, $l \in \mathbf{Z}$. Note that the $\overline{q}$ dependence has dropped out – only the $\overline{L}_0 = 0$ states contribute to the modified elliptic genus. Of course, the index must have this property in order to count BPS states, since the BPS condition is equivalent to requiring $\overline{L}_0 = 0$.

It was furthermore shown in [55] that the integers $\widehat{c}(nm,l)$ are the coefficients in the following Fourier expansion

$$Z(q,y) \equiv -\eta(q)^{-6}\vartheta_1(y|q)^2 = \sum_{n,l} \widehat{c}(n,l) q^n y^l \tag{3.8}$$

where $\eta(q)$ is the usual Dedekind eta function, and $\vartheta_1(y|q)$ is defined by the product formula

$$\vartheta_1(y|q) = i(y^{1/2} - y^{-1/2})q^{1/8} \prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n) \tag{3.9}$$

Finally, it was observed in [55] that $\widehat{c}(n,l)$ actually only depends on a single combination of parameters $4n - l^2$:

$$\widehat{c}(n,l) = \widehat{c}(4n - l^2) \tag{3.10}$$

Using (3.10) in (3.7) yields

$$\mathcal{E}_2(p,q,y) = \sum_{s,k,n,l} s(p^k q^n y^l)^s \widehat{c}(4nk - l^2) \tag{3.11}$$

When (k,n,l) are coprime, $\widehat{c}(4nk - l^2)$ counts BPS black holes with $k = Q_1 Q_5$, S^1 momentum n and spin $J^3_L = \frac{l}{2}$, multiplied by an overall $(-)^l$ and summed over J^3_R

weighted by $2(J_R^3)^2(-)^{2J_R^3}$:

$$\widehat{c}(4nk-l^2)\Big|_{(k,n,l)\,\mathrm{coprime}} = (-)^l \sum_{J_R,BPS\ states} 2(J_R^3)^2(-)^{2J_R^3} \tag{3.12}$$

When they are not coprime, the black hole can fragment, and the situation is more complicated due to multiple contributions in $\mathcal{E}_2$ [55]. In the present discussion we will always avoid this complication by choosing coprime charges.

We should note that $Z(q,y)$ is also the modified elliptic genus of T^4, i.e.

$$\mathcal{E}_2^{(1)} = \sum_{n,l} \widehat{c}(n,l) q^n y^l = Z(q,y). \tag{3.13}$$

This corresponds to the coprime D1-D5 system with $k = 1 = Q_1 = Q_5$. By writing

$$Z(q,y) = \sum_m \widehat{c}(4m) q^m \sum_k q^{k^2} y^{2k} + \sum_m \widehat{c}(4m-1) q^m \sum_k q^{k^2+k} y^{2k+1} \tag{3.14}$$

and using (3.8) along with the standard Fourier expansion of the theta function

$$\vartheta_1(y|q) = i \sum_{n\in\mathbf{Z}} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \tag{3.15}$$

one can reorganize the generating functions for $\widehat{c}$ as

$$\begin{aligned} \textstyle\sum_m \widehat{c}(4m) q^m &= -q^{\frac{1}{4}} \eta(q)^{-6} \textstyle\sum_{m\in\mathbf{Z}} q^{m^2+m}, \\ \textstyle\sum_m \widehat{c}(4m-1) q^m &= q^{\frac{1}{4}} \eta(q)^{-6} \textstyle\sum_{m\in\mathbf{Z}} q^{m^2}. \end{aligned} \tag{3.16}$$

These expressions will analyzed microscopically below in section 4.

3.2 Counting 4D $\mathcal{N} = 4$ black holes

A general D0-D2-D4-D6 black hole in a 4D IIA string compactification has an M-theory lift to a 5D black hole configuration in a multi-Taub-NUT geometry. This

observation was used in [37] to derive a simple relation between 5D and 4D BPS black hole degeneracies. For the case of $K3 \times T^2$ compactification, corresponding to $\mathcal{N} = 4$ string theory, the relevant 5D black holes were found in [66, 17] and the degeneracies are well known. In this section we translate this into an exact expression for the 4D degeneracies, which turn out to be Fourier expansion coefficents of a well-studied weight 10 automorphic form Φ of the modular group of a genus 2 Riemman surface [16, 31].

A decade ago an inspired conjecture was made [31] by Dijkgraaf, Verlinde and Verlinde for the 4D degeneracies of $\mathcal{N} = 4$ black holes, and this was shown to pass several consistency checks. We will see that our analysis precisely confirms their old conjecture.

$\mathcal{N} = 4$ string theory in four dimensions can be obtained from IIA compactification on $K3 \times T^2$. The duality group is conjectured to be

$$SL(2;\mathbf{Z}) \times \mathbf{SO(6,22;Z)}. \tag{3.17}$$

The first factor may be described as an electromagnetic S-duality which acts on electric charges $q_{e\Lambda}$ and magnetic charges q_m^Λ, $\Lambda = 0, ..27$ transforming in the 28 of the second factor. For the electric objects, we may take

$$q_e = (q_0; q_A; q_{23}; q_j), \tag{3.18}$$

where q_0 is D0-charge, $q_A, A = 1, ...22$ is $K3$-wrapped D2 charge, q_{23} is $K3$-wrapped D4 charge, and $q_i, i = 24, ...27$ are momentum and winding modes of $K3 \times S^1$-wrapped NS5 branes. The magnetic objects are 24 types of D-branes which wrap $T^2 \times$ (K3 cycle) and 4 types of F-string T^2 momentum/winding modes.

Now consider a black hole corresponding to a bound state of a single D6 brane with D0 charge q_0, $K3$-wrapped D2 charge q_A, and T^2-wrapped D2 charge q^{23}:

$$q_m = (1; q^A = 0; q^{23}; q^i = 0), \quad q_e = (q_0; q_A; q_{23} = 0; q_i = 0) \tag{3.19}$$

The duality invariant charge combinations are

$$\frac{1}{2}q_e^2 = \frac{1}{2}C^{AB}q_A q_B, \quad \frac{1}{2}q_m^2 = q^{23}, \quad q_e \cdot q_m = q_0 \tag{3.20}$$

where C^{AB} is the intersection matrix on $H^2(K3; \mathbf{Z})$.

By lifting this to M-theory on Taub-NUT, it was argued in [37] that the BPS states of this system are the same as those of a 5D black hole in a $K3 \times T^2$ compactification, with T^2-wrapped M2 charge $\frac{1}{2}q_m^2$, $K3$-wrapped M2 charge q_A and angular momentum $J_L = q_0/2$. We now use one of the compactification circles to interpret the configuration as IIA on $K3 \times S^1$ with $\frac{1}{2}q_m^2$ F-strings winding S^1 and q_A D2-branes. T-dualizing the S^1 yields q_A D3-branes carrying momentum $\frac{1}{2}q_m^2$. This is then U-dual to a Q_1 D1 branes and Q_5 D5 branes on $K3 \times S^1$ with

$$N \equiv Q_1 Q_5 = \frac{1}{2}q_e^2 + 1 \tag{3.21}$$

angular momentum[2]

$$J_L = \frac{1}{2}q_e \cdot q_m \tag{3.22}$$

and left-moving momentum along the S^1:

$$\ell_0 = \frac{1}{2}q_m^2. \tag{3.23}$$

[2]One should keep in mind that J_L is half the R-charge F_L [17], and is hence takes values in $\frac{1}{2}\mathbf{Z}$.

Hence, with the above relations between parameters, according to [37] the 4D degeneracy of states with charges (3.19) and 5D degeneracies are related by

$$d_4(1;0;q^{23};0|q_0;q_A;0;0) = d_5\left(q^{23}, q_A; \frac{q_0}{2}\right). \tag{3.24}$$

Since the degeneracies are U-dual we may also write[3]

$$d_4(q_m^2, q_e^2, q_e \cdot q_m) = d_5(\ell_0, N, J_L) = d_5\left(\frac{1}{2}q_m^2, \frac{1}{2}q_e^2 + 1, \frac{1}{2}q_e \cdot q_m\right). \tag{3.25}$$

Here and elsewhere in this chapter by "degeneracies," in a slight abuse of language, we mean the number of bosons minus the number of fermions of a given charge, and the center-of-mass multiplet is factored out.

Of course these microscopic BPS degeneracies d_5 of the D1-D5 system are well known [66, 17]. Their main contribution comes from the coefficients in the Fourier expansion of the elliptic genus of $\mathrm{Hilb}^N(K3)$:

$$\chi_N(\rho, \nu) = \sum_{\ell_0, J_L} d_5'(\ell_0, N, J_L) e^{2\pi i(\ell_0\rho + 2J_L\nu)} \tag{3.26}$$

as reviewed in the previous section. Equation (3.2) is the generating function for BPS states of CFTs on $\mathrm{Hilb}^N(K3)$ in the D5 worldvolume. However it does not quite give the degeneracies needed in (3.25) because it leaves out the decoupled contribution from the elliptic genus of a single fivebrane. This remains even when $N = 0$ and there are no D1 branes at all. (By U-duality, we are free to view the system as a single fivebrane and N D1 branes.) Using the U-dual relation of a $K3$ -wrapped D5 brane to a fundamental heterotic string, the elliptic genus, not including the center

[3]Note that d_n denotes fixed-charge degeneracies and does not involve a sum over U-duality orbits.

of mass contribution, is [5, 41]

$$Z_0(\nu,\rho) = (e^{\pi i\nu} - e^{-\pi i\nu})^{-2} e^{-2\pi i\rho} \prod_{n\geq 1} (1 - e^{2\pi i(n\rho+\nu)})^{-2} (1 - e^{2\pi i(n\rho-\nu)})^{-2} (1 - e^{2\pi i n\rho})^{-20}. \tag{3.27}$$

This shifts Φ' to

$$\frac{1}{\Phi'(\Omega)} \to \frac{Z_0(\nu,\rho)}{\Phi'(\Omega)} = \frac{e^{2\pi i\sigma}}{\Phi(\Omega)} \tag{3.28}$$

where $\Phi(\Omega)$ has a product representation

$$\Phi(\Omega) = e^{2\pi i(\rho+\sigma+\nu)} \prod_{(k,l,m)>0} \left(1 - e^{2\pi i(k\rho+l\sigma+m\nu)}\right)^{c(4kl-m^2)} \tag{3.29}$$

where $(k,l,m) > 0$ means that $k,l \geq 0$, $m \in \mathbf{Z}$ and in the case $k = l = 0$, the product is only over $m < 0$. $\Phi(\Omega)$ is the unique automorphic form of weight 10 of the modular group $Sp(2,\mathbf{Z})$ and was studied in [16]. The 5D BPS degeneracies are then the Fourier coefficients in

$$\sum_{\ell_0,N,J_L} d_5(\ell_0,N,J_L) e^{2\pi i(\ell_0\rho+(N-1)\sigma+2J_L\nu)} = \frac{1}{\Phi(\rho,\sigma,\nu)}. \tag{3.30}$$

Inserting the 4D-5D relation (3.25)(3.30) agrees exactly with the formula proposed in [31] for the microscopic degeneracy of BPS black holes of $\mathcal{N} = 4$ string theory.

3.3 Counting 4D $\mathcal{N} = 8$ black holes

3.3.1 5D $\to$ 4D

Let us now apply the 4D/5D connection to transform the 5D degeneracies (3.7) into 4D ones. The fact that the $\widehat{c}$ coefficients depend only on the combination $4nk - l^2$ is very encouraging, for the following reason. We expect the 1/8 BPS 5D degeneracies

to be related to degeneracies of 1/8 BPS black holes in 4D, and in 4D U-duality implies [46] that the black hole entropy must depend on the unique quartic invariant of $E_{7,7}$, the so-called Cremmer-Julia invariant [20]. In an $\mathcal{N} = 4$ language, this invariant takes the form

$$\mathcal{J} = q_e^2 q_m^2 - (q_e \cdot q_m)^2 \tag{3.31}$$

where q_e and q_m are the electric and magnetic charge vectors for $\mathcal{N} = 4$ BPS states. (See e.g. [61] for details on the notation.) This is precisely the dependence of $\widehat{c}$ on n, m, l, provided we identify

$$k = \frac{1}{2} q_e^2, \qquad n = \frac{1}{2} q_m^2, \qquad l = q_e \cdot q_m. \tag{3.32}$$

Note that from the purely 5D point of view, there was no obvious reason that $\widehat{c}$ should depend only on the combination $4nk - l^2$ as there is no 5D U-duality which mixes spins with charges.

Let us now derive the identification (3.32) from the dictionary of [37], beginning from the IIB spinning 5D D1-D5-n black hole of the previous section. First we T-dual on S^1 to obtain a black hole with spin $\frac{l}{2}$, F-string winding n, Q_1 D0-branes, and Q_5 D4-branes. Now T-dual so that there are $Q_1 + Q_5$ D2 branes with intersection number $Q_1 Q_5 = k$ on the T^4. Next we compactify on a single center Taub-NUT, whose asymptotic circle we identify as the the new M-theory circle. The result is three orthogonal sets of (n, Q_1, Q_5) D2-branes on T^6, l D0-branes, and one D6-brane. For IIA D-brane configurations with D0, D2, D4, D6 charges $(q_0, q_{ij}, p^{ij}, p^0)$, where

$i = 1, ...6$ runs over the T^6 cycle and $p^{ij} = -p^{ji},\ q_{ij} = -q_{ji}$ $\mathcal{J}$ reduces to[4]

$$\begin{aligned}\mathcal{J} &= \frac{1}{12}(q_0\epsilon_{ijklmn}p^{ij}p^{kl}p^{mn} + p^0\epsilon^{ijklmn}q_{ij}q_{kl}q_{mn}) \\ &\quad -p^{ij}q_{jk}p^{kl}q_{li} + \frac{1}{4}p^{ij}q_{ij}p^{kl}q_{kl} - (p^0q_0)^2 + \frac{1}{2}p^0q_0p^{ij}q_{ij}.\end{aligned} \tag{3.33}$$

For our D0-D2-D6 configuration, we can pick a basis of cycles without loss of generality such that the nonzero charges are

$$p^0 = 0, \quad q_0 = l, \quad q_{12} = -q_{21} = n, \quad q_{34} = -q_{43} = Q_1, \quad q_{56} = -q_{65} = Q_5 \tag{3.34}$$

Then (3.33) reduces to

$$\mathcal{J} = 4nk - l^2, \tag{3.35}$$

which, as stated above, is exactly the argument of (3.7).

According to [37] the weighted degeneracy of the 4D black hole resulting from U-duality and Taub-NUT compactification equals that of the original 5D black hole, when J^3_R in (3.12) is identified with the generator J^3 of $\mathbf{R}^3$ rotations in 4D. Note that, since $\mathcal{J}$ is odd if and only if l is, we may trade $(-)^l$ for $(-)^{\mathcal{J}}$ in (3.12). Therefore, for fixed coprime charges, the weighted 4D BPS degeneracy depends only on the the Cremmer-Julia invariant and is given by

$$\sum_{J^3, BPS\ states} 2(J^3)^2(-)^{2J^3} = (-)^{\mathcal{J}}\widehat{c}(\mathcal{J}). \tag{3.36}$$

Note that, although this formula for the 4D BPS degeneracy was derived assuming a specific D6-D2-D0 configuration, it applies to all D-brane configurations by U-duality.

As a first check on this conjecture, we note that for large charges $\widehat{c}(\mathcal{J}) \sim e^{\pi\sqrt{J}}$. From the supergravity solutions Area $= 4\pi\sqrt{J}$, so there is agreement with the Bekenstein-Hawking entropy.

[4]See e.g. [34], equation (66), and take $p^0 = p_{87},\ \ p_{8i} = 0$, etc. Our definition of $\mathcal{J}$ differs from that of [34] by a sign.

As an example, let's consider the modified elliptic genus for the D4-D0 black hole on T^6, in which we fix the D4 charges and sum over D0 charge q_0. Consider the T^6 of the form $T^2 \times T^2 \times T^2$ with $\alpha_1, \alpha_2, \alpha_3$ being the three 2-cycles associated with the T^2's. Let A^1, A^2, A^3 be the dual 4-cycles. We shall consider the D4-brane wrapped on the cycle $[P] = A^1 + A^2 + A^3$. Its triple self-intersection number is $D = P \cdot P \cdot P = 6$. From (3.33) we have

$$\mathcal{J} = 4q_0. \tag{3.37}$$

We then have

$$\mathcal{E}_2(q) = \sum_{q_0 \in \mathbf{Z}} \widehat{c}(4q_0) q^{q_0} = -q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbf{Z}} q^{m^2+m}. \tag{3.38}$$

according to (3.16).

A straightforward generalization of this example is the D4-D2-D0 system, where we wrap (q_1, q_2, q_3) D2 branes on the 2-cycles $(\alpha_1, \alpha_2, \alpha_3)$. In this case, the Cremmer-Julia invariant becomes

$$\mathcal{J} = 4(q_0 + q_1 q_2 + q_1 q_3 + q_2 q_3) - (q_1 + q_2 + q_3)^2 \tag{3.39}$$

and the sum over q_0 produces

$$\mathcal{E}_2(q) = \sum_{q_0 \in \mathbf{Z}} (-1)^{\mathcal{J}} \widehat{c}(\mathcal{J}) q^{q_0} = \begin{cases} -q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbf{Z}} q^{m^2+m-\frac{1}{4}\tilde{\mathcal{J}}} & q_1 + q_2 + q_3 \text{ even} \\ -q^{\frac{1}{4}} \eta(q)^{-6} \sum_{m \in \mathbf{Z}} q^{m^2-\frac{1}{4}\tilde{\mathcal{J}}-\frac{1}{4}} & q_1 + q_2 + q_3 \text{ odd} \end{cases} \tag{3.40}$$

where $\tilde{\mathcal{J}} = 4(q_1 q_2 + q_1 q_3 + q_2 q_3) - (q_1 + q_2 + q_3)^2$. Now let us turn to the 4D derivation of (3.38) and (3.40).

3.3.2 D4-D2-D0 bound states on T^6

In this section we sketch a derivation of (3.38) and (3.40) using a 4D microscopic analysis. The derivation is not complete because, as we will discuss below, we ignore some potential subtleties associated to the fact that P is not simply connected. In principle it should be possible to close this gap. A microscopic description of T^6 black holes using the M-theory picture of wrapped fivebranes has been given in [14], adapting the description given in [56] for a general Calabi-Yau, in terms of a $(0,4)$ 2D CFT living on the M-theory circle. For uniformity and simplicity of presentation, we here will use the IIA description in which fivebrane momenta around the M-theory circle become bound states of D0 branes to D4 branes.

As above (3.37) we examine the special case of the D4-D0 system wrapped on $[P] = A^1 + A^2 + A^3$. The D4-D0 system can be described in terms of the quantum mechanics of q_0 D0-branes living on the D4-brane world volume P. There is a family of complex structures on T^6 such that P is holomorphically embedded in T^6. In this case, one can compute its Euler character, $\chi(P) = 6$. It follows from the Riemann-Roch formula that the only modulus of P is the overall translation in T^6.[5] Since $\chi(P) = 6$, P has $4 + 2b_1$ 2-cycles. By the Lefschetz hyperplane theorem we have $b_1(P) = b_1(T^6) = 6$, and therefore $b_2(P) = 16$. All but one of the 2-cycles come from the intersection of P with $\binom{6}{4} = 15$ 4-cycles in T^6. We will be mostly interested in 3 of these, denoted by $\tilde{\alpha}_i$, corresponding to intersections of A^i with P. Turning on fluxes along these three 2-cycles corresponds to having charges of D2-branes wrapped

[5]The dual line bundle $\mathcal{L}_P$ of the divisor P has only one holomorphic section. However as T^6 is not simply connected, the line bundle $\mathcal{L}_P$ is not only determined by $c_1(\mathcal{L}_P) = [P]$. In fact the translation of T^6 takes it to a different line bundle.

on the α_i's. Their intersection numbers are

$$\tilde{\alpha}_i \cdot \tilde{\alpha}_j = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases} \tag{3.41}$$

There is, however, one extra 2-cycle in P, which we shall denote by β, that does not correspond to any cycle in the T^6.

One can show from the adjunction formula that $c_1(P)$ is Poincaré dual to $-(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3)$. It then follows from Hirzebruch signature theorem that

$$\sigma(P) = -\frac{2}{3}\chi(P) + \frac{1}{3}\int_P c_1^2 = -2. \tag{3.42}$$

We conclude that the intersection form on P is odd (and that P is not a spin manifold). Essentially the unique way to extend (3.41) to an odd rank 4 unimodular quadratic form is to have an extra 2-cycle γ with

$$\gamma \cdot \tilde{\alpha}_i = 1, \quad \gamma \cdot \gamma = 1. \tag{3.43}$$

Now if we choose $\beta = 2\gamma - \sum \tilde{\alpha}_i$, we have

$$\beta \cdot \tilde{\alpha}_i = 0, \quad \beta \cdot \beta = -2. \tag{3.44}$$

Note that $(\tilde{\alpha}_i, \beta)$ is not an integral basis for $H_2(P, \mathbf{Z})$, yet β is the smallest 2-cycle that doesn't intersect $\tilde{\alpha}_i$. The total intersection form on P is the sum of this rank 4 form together with 6 copies of σ_1 coming from the 12 other 2-cycles in P.

In fact, when P is smoothly embedded in the T^6, the T^6 is a principally polarized Abelian variety, and is the Jacobian variety of a certain genus 3 curve S. There is a canonical embedding $S \to T^6 \simeq \mathcal{J}(S)$. Up to translation, P is given by $S + S \subset T^6$. One can choose the image of S to coincide with P, and then the class of S in P is nothing but the 2-cycle γ above.

Now one can turn on gauge field flux on the D4-brane world volume along β, which does not correspond to any D2-brane charge. This flux nevertheless induces D0-brane charge. There is a subtlety in the quantization of this flux. As well known, the curvature of the D4-brane world volume induces an anomalous D0-brane charge $-\chi(P)/24 = -\frac{1}{4}$. In order that the total D0 charge be integral the flux along the cycle β on the D4-brane must be half-integer, i.e. of the form $(m+\frac{1}{2})\beta$. The total induced D0-brane charge is $\Delta q_0 = -\frac{1}{2}(m+\frac{1}{2})^2\beta\cdot\beta - \frac{1}{4} = m^2+m$, which is indeed an integer.[6]

We ignore here the facts arising form nonzero $b_1(P)$ that there is a moduli space of flat connections as well as overall T^6 translations which must be quantized. These factors are treated in the language of the 2D CFT in [14]. They are found to lead to extra degrees of freedom which are however eliminated by extra gauge constraints. A complete microscopic derivation, not given here, would have to show that a careful accounting of these factors give a trivial correction to our result.

It is now straightforward to reproduce (3.38). Each D0-D4 bound state is in a hypermultiplet which contributes minus one to $Tr[2(J^3)^2(-)^{2J^3}]$. Counting the number of ways of distributing n D0-branes among the $\chi(P)=6$ ground states of the supersymmetric quantum mechanics, and then summing over n, gives the factor of $q^{1/4}\eta(q)^{-6} = \prod_{k=1}^{\infty}(1-q^k)^{-6}$ in (3.38). Including finally the sum over fluxes on β, we precisely reproduce the degeneracy (3.38)!

Let us now consider the more general case of D4-D2-D0 system. Again we shall assume $(p^1,p^2,p^3)=(1,1,1)$. The D2-brane charges are labelled by (q_1,q_2,q_3). The

[6]In the M-theory picture the anomalous D0 charge is the left-moving zero point energy $-\frac{c_L}{24} = -\frac{1}{4}$, and the 2-cycle fluxes correspond to momentum zero modes of scalars on a Narain lattice.

bound state is described by the D4-brane with D2-brane dissolved in its world volume. We end up with the gauge flux

$$F = (m+1/2)\beta + \sum_{i=1}^{3} q_i \delta_i, \qquad \delta_i \cdot \tilde{\alpha}_j = \delta_{ij}. \tag{3.45}$$

In above expression δ_i is defined up to a shift of an integer multiple of β. Since we are summing over m, this ambiguity is irrelevant. We can choose $\delta_i = \gamma - \tilde{\alpha}_i$. The total induced D0-brane charge is then

$$\begin{aligned}
\Delta q_0 &= -\int \frac{1}{2}F^2 - \frac{1}{4} \\
&= (m+1/2)^2 + (m+1/2)\sum q_i + \frac{1}{2}\sum q_i^2 - \frac{1}{4} \\
&= \left(m + \frac{1}{2} + \frac{1}{2}\sum q_i\right)^2 - \frac{1}{12}D^{AB}q_A q_B - \frac{1}{4},
\end{aligned} \tag{3.46}$$

where D^{AB} is the inverse matrix of $D_{AB} \equiv D_{ABC}p^C$,

$$D^{AB}q_A q_B = 3(2q_1q_2 + 2q_2q_3 + 2q_3q_1 - q_1^2 - q_2^2 - q_3^2). \tag{3.47}$$

Note that $\frac{1}{3}D^{AB}q_Aq_B = 0$ mod 4 if $\sum q_i$ is even, and $\frac{1}{3}D^{AB}q_Aq_B = -1$ mod 4 if $\sum q_i$ is odd. Therefore Δq_0 is always an integer, as expected. The Cremmer-Julia invariant is in this case

$$\mathcal{J} = 4\left(q_0 + \frac{1}{12}D^{AB}q_Aq_B\right). \tag{3.48}$$

The counting of D0-brane states as before gives the generating function

$$\sum_{q_0}(-)^{\mathcal{J}}c(\mathcal{J})q^{q_0} = -\prod_{k=1}^{\infty}(1-q^k)^{-6}\sum_{m\in\mathbf{Z}} q^{m^2+m-\frac{1}{12}D^{AB}q_Aq_B} \tag{3.49}$$

in the case $\sum q_i \in 2\mathbf{Z}$ and $\mathcal{J} \equiv 0$ mod 4, and

$$\sum_{q_0}(-)^{\mathcal{J}}c(\mathcal{J})q^{q_0} = -\prod_{k=1}^{\infty}(1-q^k)^{-6}\sum_{m\in\mathbf{Z}} q^{m^2-\frac{1}{12}D^{AB}q_Aq_B-\frac{1}{4}} \tag{3.50}$$

in the case $\sum q_i \in 2\mathbf{Z}+1$ and $\mathcal{J} \equiv -1$ mod 4. These are precisely the degeneracies (3.40) we derived from 5D earlier!

Chapter 4

Probing $AdS_2 \times S^2$

In this chapter we will analyze the supersymmetric D-branes in type IIA string theory compactified on $AdS_2 \times S^2 \times X$ as classical probes, where X is a Calabi-Yau threefold.

The relevant near horizon attractor geometry was reviewed in chapter 1. In section 1 the problem of supersymmetric branes is analyzed from the viewpoint of the four dimensional effective $\mathcal{N} = 2$ theory on $AdS_2 \times S^2$. This analysis is facilitated by the recent construction [15] of the κ-symmetric superparticle action carrying general electric and magnetic charges (u^Λ, v_Λ) in such theories. It is found that there is always a supersymmetric trajectory whose position is determined by the phase of the central charge $Z(u^\Lambda, v_\Lambda)$. In global AdS_2 coordinates

$$ds^2 = R^2(-\cosh^2\chi d\tau^2 + d\chi^2 + d\theta^2 + \sin^2\theta d\phi^2) \tag{4.1}$$

the supersymmetric trajectory is at

$$\tanh\chi = \frac{\mathrm{Re}Z}{|Z|}. \tag{4.2}$$

For the general case $\chi \neq 0$ this trajectory is accelerated by the electromagnetic forces. We further consider n-particle configurations with differing charges and differing central charges $Z_i, \quad i = 1, ..n$, constrained only by the condition that they all have the same sign for $\mathrm{Re}Z_i$. Surprisingly if the positions of the charges are each determined by (4.2), a common supersymmetry is preserved for the entire multiparticle configuration. This is quite different than the case of fluxless Calabi-Yau-Minkowski compactifications, where there is a common supersymmetry only if the charges are aligned. Supersymmetry preservation is possible only because of the enhanced near-horizon superconformal group. This phenomena should have a counterpart in higher dimensional AdS spaces and may be of interest for braneworld scenarios.

In section 2 we consider the problem from the ten-dimensional perspective. For simplicity we consider only the $AdS_2 \times S^2 \times CY_3$ geometries arising from $D0 - D4$ Calabi-Yau black holes. Adapting the analysis of [57] to this context, we allow the wrapped branes to induce lower brane charges by turning on worldvolume field strengths. We will find that there are no static, supersymmetric D0-branes in global coordinates because they want to accelerate off to the boundary of AdS_2 (there are static BPS configurations in Poincaré coordinates) . For a D2-brane embedded holomorphically in the Calabi-Yau, we will find that it is half BPS and sits at $\chi = \tanh^{-1}(\sin \beta_{CY})$. Here, β_{CY} is related to the amount of magnetic flux on the worldvolume. All D2-brane that are static with respect to a common global time in AdS_2 preserve the same set of half of the supersymmetries regardless of β_{CY}. Similar conclusions hold for D4, D6-branes wrapped on the Calabi-Yau. We also consider a D2-brane wrapped on the S^2 of the $AdS_2 \times S^2$ product and find that it is once again

half BPS and sits at $\chi = \tanh^{-1}(\sin \beta_{S^2})$.

In section 3 we will consider the gravity solution corresponding to the superparticles in $AdS_2 \times S^2$. These solutions are constructed using the ansatz described in chapter 2. The problem of finding black hole solutions will be related to finding a certain harmonic function on a double cover of $\mathbf{R}^3$, and explicit solutions are found. In particular, it is very natural to write down the solution that describes a pair of black hole and anti-black hole sitting at the north and south poles of the S^2, which are mutually supersymmetric. All the black hole solutions, however, have regions of closed timelike curves. It would be nice to understand whether/how these CTCs can be removed and what it means for states in $AdS_2 \times S^2$, although we do not have the answer in this thesis.

4.1 Superparticles in $AdS_2 \times S^2$

Flux compactifications on a Calabi-Yau threefold are described by an effective $d = 4$, $\mathcal{N} = 2$ supergravity with an $AdS_2 \times S^2$ vacuum solution whose moduli are at the attractor point with charges (p^Λ, q_Λ). This theory contains zerobranes[1] with essentially arbitrary charges (u^Λ, v_Λ) arising from various wrapped brane configurations. The κ-symmetric worldline action of these zerobranes was determined in [15]. In this section we use the results of [15] to determine the possible supersymmetric worldline trajectories.

[1] We use the term zerobrane in a general sense and do not specifically refer here to a ten-dimensional D0-brane.

The Killing spinor equation is

$$\nabla_\mu \epsilon_A - \frac{i}{2}\epsilon_{AB} T^-_{\mu\nu}\gamma^\nu \epsilon^B = 0, \tag{4.3}$$

where ϵ^A, $\epsilon_A = (\epsilon^A)^*$ $(A = 1, 2)$ are chiral and anti-chiral R-symmetry doublets of spinors. T^- is the anti-self-dual part of the graviphoton field strength, satisfying

$$Z_{BH} = \frac{1}{4\pi}\int_{S^2} T^- = e^{-\mathcal{K}/2}\left(F_\Lambda p^\Lambda - X^\Lambda q_\Lambda\right) \tag{4.4}$$

where $\mathcal{K} = -\ln i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda)$ is the Kähler potential. Define the phase of the central charge $e^{i\alpha} = Z_{BH}/|Z_{BH}|$. Then we can write $T^- = -ie^{i\alpha}(1 + i*)F$, where $F = \frac{1}{R}\omega_{AdS}$. In terms of the the doublet of spinors (ϵ_1, ϵ^2) and (ϵ^1, ϵ_2), the Killing spinor equation can be written as

$$\nabla_\mu \epsilon + \frac{i}{2} e^{-i\alpha\gamma_5} \not{F} \gamma_\mu \sigma^2 \epsilon = 0. \tag{4.5}$$

Note that there is an ambiguity in choosing the overall phase of the moduli fields and the central charge,

$$X^\Lambda \to e^{i\theta} X^\Lambda, \quad F_\Lambda \to e^{i\theta} F_\Lambda, \quad \epsilon \to e^{\frac{i}{2}\theta\gamma_5}\epsilon, \tag{4.6}$$

so we are free to set $\alpha = 0$.

The solutions to the Killing spinor equation in global $AdS_2 \times S^2$ coordinates (4.1) are [3]

$$\begin{aligned} \epsilon &= \exp\left(-\tfrac{i}{2}\chi\gamma^0\sigma^2\right)\exp\left(\tfrac{i}{2}\tau\gamma^1\sigma^2\right)R(\theta,\phi)\epsilon_0 \\ R(\theta,\phi) &\equiv \exp\left(-\tfrac{i}{2}(\theta - \pi/2)\gamma^{012}\sigma^2\right)\exp\left(-\tfrac{i}{2}\phi\gamma^{013}\sigma^2\right) \end{aligned} \tag{4.7}$$

where ϵ_0 is a doublet of arbitrary constant spinors. Alternatively, in the Poincare metric (1.15), the Killing spinors are [53]

$$\epsilon = \sigma^{-1/2}R(\theta,\phi)\epsilon_0^+ \quad \text{and} \quad \epsilon = (\sigma^{1/2} + i\sigma^{-1/2}t\gamma^1\sigma^2)R(\theta,\phi)\epsilon_0^-, \tag{4.8}$$

where $\epsilon_0^\pm$ are constant spinors satisfying $-i\gamma^0\sigma^2\epsilon_0^\pm = \pm\epsilon_0^\pm$, and $R(\theta, \phi)$ denotes the rotation on the S^2 as in (4.7). Note that γ^μ are the *normalized* gamma matrices in the corresponding frame.

The zerobrane action constructed in [15] has a local κ-symmetry parameterized by a four-dimensional spinor doublet κ_A on the worldline. In addition the spacetime supersymmetries ϵ_A act non-linearly in Goldstone mode on the worldline fermions. In general [8], a brane configuration trajectory will preserve a spacetime supersymmetry generated by ϵ if the action on the worldvolume fermions can be compensated for by a κ transformation. This condition can typically be written

$$(1 - \Gamma)\epsilon = 0 \tag{4.9}$$

where Γ is a matrix appearing in the κ-transformations. For the case at hand it follows from the results of [15] that the condition is

$$\begin{aligned} \epsilon_A + e^{i\varphi}\Gamma_{(0)}\epsilon_{AB}\epsilon^B &= 0 \\ \epsilon^A + e^{-i\varphi}\Gamma_{(0)}\epsilon^{AB}\epsilon_B &= 0 \end{aligned} \tag{4.10}$$

where $\Gamma_{(0)}$ is the gamma matrix projected to the zerobrane worldline, and $e^{i\varphi}$ is the phase of the central charge Z of the zerobrane,

$$Z = e^{-\mathcal{K}/2}\left(u^\Lambda F_\Lambda - v_\Lambda X^\Lambda\right) = e^{i\varphi}|Z|, \tag{4.11}$$

where (u^Λ, v_Λ) are its magnetic and electric charges. In terms of the spinor doublet, one can write (4.10) as

$$-ie^{-i\varphi\gamma_5}\Gamma_{(0)}\sigma^2\epsilon = \epsilon. \tag{4.12}$$

Let us solve the condition for (4.12) to hold along the world line of a zerobrane sitting

at constant (χ, θ, ϕ). Writing the Killing spinor as

$$\epsilon = \exp\left(-\frac{i}{2}\chi\gamma^0\sigma^2\right)\exp\left(\frac{i}{2}\tau\gamma^1\sigma^2\right)\epsilon_0' \tag{4.13}$$

where $\epsilon_0' = R(\theta, \phi)\epsilon_0$, it suffices to solve

$$\begin{aligned} -ie^{-i\varphi\gamma_5}\gamma^0\sigma^2 \exp\left(-\tfrac{i}{2}\chi\gamma^0\sigma^2\right)\epsilon_0' &= \exp\left(-\tfrac{i}{2}\chi\gamma^0\sigma^2\right)\epsilon_0' \\ -ie^{-i\varphi\gamma_5}\gamma^0\sigma^2 \exp\left(-\tfrac{i}{2}\chi\gamma^0\sigma^2\right)\gamma^1\sigma^2\epsilon_0' &= \exp\left(-\tfrac{i}{2}\chi\gamma^0\sigma^2\right)\gamma^1\sigma^2\epsilon_0'. \end{aligned} \tag{4.14}$$

Some straightforward algebra simplifies the above equations to

$$\begin{aligned} -i\gamma^0\sigma^2\left(\cos\varphi + i\cosh\chi\sin\varphi\gamma_5 + \sinh\chi\sin\varphi\gamma_5\gamma^0\sigma^2\right)\epsilon_0' &= \epsilon_0' \\ i\gamma^0\sigma^2\left(\cos\varphi - i\cosh\chi\sin\varphi\gamma_5 + \sinh\chi\sin\varphi\gamma_5\gamma^0\sigma^2\right)\epsilon_0' &= \epsilon_0'. \end{aligned} \tag{4.15}$$

A solution exists only when

$$\tanh\chi = \cos\varphi, \tag{4.16}$$

and therefore $\cosh\chi\sin\varphi = \pm 1$. Correspondingly the constraints on ϵ_0' become

$$\gamma_5\gamma^0\sigma^2\epsilon_0' = \mp\epsilon_0', \tag{4.17}$$

where the sign on the RHS depends on the sign of $\sin\varphi$. This may be written as a condition on ϵ_0,

$$\left(1 \pm e^{\frac{i}{2}\phi\gamma^{013}\sigma^2} e^{i(\theta-\pi/2)\gamma^{012}\sigma^2} e^{\frac{i}{2}\phi\gamma^{013}\sigma^2}\gamma_5\gamma^0\sigma^2\right)\epsilon_0 = 0\,, \tag{4.18}$$

which makes it clear that zerobranes sitting at antipodal points on the S^2 will preserve opposite halves of the spacetime supersymmetries.

We conclude that a zerobrane following its charged geodesic in $AdS_2 \times S^2$ is half BPS. The "extremal" case $\varphi = 0$ and π corresponds to the probe zerobrane with its

charge aligned or anti-aligned with the charge of the original black hole. They cannot be stationary with respect to global time in the AdS_2. Using the Killing spinors on the Poincaré patch (4.8), it is clear that the "extremal" zerobranes following their charged geodesics (static on the Poincaré patch) are also half BPS. In the special case $\varphi = \pi/2$ in (4.16) the zerobrane moves along an uncharged geodesic and experiences no electromagnetic forces . This corresponds to the case when the zerobrane charge is orthogonal to all the black hole charges.

A somewhat surprising feature is that there are supersymmetric *multiparticle* configurations of zerobranes with *unaligned* charges. All "positively-charged" zerobranes with $0 < \varphi < \pi$ preserve the same set of half of the supersymmetries, and all "negatively-charged" zerobranes with $-\pi < \varphi < 0$ preserve the other set. Using the attractor equations the positive charge condition can be written in terms of the symplectic product of the black hole and zerobrane charges as

$$u^\Lambda q_\Lambda - p^\Lambda v_\Lambda > 0. \tag{4.19}$$

Given an arbitrary collection of zerobranes obeying (4.19) there is a half BPS configuration with the position of each trajectory determined in terms of the charges of the zerobrane by (4.16). Of course, such a supersymmetric configuration of particles with unaligned charges is not possible in the full black hole geometry prior to taking the near horizon limit. The preserved supersymmetry is part of the enhanced near-horizon supergroup.

This result is consistent with the expectation from the BPS bound. The energy

of a charged zerobrane sitting at position χ the AdS_2 is given by

$$H = |Z| \cosh\chi - \frac{\mathrm{Re}(Z\overline{Z}_{BH})}{|Z_{BH}|} \sinh\chi = |Z| \left(\cosh\chi - \cos\varphi \sinh\chi\right). \tag{4.20}$$

where the first term comes from the gravitational warping, and the second term comes from the coupling to the gauge field potential. At the stationary point $\tanh\chi = \cos\varphi$, the energy of the zerobrane is

$$|Z \sin\varphi| = \frac{|\mathrm{Im} Z\overline{Z}_{BH}|}{|Z_{BH}|}. \tag{4.21}$$

Therefore, as long as $\mathrm{Im}(Z\overline{Z}_{BH})$ is always positive (or negative), the BPS energy for multiple zerobranes is additive, in agreement with the supersymmetry analysis above.

4.2 Supersymmetric D-branes in $AdS_2 \times S^2 \times CY_3$

In this section we analyze supersymmetric brane configurations from the point of view of the ten-dimensional IIA theory on $AdS_2 \times S^2 \times CY_3$. For simplicity we will focus on specific examples rather than the most general solution.

The extremal black hole in type IIA string theory compactified on a Calabi-Yau manifold M preserves four supersymmetries. After we take the near horizon limit, the number of preserved supersymmetries doubles to eight. We consider a background with only D0 and D4-brane charges, i.e. $q_A = p^0 = 0$, so that according to the attractor equations there is no B-field. The RR field strengths in the resulting $AdS_2 \times S^2 \times M_6$ are given as in (1.17). As shown in Appendix A, the ten-dimensional Killing spinor doublet is of the form

$$\varepsilon_1 = \epsilon_1 \otimes \eta_+ + \epsilon^1 \otimes \eta_-,$$

$$\varepsilon_2 = \epsilon^2 \otimes \eta_+ + \epsilon_2 \otimes \eta_-, \tag{4.22}$$

where $\eta_+, \eta_- = \eta_+^*$ are the chiral and anti-chiral covariantly constant spinors on M; $\epsilon_A = (\epsilon^A)^*$, $\epsilon^{1,2}$ are four-dimensional chiral spinors satisfying the four-dimensional Killing spinor equation

$$\nabla_\mu \epsilon_A + \frac{i}{2} \not{F}^{(2)} \gamma_\mu (\sigma^2)_{AB} \epsilon^B = 0. \tag{4.23}$$

This is the same equation as (4.5) with $\alpha = 0$, and the solutions are given by (4.7)(4.8).

We want to find all the BPS configurations of D-branes that are wrapped on compact portions of our background, and are pointlike in the AdS_2. In order for the D-brane to be supersymmetric, we only need to check that the κ-symmetry constraint

$$\Gamma \varepsilon = \varepsilon \tag{4.24}$$

is satisfied, where ε is the Killing spinor corresponding to the unbroken supersymmetry (more precisely, the pullback onto the brane world volume). The κ projection matrix is given by [18, 2, 12, 11]

$$\begin{aligned}
\Gamma &= \frac{\sqrt{\det G}}{\sqrt{\det(G+\mathcal{F})}} \sum_n \frac{1}{2^n n!} \Gamma^{\widehat{\mu}_1 \widehat{\nu}_1 \cdots \widehat{\mu}_n \widehat{\nu}_n} \mathcal{F}_{\widehat{\mu}_1 \widehat{\nu}_1} \cdots \mathcal{F}_{\widehat{\mu}_n \widehat{\nu}_n} \Gamma_{(10)}^{n + \frac{p-2}{2}} \Gamma_{(0)} \sigma^1, \\
\Gamma_{(0)} &= \frac{1}{(p+1)! \sqrt{\det G}} \epsilon^{\widehat{\mu}_0 \cdots \widehat{\mu}_p} \Gamma_{\widehat{\mu}_0 \cdots \widehat{\mu}_p}.
\end{aligned} \tag{4.25}$$

where the hatted indices label coordinates on the brane world-volume, G is the pull-back of the spacetime metric, and $\mathcal{F} = F + f^*(B)$ (the B-field is zero in our discussion). See Appendix A for conventions on 10D gamma matrices.

Unless otherwise noted we will work in global coordinates (4.1).

4.2.1 D0-brane

For a static D0-brane in global coordinates, we have $\Gamma_{(0)} = \gamma^0$. The κ-symmetry matrix is

$$\Gamma = \Gamma_{(10)}\gamma^0\sigma^1 \tag{4.26}$$

Writing the doublet ε in terms of the 4-dimensional spinor doublet ϵ

$$\varepsilon = \epsilon \otimes \eta_+ + \epsilon^* \otimes \eta_- \,, \tag{4.27}$$

The matrix Γ acts on ε as $\gamma^0\sigma^1\sigma^3 = -i\gamma^0\sigma^2$. The κ-symmetry constraint (4.24) becomes

$$(1 + i\gamma^0\sigma^2)\epsilon = 0\,. \tag{4.28}$$

Using the explicit solutions of the Killing spinors in global AdS (4.7), we see that (4.28) cannot be satisfied at all τ, so a D0-brane static in global AdS can never be BPS. This is of course expected since the charged geodesic cannot be static in global coordinates. On the other hand, using (4.8) we see that a D0-brane static with respect to the Poincaré time is always half BPS, as expected.

4.2.2 D2 wrapped on Calabi-Yau, $F = 0$

Now let us consider a D2-brane wrapped on M and static in global $AdS_2 \times S^2$, without any world-volume gauge fields turned on. The κ-symmetry matrix is

$$\Gamma = \frac{1}{2\sqrt{\det' G}}\gamma^0\epsilon^{\widehat{ab}}\Gamma_{\widehat{ab}}\sigma^1 \tag{4.29}$$

where $\det'$ takes the determinant of the spatial components of the world volume metric. Acting on ε, we have

$$\Gamma_{\widehat{ab}}\varepsilon \;=\; \partial_{\widehat{a}}X^I\partial_{\widehat{b}}X^J\gamma_{IJ}\varepsilon$$

$$
\begin{aligned}
&= 2\partial_{\widehat{a}} X^i \partial_{\widehat{b}} X^{\bar{j}} \gamma_{i\bar{j}} \varepsilon + \partial_{\widehat{a}} X^i \partial_{\widehat{b}} X^j \gamma_{ij} \varepsilon + \partial_{\widehat{a}} X^{\bar{i}} \partial_{\widehat{b}} X^{\bar{j}} \gamma_{\bar{i}\bar{j}} \varepsilon \\
&= 2\partial_{\widehat{a}} X^i \partial_{\widehat{b}} X^{\bar{j}} \left(-g_{i\bar{j}} \gamma_{(6)}\right) \varepsilon + \left(\frac{1}{2} \partial_{\widehat{a}} X^i \partial_{\widehat{b}} X^j \Omega_{ijk} \epsilon \otimes \gamma^k \eta_- + c.c.\right). \qquad (4.30)
\end{aligned}
$$

Firstly, the κ-symmetry constraint $\Gamma\varepsilon = \varepsilon$ implies $\epsilon^{\widehat{a}\widehat{b}} \partial_{\widehat{a}} X^i \partial_{\widehat{b}} X^j \Omega_{ijk} = 0$, which means that the D2-brane must wrap a holomorphic 2-cycle. It then follows that Γ acts on ε as $\Gamma\varepsilon = i\gamma^0 \gamma_{(6)} \sigma^1 \varepsilon = \gamma_{(4)} \gamma^0 \sigma^2 \varepsilon$. Therefore (4.24) becomes

$$
(1 - \gamma_{(4)} \gamma^0 \sigma^2) \epsilon = 0\,. \qquad (4.31)
$$

It is clear that the wrapped D2-brane sitting at $\chi = 0$ in AdS_2 is half BPS. Note that the D2-brane without gauge field flux doesn't feel any force due to the RR fluxes ($q_A = 0$), so its stationary position is at the center of AdS_2.

4.2.3 D2 wrapped on Calabi-Yau, $F \neq 0$

With general worldvolume gauge field strength F turned on, the matrix Γ is

$$
\Gamma = \frac{1}{\sqrt{\det'(G+F)}} \left(1 + \frac{1}{2} \Gamma^{\widehat{a}\widehat{b}} F_{\widehat{a}\widehat{b}} \Gamma_{(10)}\right) \gamma^0 \left(\frac{1}{2} \epsilon^{\widehat{c}\widehat{d}} \Gamma_{\widehat{c}\widehat{d}}\right) \sigma^1 \qquad (4.32)
$$

An argument nearly identical to the one given in [57] shows that the supersymmetric D2-brane must wrap a holomorphic 2-cycle, and the gauge flux F satisfies

$$
\frac{\sqrt{\det G}}{\sqrt{\det(G+F)}} (f^* J + iF) = e^{i\beta} \mathrm{vol}_2 \qquad (4.33)
$$

where vol_2 is the volume form on the D2-brane (which is just $f^* J$ for a holomorphically wrapped brane), and β is a constant phase determined in terms of the D0-brane charge $2\pi n = \frac{1}{2\pi\alpha'} \int F$ via

$$
\frac{\tan\beta}{2\pi\alpha'} \int J = 2\pi n \qquad (4.34)
$$

If the probe D2-brane is wrapped on the 2-cycle $[\Sigma_2] = n_A \alpha^A$, then using (1.18) we have

$$\tan\beta = \frac{n}{n_A p^A}\sqrt{\frac{D}{q_0}} \tag{4.35}$$

Note that from (4.33) we have $\cos\beta > 0$, since J is positive when restricted to holomorphic cycles. The κ-symmetry condition then becomes

$$(1 - e^{-i\beta\gamma_{(4)}}\gamma_{(4)}\gamma^0\sigma^2)\epsilon = 0 \tag{4.36}$$

These is identical to (4.12) if we set $\varphi = \beta - \pi/2$. We can immediately read off the conditions for the static D2-brane to preserve supersymmetry when it sits at $\theta = \pi/2$, $\phi = 0$ in the S^2:

$$\sin\beta = \tanh\chi\,, \qquad \cos\beta = \mathrm{sech}\chi\,, \qquad (1 - \gamma_{(4)}\gamma^0\sigma^2)\epsilon_0 = 0\,. \tag{4.37}$$

We see that for general $-\pi/2 < \beta < \pi/2$, the D2-brane sits at $\chi = \tanh^{-1}(\sin\beta)$ and is half BPS. In fact they all preserve the same half supersymmetries, as discussed in section 3. Anti-D2-branes with gauge field fluxes wrapped on holomorphic 2-cycles will preserve the other half supersymmetries.

4.2.4 Higher dimensional D-branes wrapped on the Calabi-Yau

Let us consider D4, D6-branes that are wrapped on the Calabi-Yau and sit at constant position in global $AdS_2 \times S^2$. We shall use a trick [11] to write the matrix Γ as

$$\Gamma = e^{-A/2}\Gamma_{(10)}^{\frac{p-2}{2}}\Gamma_{(0)}e^{A/2}\sigma^1 \tag{4.38}$$

where

$$A = -\frac{1}{2} Y_{\widehat{ab}} \Gamma^{\widehat{ab}} \Gamma_{(10)} \tag{4.39}$$

and $Y_{\widehat{ab}}$ is an anti-symmetric matrix (analogous to the phase β in the previous subsection), related to the gauge field strength matrix $F_{\widehat{ab}}$ by

$$F = \tanh Y \tag{4.40}$$

By the same arguments as before, one can show that the BPS D-branes must wrap holomorphic cycles. Note that A acts on the Killing spinor ε as $A\varepsilon = -iY_{\widehat{ab}}(f^* J)^{\widehat{ab}} \gamma_{(4)}\varepsilon$, and $\Gamma_{(0)}$ acts as $\gamma^0(i\gamma_{(6)})^{p/2}$ (see Appendix). Let us define $\beta = -Y_{\widehat{ab}}(f^* J)^{\widehat{ab}}$. The κ-symmetry constraint can be written as

$$\Gamma\varepsilon = e^{-i\beta\gamma_{(4)}/2} \Gamma_{(10)}^{\frac{p-2}{2}} \gamma^0 (i\gamma_{(6)})^{p/2} e^{i\beta\gamma_{(4)}/2} \sigma^1 \varepsilon = \varepsilon \,. \tag{4.41}$$

We can simplify this to

$$-ie^{-i(\beta - p\pi/2)\gamma_{(4)}} \gamma^0 \sigma^2 \epsilon = \epsilon \,. \tag{4.42}$$

This equation indeed agrees with (4.28)(4.36) in the cases $p = 0, 2$. It is also identical to (4.12) provided we set $\varphi = \beta - p\pi/2$. So we conclude that a general Dp-brane (p even) wrapped on a holomorphic cycle in the Calabi-Yau, possibly with world-volume gauge fields turned on, static in the S^2 and following its charged geodesic in the AdS_2 is half BPS. As in [57] there is a deformation of the supersymmetry condition on the worldvolume gauge field F.

4.2.5 D2 wrapped on S^2, $F = 0$

Now let us turn to D2-branes wrapped on the S^2 appearing in the the $AdS_2 \times S^2 \times M$ product. The κ-symmetry matrix is $\Gamma = \Gamma_{(0)}\sigma^1 = \gamma^{023}\sigma^1$. (4.24) can be written as

$$(1 - \gamma^{023}\sigma^1)\epsilon = 0\,. \tag{4.43}$$

Defining $R(\theta, \phi)$ to be the S^2-dependent factors in (4.7), this condition becomes

$$\begin{aligned}(1 - \gamma^{023}\sigma^1)\exp\left(-\tfrac{i}{2}\chi\gamma^0\sigma^2\right) R(\theta,\phi)\epsilon_0 = 0,\\ (1 - \gamma^{023}\sigma^1)\exp\left(-\tfrac{i}{2}\chi\gamma^0\sigma^2\right)\gamma^1\sigma^2 R(\theta,\phi)\epsilon_0 = 0.\end{aligned} \tag{4.44}$$

A little algebra reduces these to

$$\cosh\frac{\chi}{2}(1 - \gamma^{023}\sigma^1)R(\theta,\phi)\epsilon_0 = \sinh\frac{\chi}{2}(1 + \gamma^{023}\sigma^1)R(\theta,\phi)\epsilon_0 = 0\,. \tag{4.45}$$

The only way to satisfy both equations is to set $\chi = 0$. Since $\gamma^{023}\sigma^1$ commutes with $R(\theta, \phi)$, we end up with the condition

$$(1 - \gamma^{023}\sigma^1)\epsilon_0 = 0 \tag{4.46}$$

We conclude that the D2-brane sitting at the center of AdS and wrapped on the S^2 is half BPS.

4.2.6 D2 wrapped on S^2, $F \neq 0$

With gauge field strength $F = f\omega_{S^2}$ turned on, the κ-symmetry matrix acts on ε as

$$\Gamma\varepsilon \;=\; \frac{\sqrt{\det G}}{\sqrt{\det(G+F)}}\left(1 + \frac{1}{2}\Gamma^{\widehat{ab}}F_{\widehat{ab}}\Gamma_{(10)}\right)\Gamma_{(0)}\sigma^1\varepsilon$$

$$\begin{aligned} &= \frac{1}{\sqrt{1+f^2}}\left(1+\gamma^{23}f\Gamma_{(10)}\right)\gamma^{023}\sigma^1\varepsilon \\ &= \exp\left(\beta\gamma^{23}\Gamma_{(10)}\right)\gamma^{023}\sigma^1\varepsilon = \gamma^{023}\sigma^1\exp\left(\beta\gamma^{23}\sigma^3\right)\varepsilon\,, \end{aligned} \tag{4.47}$$

where $f \equiv \tan\beta$ ($\cos\beta > 0$). The condition (4.24) then becomes

$$\begin{aligned} &(1-\cos\beta\gamma^{023}\sigma^1 - i\sin\beta\gamma^0\sigma^2)\exp\left(-\frac{i}{2}\chi\gamma^0\sigma^2\right)R(\theta,\phi)\epsilon_0 = 0, \\ &(1-\cos\beta\gamma^{023}\sigma^1 + i\sin\beta\gamma^0\sigma^2)\exp\left(\frac{i}{2}\chi\gamma^0\sigma^2\right)R(\theta,\phi)\epsilon_0 = 0, \end{aligned} \tag{4.48}$$

A little algebra yields

$$\begin{aligned} &(1+\sin\beta\coth\chi)\,\epsilon_0 = 0\,, \\ &\left(1+\gamma^{023}\sigma^1\cot\beta\sinh\chi\right)\epsilon_0 = 0\,. \end{aligned} \tag{4.49}$$

This means that $\sin\beta = -\tanh\chi$. In particular β, hence f, is constant on the world-volume. The condition on ϵ_0 becomes

$$(1-\gamma^{023}\sigma^1)\epsilon_0 = 0\,. \tag{4.50}$$

These D-brane configurations are again half BPS.

4.2.7 D-branes wrapped on S^2 and the Calabi-Yau

In general for a Dp-branes wrapped on S^2 times some $(p-2)$-cycle in the Calabi-Yau, and static in global AdS_2, the matrix Γ is essentially the product of the piece on S^2 and the piece on Calabi-Yau,

$$\Gamma\varepsilon = \exp\left(-\beta_{S^2}\gamma^{23}\sigma^3\right)\exp\left(-i\beta_{CY}\gamma_{(4)}\right)\left(i\gamma_{(4)}\right)^{\frac{p-2}{2}}\gamma^{023}\sigma^1\varepsilon \tag{4.51}$$

where β_{CY} and β_{S^2} are the phases related to the world-volume gauge flux along the Calabi-Yau and S^2 directions as before. Define $\varphi_{CY} = \beta_{CY} - (p-2)\pi/2$, $\varphi_{S^2} =$

$\beta_{S^2} + \pi/2$. The κ-symmetry constraint can be written as

$$-i \exp\left(-\varphi_{S^2}\gamma^{23}\sigma^3 - i\varphi_{CY}\gamma_{(4)}\right)\gamma^0\sigma^2\epsilon = \epsilon \tag{4.52}$$

This is equivalent to

$$\begin{aligned}
&\left[1 + i\exp\left(-\varphi_{S^2}\gamma^{23}\sigma^3 - i\varphi_{CY}\gamma_{(4)}\right)\gamma^0\sigma^2\right]\exp\left(-\frac{i}{2}\chi\gamma^0\sigma^2\right)R(\theta,\phi)\epsilon_0 = 0,\\
&\left[1 - i\exp\left(\varphi_{S^2}\gamma^{23}\sigma^3 + i\varphi_{CY}\gamma_{(4)}\right)\gamma^0\sigma^2\right]\exp\left(\frac{i}{2}\chi\gamma^0\sigma^2\right)R(\theta,\phi)\epsilon_0 = 0.
\end{aligned} \tag{4.53}$$

A little algebra yields

$$\begin{aligned}
&\left[\sinh\chi - \cosh\chi\cos(\varphi_{S^2} - i\gamma_{(4)}\gamma^{23}\sigma^3\varphi_{CY})\right]R(\theta,\phi)\epsilon_0 = 0\,,\\
&\Big[\cosh\chi - \sinh\chi\cos(\varphi_{S^2} - i\gamma_{(4)}\gamma^{23}\sigma^3\varphi_{CY})\\
&\qquad -\gamma^{023}\sigma^1\sin(\varphi_{S^2} - i\gamma_{(4)}\gamma^{23}\sigma^3\varphi_{CY})\Big]R(\theta,\phi)\epsilon_0 = 0\,,
\end{aligned} \tag{4.54}$$

If φ_{CY} and φ_{S^2} are both nonzero, the first equation can be satisfied only if

$$i\gamma_{(4)}\gamma^{23}\sigma^3 R(\theta,\phi)\epsilon_0 = mR(\theta,\phi)\epsilon_0, \qquad m = \pm 1. \tag{4.55}$$

However, since $\gamma_{(4)}\gamma^{23}\sigma^3$ does not commute with $R(\theta,\phi)$ at generic points on the S^2, (4.55) can never be satisfied. Therefore such wrapped D-branes cannot be BPS.

If $\varphi_{S^2} = 0$, $\varphi_{CY} \neq 0$, we have

$$\tanh\chi = \cos\varphi_{CY} \tag{4.56}$$

and

$$(1 - \gamma_{(4)}\gamma^0\sigma^2)R(\theta,\phi)\epsilon_0 = 0 \tag{4.57}$$

However, in this case again $\gamma_{(4)}\gamma^0\sigma^2$ does not commute with $R(\theta,\phi)$ for generic (θ,ϕ), and hence (4.57) has no solution.

If $\varphi_{S^2} \neq 0$, $\varphi_{CY} = 0$, we find

$$\tanh \chi = \cos \varphi_{S^2} \tag{4.58}$$

and the second equation in (4.54) becomes

$$(1 - \gamma^{023}\sigma^1)\epsilon_0 = 0 \tag{4.59}$$

We see that such D-branes are half BPS.

So far we have neglected an important subtlety. For D4 or D6-branes wrapped on S^2 times some cycle in the Calabi-Yau, the RR flux $F_{(4)}$ induces couplings of gauge fields on the brane world-volume

$$\begin{aligned} &\textstyle\int_{D4} A \wedge F_{(4)} \,, \\ &\textstyle\int_{D6} A \wedge F \wedge F_{(4)} \,, \end{aligned} \tag{4.60}$$

Since $F_{(4)} = \frac{1}{R}\omega_{S^2} \wedge J$, we see that for the D4-brane wrapped on $S^2 \times \Sigma_2$ ($[\Sigma_2] = n_A \alpha^A$), the RR flux induces an electric charge density on the brane world-volume, of total charge

$$Q = \frac{1}{2\pi g_s} \int_{S^2 \times \Sigma_2} F_{(4)} = \sum n_A p^A \tag{4.61}$$

Since the world-volume is compact, the Gauss law constraint requires the total charge to vanish. So we cannot wrap only a single D4-brane on $S^2 \times \Sigma$. One must introduce fundamental strings ending on the brane to cancel the electric charges. We then have $\sum n_A p^A$ fundamental strings ending on the D4-brane, and runoff to the boundary of AdS. This is interpreted as a classical "baryon" in the dual CFT.

Similarly for the D6-brane wrapped on $S^2 \times \Sigma_4$, one would have nonzero total electric charge on the world-volume if $\int_{\Sigma_4} F \wedge J \neq 0$. This again corresponds to

certain "baryons" in the dual CFT. In this case, $\varphi_{CY} \neq 0$, and we saw earlier that such branes are not BPS anyway.

Finally, a D6-brane wrapped on $S^2 \times \Sigma_4$ with general gauge field flux in the S^2 is half BPS, as shown in (4.58)(4.59).

4.3 Black holes in $AdS_2 \times S^2$

4.3.1 The ansatz

In this subsection we rewrite the ansatz (2.37,2.38,2.39) in chapter 2 for a general class of supersymmetric solutions of $d = 4, \mathcal{N} = 2$ supergravity in the special case where all the vector multiplet moduli fields are constant. Namely all X^Λ's are assumed to be proportional to a single complex harmonic function $h(\vec{x})$ on $\mathbf{R}^3$,

$$X^\Lambda = (P^\Lambda + iU^\Lambda)h(\vec{x}) \tag{4.62}$$

and then by homogeneity,

$$F_\Lambda = (Q_\Lambda + iV_\Lambda)h(\vec{x}) \tag{4.63}$$

where $(P^\Lambda, U^\Lambda, Q_\Lambda, V_\Lambda)$ are real constants. Unlike the examples discussed in chapter 2, however, we no longer require $h(\vec{x})$ to approach a constant at infinity. One obtains the harmonic functions (H^Λ, H_Λ) as in (2.35),

$$(H^\Lambda, H_\Lambda) = (P^\Lambda \mathrm{Re}h - U^\Lambda \mathrm{Im}h, Q_\Lambda \mathrm{Re}h - V_\Lambda \mathrm{Im}h) \tag{4.64}$$

The entropy function $\Sigma = \pi^{-1} S_{bh}(H^\Lambda, H_\Lambda)$ is then given by

$$\Sigma = \frac{S_{bh}(P^\Lambda, Q_\Lambda)}{\pi} |h(\vec{x})|^2, \tag{4.65}$$

The solution for the metric is

$$ds^2 = -\Sigma^{-1}(dt + \omega)^2 + \Sigma d\vec{x}^2, \tag{4.66}$$

where $\vec{x} = (x, y, z)$, and the gauge field strengths

$$dA^\Lambda = d\left[\Sigma^{-1}\frac{\partial \Sigma}{\partial H_\Lambda}(dt + \omega)\right] + *_3 dH^\Lambda, \tag{4.67}$$

The 1-form ω is determined by

$$*_3 d\omega = \frac{2S_{bh}(P^\Lambda, Q_\Lambda)}{\pi}\mathrm{Im}(\overline{h}dh) \tag{4.68}$$

The harmonic function $h(\vec{x})$ can have pointlike singularities (sources), but it needs to satisfy the integrability condition

$$\mathrm{Im}(\overline{h}\nabla^2 h) = 0 \tag{4.69}$$

Furthermore, in order to avoid naked singularities, $h(\vec{x})$ must never be zero.

4.3.2 The black hole solutions

To obtain $AdS_2 \times S^2$, it is convenient to use the oblate spheroidal coordinates on $\mathbf{R}^3$,

$$\begin{aligned} x &= \cosh\xi \cos\eta \cos\phi, \\ y &= \cosh\xi \cos\eta \sin\phi, \\ z &= \sinh\xi \sin\eta. \end{aligned} \tag{4.70}$$

The metric on $\mathbf{R}^3$ is written

$$d\vec{x}^2 = (\cosh^2\xi - \cos^2\eta)(d\xi^2 + d\eta^2) + \cosh^2\xi \cos^2\eta d\phi^2 \tag{4.71}$$

With the choice of harmonic function

$$h(\vec{x}) = \frac{1}{\sinh\xi + i\sin\eta}, \tag{4.72}$$

ω as (4.68) can be solved to be

$$\omega = -\frac{S}{\pi}\frac{\cos^2\eta}{\cosh^2\xi - \cos^2\eta}d\phi \tag{4.73}$$

The metric is now

$$\begin{aligned} ds^2 &= -\frac{\pi}{S}|h|^{-2}(dt+\omega)^2 + \frac{S}{\pi}|h|^2 d\vec{x}^2 \\ &= -\frac{\pi}{S}(\cosh^2\xi - \cos^2\eta)(dt - \frac{S}{\pi}\frac{\cos^2\eta}{\cosh^2\xi - \cos^2\eta}d\phi)^2 \\ &\quad + \frac{S}{\pi}(d\xi^2 + d\eta^2 + \frac{\cosh^2\xi\cos^2\eta}{\cosh^2\xi - \cos^2\eta}d\phi^2) \\ &= -\frac{\pi}{S}\cosh^2\xi dt^2 + \frac{S}{\pi}\left[d\xi^2 + d\eta^2 + \cos^2\eta(d\phi + \frac{\pi}{S}dt)^2\right] \end{aligned} \tag{4.74}$$

This is precisely the metric on global $AdS_2 \times S^2$ of radius $R = R_{bh}/2$ in a rotating frame. The Killing vector $\partial/\partial t$ corresponds to the conformal generator $H + K - J^3$.

There is an important subtlety with this $AdS_2 \times S^2$ solution. The coordinate ξ has range $(-\infty, \infty)$ and η has range $(-\pi/2, \pi/2)$. However by (4.70) $\mathbf{R}^3$ is covered by only half of the full range of (ξ, η, ϕ). In other words, the 3-dimensional "base space" is a double cover of $\mathbf{R}^3$, obtained by joining two copies of $\mathbf{R}^3$ with "cuts", parameterized by the spheroidal coordinates as

$$\begin{aligned} \mathbf{H}_+ : &\quad \xi \in (-\infty, +\infty),\ \eta \in (0, \pi/2],\ \phi \in [0, 2\pi) \\ \mathbf{H}_- : &\quad \xi \in (-\infty, +\infty),\ \eta \in [-\pi/2, 0),\ \phi \in [0, 2\pi) \end{aligned} \tag{4.75}$$

along the locus $\eta = 0$, corresponding to the xy plane with a hole, $\{x^2 + y^2 > 1, z = 0\}$.

Let us consider a modified harmonic function

$$h(\xi, \eta, \phi) = \frac{1}{\sinh\xi + i\sin\eta} - i\tilde{h}(\xi, \eta, \phi) \tag{4.76}$$

where $\tilde{h}$ is a real harmonic function with point-like sources. Such harmonic function can be produced by gluing two harmonic functions in $\mathbf{R}^3$, $h_+(\vec{x})$ and $h_-(\vec{x})$, living on $\mathbf{H}_+$ and $\mathbf{H}_-$ respectively. They are not necessarily continuous across the locus $\{x^2 + y^2 > 1, z = 0\}$: as one jumps across this surface in $\mathbf{H}_\pm$, the coordinate ξ changes sign. h_+ and h_- thus must satisfy the gluing condition along the cut,

$$\begin{aligned} h_+(x, y, z \to 0^+) &= h_-(x, y, z \to 0^-), \\ h_+(x, y, z \to 0^-) &= h_-(x, y, z \to 0^+), \\ \partial_z h_+(x, y, z \to 0^+) &= \partial_z h_-(x, y, z \to 0^-) \\ \partial_z h_+(x, y, z \to 0^-) &= \partial_z h_-(x, y, z \to 0^+), \qquad x^2 + y^2 > 1. \end{aligned} \tag{4.77}$$

Now $\tilde{h}$ is simply given by

$$\begin{aligned} \tilde{h}(\xi, \eta, \phi) &= h_+(\xi, \eta, \phi), \quad \eta > 0, \\ \tilde{h}(\xi, \eta, \phi) &= h_-(\xi, \eta, \phi), \quad \eta < 0, \end{aligned} \tag{4.78}$$

In particular, if we demand $h_+ > 0$ and $h_- < 0$, then $h(\xi, \eta, \phi)$ is never zero. An easy way to satisfy (4.77) is to set

$$h_+(x, y, z \to 0^\pm) = 0, \quad h_-(x, y, z) = -h_+(x, y, -z). \tag{4.79}$$

Now h_+ is nothing but the electric potential in $\mathbf{R}^3$ at the presence of an infinite sheet of conductor at $\{x^2 + y^2 > 1, z = 0\}$ and some point charges q_i at $\vec{x}_i$. h_- would then corresponds to the potential due to the opposite point charges at $z_i' = -z_i$. The integrability condition (4.69), however, demands $z_i = 0$. With a point charge at $x = y = z = 0$, the gravity solution obtained from $h(\vec{x})$ corresponds to a pair of black holes with opposite charges, sitting at $\xi = 0$ in global AdS_2 and antipodal points $\eta = \pm\pi/2$ on the S^2.

Figure 4.1: The solution describing a pair of black hole and anti-black hole at the north and south poles of the S^2 is related to the electric potential due to a charge at the presence of a conductor sheet with a hole at the center.

Note that if we choose the point charges to be away from the origin on the disc $\{x^2 + y^2 < 1, z = 0\}$, the pair of black holes are no longer at antipodal points on the S^2. However they will be rotating with large angular momenta, and the configuration is still supersymmetric.

4.3.3 A pair of black holes

It is in fact possible to write down the explicit solution corresponding to a pair of black holes at $\xi = 0, \eta = \pm\pi/2$, which corresponds to the harmonic function

$$h(\xi, \eta) = \frac{1}{\sinh \xi + i \sin \eta} - \frac{2iq}{\pi r} \tan^{-1}\left(\frac{\sin \eta}{r}\right), \qquad r = \sqrt{\cosh^2 \xi - \sin^2 \eta} \tag{4.80}$$

One can solve for the 1-form ω,

$$\omega = -\frac{S}{\pi}\left\{\frac{\cos^2\eta}{\cosh^2\xi - \cos^2\eta}\left[1 + \frac{4q\sin\eta}{\pi r}\tan^{-1}\left(\frac{\sin\eta}{r}\right)\right] + \frac{2q}{\pi}\ln\frac{\cosh^2\xi}{\cosh^2\xi - \cos^2\eta}\right\}d\phi \tag{4.81}$$

and hence obtain the explicit analytic solution of the metric and gauge fields.

Furthermore, one can write the explicit solution corresponding to a pair of black holes (of opposite central charges) at $\xi = \xi_0, \eta = \pi/2$ and $\xi = -\xi_0, \eta = -\pi/2$. This is given by the harmonic function

$$h(\xi,\eta) = \frac{1}{\sinh\xi + i\sin\eta} + \frac{2Z}{\pi\rho}\tan^{-1}\left(\frac{\sin\eta + \sinh\xi_0\sinh\xi}{\rho}\right) \tag{4.82}$$

where

$$\rho \equiv \sqrt{\cosh^2\xi\cos^2\eta + (\sinh\xi\sin\eta - \sinh\xi_0)^2} \tag{4.83}$$

is the distance from the point $(0, 0, z_0 = \sinh\xi_0)$ in $\mathbf{R}^3$. $Z = a - bi$ is the central charge of the black hole relative the flux in $AdS_2 \times S^2$. It determines the position ξ_0 via the integrability condition (4.69),

$$\sinh\xi_0 = \frac{a}{b} \tag{4.84}$$

4.3.4 A single black hole

One can now also construct the solution corresponding to a single black hole in $AdS_2 \times S^2$, by adding the harmonic functions $1/r$ (or $1/\rho$) to (4.80) (or (4.82)) to cancel one of the point charges. The function $1/|\vec{x} - \vec{x}_0|$ is simply the harmonic potential sourced by two point charges, one at (ξ_0, η_0, ϕ_0) and the other at $(-\xi_0, -\eta_0, \phi_0)$. The solution of a single black hole at $\xi = \xi_0, \eta = \pi/2$, for example, is determined by the

harmonic function

$$h(\xi,\eta) = \frac{1}{\sinh\xi + i\sin\eta} + \frac{Z}{\pi\rho}\left[\tan^{-1}\left(\frac{\sin\eta + \sinh\xi_0\sinh\xi}{\rho}\right) + \frac{\pi}{2}\right] \tag{4.85}$$

We must still demand h to be nonvanishing to avoid naked singularities. It is a straightforward exercise to show that (4.85) vanishes only when

$$\begin{aligned} &\sinh\xi = \frac{a}{b}\sin\eta, \\ &\tan\theta = \frac{b}{\pi}\theta, \quad \theta \equiv \tan^{-1}(\sqrt{1+(a/b)^2}\tan\eta) + \frac{\pi}{2} \in [0,\pi] \end{aligned} \tag{4.86}$$

with a special case $\eta = -\pi/2$, or $\theta = 0$, where h vanishes only if $b = \pi$. We see that if $0 \leq b < \pi$, then h can never vanish. This is a bound on the magnetic charge of the black hole.

The 1-form ω corresponding to (4.85) is

$$\begin{aligned} \omega = -\frac{S}{\pi}\Bigg\{\frac{\cos^2\eta}{\cosh^2\xi - \cos^2\eta}\Bigg[1 + \frac{4q(\sin\eta + \sinh\xi_0\sinh\xi)}{\pi\rho}\tan^{-1}\left(\frac{\sin\eta + \sinh\xi_0\sinh\xi}{\rho}\right) \\ +2q\frac{(\sin\eta + \sinh\xi_0\sinh\xi)}{\rho}\Bigg] + \frac{2q}{\pi}\ln\frac{\cosh^2\xi}{\cosh^2\xi - \cos^2\eta}\Bigg\}d\phi \end{aligned} \tag{4.87}$$

where q is related to Z by $Z = 2q(\sinh\xi_0 - i)$.

4.3.5 Closed timelike curves and a modified solution

Let us examine the solution given by (4.80) and (4.81) in more detail. It is not obvious that the metric would be smooth at the equator of the sphere at the center of AdS_2, $\xi = \eta = 0$, where the coordinates are singular. Near this ring, we can approximate

$$h \simeq \frac{1}{\xi + i\eta} - \frac{2iq}{\pi}\eta,$$

$$\omega \simeq -\frac{S}{\pi}\left[\frac{1}{\xi^2+\eta^2}(1+\frac{4q}{\pi}\eta^2) - \frac{2q}{\pi}\ln(\xi^2+\eta^2)\right] d\phi \tag{4.88}$$

and the metric becomes

$$ds^2 \to 2dtd\phi + \frac{S}{\pi}\left[\left(\frac{4q}{\pi}\ln(\xi^2+\eta^2)+1\right)d\phi^2 + d\xi^2 + d\eta^2\right] \tag{4.89}$$

to leading order in ξ, η. We see that $\partial/\partial\phi$ generates a closed timelike curve at small ξ, η. In the limit $q \ll 1$, the region of CTCs is approximately $\xi^2 + \eta^2 < \exp(-\pi/4q)$. Note that this region is exponentially small for $q \ll 1$.

The ring $\xi = \eta = 0$ is not a curvature singularity, nevertheless the existence of CTCs near the ring is problematic. In particular the CTCs near the ring are present for arbitrarily small black holes. In attempt to cure this problem, we can modify the harmonic function (4.80) to

$$h(\xi,\eta) = \frac{1}{\sinh\xi + i\sin\eta} - \frac{2iq}{\pi r}\tan^{-1}\left(\frac{\sin\eta}{r}\right) - 2iq\phi_0(\xi,\eta) \tag{4.90}$$

where ϕ_0, when restricted to $\mathbf{H}_+$ ($\eta > 0$), is the harmonic potential at the presence of asymptotically uniform electric field along z direction as $z \to -\infty$, and zero electric field as $z \to \infty$. It is given by

$$\phi_0(\xi,\eta) = \begin{cases} -\frac{\sin\eta}{\pi}\left[1-\sinh\xi\tan^{-1}\left(\frac{1}{\sinh\xi}\right)\right], & \xi > 0 \\ \sinh\xi\sin\eta - \frac{\sin\eta}{\pi}\left[1-\sinh\xi\tan^{-1}\left(\frac{1}{\sinh\xi}\right)\right], & \xi < 0. \end{cases} \tag{4.91}$$

The 1-form ω is then solved to be

$$\begin{aligned}\omega = &-\frac{S}{\pi}\frac{\cos^2\eta}{\cosh^2\xi - \cos^2\eta}\left\{1+\frac{4q}{\pi}\left[\frac{\sin\eta}{r}\tan^{-1}\left(\frac{\sin\eta}{r}\right)\right.\right.\\ &\left.\left.+\sinh^2\xi\left(1-\sinh\xi\tan^{-1}\left(\frac{1}{\sinh\xi}\right)\right)\right]\right\}d\phi, \qquad \xi > 0\\ \omega = &-\frac{S}{\pi}\frac{\cos^2\eta}{\cosh^2\xi - \cos^2\eta}\left\{1+\frac{4q}{\pi}\left[\frac{\sin\eta}{r}\tan^{-1}\left(\frac{\sin\eta}{r}\right)\right.\right.\end{aligned}$$

$$+\sinh^2\xi\left(1-\sinh\xi\tan^{-1}\left(\frac{1}{\sinh\xi}\right)\right)\Bigg]-4q\sinh^3\xi\Bigg\}d\phi, \qquad \xi<0 \tag{4.92}$$

(4.90) and (4.92) give a solution that is asymptotically $AdS_2 \times S^2$ on the right boundary. It is straightforward to check that near $\xi=\eta=0$, the metric is indeed smooth and free of CTCs, $ds^2 \to 2dtd\phi+\frac{S}{\pi}(d\xi^2+d\eta^2)$, provided that $q<\pi/8$. However there are new regions of CTCs for $\xi<0$, due to the extra term ϕ_0 in (4.90).

Let us examine the asymptotic behavior of the solution given by (4.90). In the limit $\xi \to \pm\infty$, we have

$$\begin{aligned}
h(\xi,\eta) &\to 2e^{-|\xi|}, \quad \omega \to -\tfrac{S}{\pi}(1+\tfrac{4q}{3\pi})4\cos^2\eta e^{-2|\xi|}d\phi, \qquad \xi\to+\infty;\\
h(\xi,\eta) &\to iqe^{|\xi|}\sin\eta, \quad \omega \to -\tfrac{S}{\pi}2q\cos^2\eta e^{|\xi|}d\phi, \qquad \xi\to-\infty, \quad \eta\neq 0 \text{ fixed};\\
h(\xi,\eta) &\to -2e^{-|\xi|}, \quad \omega\to -\tfrac{S}{\pi}2qe^{|\xi|}, \qquad \xi\to-\infty, \quad \eta=0.
\end{aligned} \tag{4.93}$$

and the metric behaves like

$$\begin{aligned}
ds^2 &\to -\tfrac{\pi}{S}\tfrac{e^{2|\xi|}}{4}dt^2+\tfrac{S}{\pi}\left[d\xi^2+d\eta^2+\cos^2\eta\left(d\phi+\tfrac{\pi}{S}\left(1+\tfrac{4q}{3\pi}\right)dt\right)^2\right], \quad \xi\to+\infty;\\
ds^2 &\to -\tfrac{\pi}{S}\tfrac{dt^2}{4q^2z^2}+\tfrac{S}{\pi}4q^2z^2(dx^2+dy^2+dz^2), \qquad \xi\to-\infty, \quad \eta\neq 0 \text{ fixed};\\
ds^2 &\to -\tfrac{\pi}{S}\tfrac{e^{2|\xi|}}{4}(dt-\tfrac{S}{\pi}2qe^{|\xi|}d\phi)^2+\tfrac{S}{\pi}\left(d\xi^2+d\eta^2+d\phi^2\right), \qquad \xi\to-\infty, \quad \eta=0
\end{aligned} \tag{4.94}$$

We see that the solution determined by (4.90) describes a pair of black holes at the center of AdS_2, that is asymptotic to $AdS_2\times S^2$ (in a rotating frame) on the right ($\xi\to+\infty$). On the left of the black holes, away from the equator of the S^2 ($\eta=0$), the asymptotic geometry is distorted as in the second line of (4.94). Near the equator there are CTCs for approximately $\xi<\ln q$, assuming $q\ll 1$. The region of CTCs in this modified solution is pushed to infinity logarithmically as $q\to 0$.

Chapter 5

From AdS$_3$/CFT$_2$ to Black Holes/Topological Strings

A four-dimensional black hole in an M-theory compactification on a Calabi-Yau threefold X times an S^1 can be constructed by wrapping an M5-brane with fluxes and S^1 momentum on a 4-cycle P in X. This black hole is dual to the R sector of the $(0,4)$ CFT which lives on the dimensionally reduced M5 worldvolume [56]. The NS sector of this same $(0,4)$ CFT is dual to supergravity on $AdS_3 \times S^2 \times X$ (as well as the 5D black ring [21]).

The elliptic genus $Z_{BH} = Z_{CFT}$ of the 4D black hole is[1]

$$Z_{BH} = \mathrm{Tr}(-)^F q^{L_0 - \frac{c_L}{24}} e^{2\pi i q_A y^A}, \tag{5.1}$$

where the trace is over chiral primary states on $AdS_3 \times S^2 \times X$, q_A is a membrane charge and y^A the conjugate potential. We work in the dilute gas expansion in which

[1] This is related by spectral flow to the Ramond sector trace. If the center of mass multiplet is included, there should be an extra insertion of F^2/τ_2, but we will suppress this herein.

(5.1) is dominated by multi-particle chiral primaries states of membranes wrapping holomorphic curves in X.

A crucial point in the following is that both membrane and anti-membrane states contribute to (5.1). This is because a membrane wrapping a holomorphic curve $Q = q_A \alpha^A$ in X and sitting at say the north pole of the S^2 preserves the the *same* set of supersymmetries as the oppositely-charged *anti*-M2-brane wrapping Q and sitting at the south pole [63], as we have seen in the last chapter. This may sound strange as we are used to the idea in flat space that branes and antibranes preserve opposite supersymmetries because they have opposite charges. However in $AdS_2 \times S^2$ the S^2 angular momentum plays the role of the central charge in stabilizing BPS states. Static wrapped branes in this background carry this angular momentum much like a static electron in the field of a monopole. Hence branes and antibranes at anitpodal points can carry the same angular momentum and preserve the same supersymmetries.

In this chapter we work out in detail the degeneracies of chiral primary wrapped membranes of all stripes and their contribution to Z_{BH}. We find the product of two complex conjugate factors, one from branes and another from antibranes. Including an additional factor from massless supergravity modes, a modular transformation factor and using the Gopakumar-Vafa relation [42] between BPS degeneracies and Gromov-Witten invariants, we recover precisely the OSV relation [59][2]

$$Z_{\rm BH} = |Z_{\rm top}|^2. \tag{5.2}$$

The factorization into topological and anti-topological string partition functions now

[2] The agreement is up to factors which depend only on q and not any of the Calabi-Yau data.

has a simple physical explanation: $Z_{\rm top}$ is the membrane contribution, while $\overline{Z}_{\rm top}$ is the anti-membrane contribution.[3] We further hope that the the framework can be used to systematically compute non-perturbative corrections to (5.2), such as perhaps arising from chiral primary wrapped M5 branes.

The relevant M-theory attractor $AdS_3 \times S^2 \times X$ geometry was introduced in chapter 1. Some basic facts about the relevant $(0,4)$ superconformal algebra are reviewed in section 1. The classical and quantum BPS states of wrapped M2-branes are described in section 2. In section 3 we compute the elliptic genus from the bulk theory, including the contribution from wrapped M2-branes, massless bulk supergravity fields and boundary singletons. The result is compared to the black hole partition function and topological strings in section 4.

5.1 The $(0,4)$ superconformal algebra

M-theory on the $AdS_3 \times S^2 \times X$ attractor geometry has super-isometry group $SL(2,\mathbf{R})_L \times SU(1,1|2)_R$. Let us describe the representations of the corresponding superalgebra.

The superconformal algebra $SU(1,1|2)_R$ has the bosonic subalgebra $SL(2,\mathbf{R})_R \times SU(2)_R$. The fermionic generators are $\overline{G}^{\alpha A}_{\pm\frac{1}{2}}$, where $\alpha = 1,2$ is an $SU(2)_R$ index, and $A = 1,2$ labels a doublet that transforms under the outer automorphism $SU(2)$. The superalgebra takes the form

$$\begin{aligned}
\{\overline{G}^{\alpha A}_{\pm\frac{1}{2}}, \overline{G}^{\beta B}_{\pm\frac{1}{2}}\} &= \epsilon^{\alpha\beta}\epsilon^{AB}\overline{L}_{\pm 1}, \\
\{\overline{G}^{\alpha A}_{\frac{1}{2}}, \overline{G}^{\beta B}_{-\frac{1}{2}}\} &= \epsilon^{\alpha\beta}\epsilon^{AB}\overline{L}_0 + \epsilon^{AB} J^{\alpha\beta}_R.
\end{aligned} \tag{5.3}$$

[3]Related discussions of this factorization appear in [68, 1, 29, 26].

Note that $(\overline{G}^{\alpha A}_{\pm\frac{1}{2}})^\dagger = \epsilon_{\alpha\beta}\epsilon_{AB}\overline{G}^{\beta B}_{\mp\frac{1}{2}}$, the bar simply indicates that these generators act on the right moving sector of the CFT. A highest weight state $|h\rangle$ is annihilated by all the $G^{\alpha A}_{\frac{1}{2}}$'s. $|h\rangle$ is a chiral primary if it is further annihilated by $G^{++}_{-\frac{1}{2}}$, or equivalently if $\overline{L}_0|h\rangle = j_R|h\rangle$. It then follows from the algebra that $|h\rangle$ is annihilated by $G^{+-}_{-\frac{1}{2}}$ as well. A short representation is obtained by acting with lowering operators on a chiral primary. Each short representation consists of four representations of $SL(2,\mathbf{R})_R \times SU(2)_R$,[23]

$$(\bar{\ell}_0 = j_R, j_R)_S \to (j_R, j_R) + 2(j_R + \frac{1}{2}, j_R - \frac{1}{2}) + (j_R + 1, j_R - 1) \tag{5.4}$$

The four highest weight states in (5.4) are obtained by acting on $|h\rangle$ with the two broken $G_{-1/2}$'s. A general long representation $(\bar{\ell}_0, j_R)_{long}$ is built out of a highest weight state that does not saturate the unitarity bound $\overline{L}_0 \geq j_R$.

The global $SL(2,R)\times SU(1,1|2)$ symmetry is enlarged to the $(0,4)$ super-Virasoro algebra in the dual CFT. The chiral primaries are NS sector states with $\overline{L}_0 = J_R$. Under the spectral flow from NS to R sector,

$$\begin{aligned} &\overline{L}_n \to \overline{L}_n - (J_R)^3_n + \frac{c_R}{24}\delta_{n,0}, \\ &(J_R)^3_n \to (J_R)^3_n - \frac{c_R}{12}\delta_{n,0}, \qquad (J_R)^\pm_n \to (J_R)^\pm_{n\mp\frac{1}{2}}, \end{aligned} \tag{5.5}$$

the chiral primaries are mapped to Ramond ground states on the right, with $\overline{L}_0 = c_R/24$. They are counted by the elliptic genus.

5.2 BPS wrapped branes

5.2.1 Classical

The geometry (1.21) has a classical supersymmetric M2-brane wrapping a holomorphic genus g curve in the class $C = q_A\alpha^A$ and sitting at the center of AdS_3, $\chi = 0$. The kappa-symmetry analysis is very similar to the analysis of D2-branes in [63] and will not be repeated here. It can sit at any point on the S^2, but the unbroken supersymmetries vary as the point moves on the S^2. We will take it to sit at the north pole with $\theta = 0$. There is also an oppositely charged configuration consisting of an anti-membrane at the south pole ($\theta = \pi$) which preserves the same supersymmetries.

Both of these configurations have non zero angular momentum corresponding to ϕ rotations of the S^2 and given by

$$J^3 = \frac{1}{2}q_A p^A. \tag{5.6}$$

This angular momentum is carried by the fields much as for a monopole-electron pair in 4D. Although the M2 and anti-M2 have opposite charges, they still carry the same sign J^3 because they sit at opposite poles. Since they are static and saturate the BPS bound $L_0 = J^3$, it follows that AdS_3 mass and angular momentum are classically

$$L_0 = \overline{L}_0 = \frac{1}{2}q_A p^A. \tag{5.7}$$

This agrees with a direct calculation of the mass $M = \ell(L_0 + \overline{L}_0)$ as the membrane tension $\frac{1}{(2\pi)^2}$ times the membrane area $\frac{(2\pi)^2}{\ell}q_A p^A$.

Figure 5.1: Wrapped M2 and anti-M2 branes sitting at center of AdS_3 and at the north and south poles of the S^2, respectively.

5.2.2 Quantum

Quantum mechanically, the M2 fluctuates over the moduli space $\mathcal{M}_C$ of the genus g curve C in X, and has a degeneracy from worldvolume fermion zero modes. The supersymmetric quantum ground states correspond to cohomology classes on $\mathcal{M}_C$ and BPS hypermultiplets in 5D. This problem was studied in the context of compactification to 5D Minkowski space in [42], where the hypermuliplets have $SO(4) \sim SU(2)_L \times SU(2)_R$ spin content

$$\sum N_{Q,j_L,j_R}([(0,\frac{1}{2}) \oplus 2(0,0)] \otimes (j_L, j_R) \oplus [(\frac{1}{2},0)] \oplus 2(0,0)] \otimes (j_R, j_L)) \tag{5.8}$$

for some integers N_{Q,j_L,j_R} which depend on X and Q, the homology class of C. The range of j_L is determined by the genus of C, and j_R is related to the weight of Lefschetz action on the moduli space of C [42].

We wish to find the supersymmetric ground states - i.e. chiral primaries- of these

hypermultiplets on $AdS_3 \times S^2$. In this case the unbroken global superalgebra is $SU(1,1|2)$ and the relevant central charge is the angular momentum J^3 rather than the (graviphoton component of the) charge q_A (this problem was considered [23, 36]). Due to its coupling to the 4-form flux (1.19), the M2-brane feels a magnetic field on the S^2 of $B = q_A p^A$ units of flux. This leads to Landau levels on the S^2 which fall into representations of the $SU(2) \in SU(1,1|2)$ rotation. The highest weight states arising from a hypermulitplet in the representaion (j_L, j_R) have total spin

$$J^3 = \frac{1}{2} q_A p^A + m_R + m_L + \frac{1}{2} + l, \tag{5.9}$$

where $l \geq 0, \quad -j_{L,R} \leq m_{L,R} \leq j_{L,R}$, and l is the orbital angular momentum on the S^2. $m_R + m_L$ appears in this expression because the $U(1) \in SU(2)$ rotations of S^2 correspond to a diagonal $U(1)$ rotation in the $SU(2)_L \times SU(2)_R$ of R^4. The shift of $\frac{1}{2}$ appears because of the tensor product with a spin half hypermultiplet appearing in the definition (5.8).

The BPS chiral primary bound implies that these states have $\overline{L}_0 = J^3$. They are multiplets under the $SL(2,R)_L$ conformal algebra which commutes with $SU(1,1|2)$ and acts on the AdS_3 component of the wavefunction. There is one lowest weight state with $L_0 = \overline{L}_0 + m_L - m_R + \frac{1}{2} = \frac{1}{2} q_A p^A + 2m_L + l + 1$ for $-j_L \leq m_L \leq j_L, \quad -j_R \leq m_R \leq j_R$. $m_L - m_R$ appears in this expression because the $U(1))$ spatial rotations of AdS_3 correspond to an anti-diagonal $U(1)$ rotation in the $SU(2)_L \times SU(2)_R$ of R^4. Each of these has a further tower of chiral primary descendants obtained by acting with L_{-1}.

In summary for every charge q_A (j_L, j_R) hypermultiplet there is one chiral primary

with

$$L_0 = \frac{1}{2}q_A p^A + 2m_L + l + 1 + J_\phi \tag{5.10}$$

for every integrally-spaced value of

$$-j_L \leq m_L \leq j_L, \quad -j_R \leq m_R \leq j_R, \quad l \geq 0, \quad J_\phi \geq 0. \tag{5.11}$$

In addition there are antimembrane chiral primaries. M-theory with no branes is invariant under parity P, which we take to interchange the north and south pole of the S^2 . When branes are added it is invariant under CP where C reverses the brane charges. L_0 and J^3 are CP invariant. Hence the action of CP on a chiral primary gives another chiral primary. In the case at hand it turns each of the above M2-brane states into an antipodally located anti-M2-brane state. However these states will contribute differently to the elliptic genus because they have different charges.

5.3 The elliptic genus on $AdS_3 \times S^2 \times X$

In this section we compute the supergravity elliptic genus from M theory. In the NS sector this is given by[4]

$$Z_{sugra}(\tau, y^A) = \mathrm{Tr}_{\overline{L}_0 = J^3_R}(-)^F e^{2\pi i \tau L_0 + 2\pi i q_A y^A} \tag{5.12}$$

We work in the dilute gas approximation in which the density of chiral primaries is low. This is the case for large $\Im\tau$ and/or large $\Im y^A$. There will be two kinds of contributions, one from wrapped membranes and one from supergravity modes, which are computed in the next two subsections.

[4]Note that we are not including the factor of $q^{-\frac{c_L}{24}}$ in Z_{sugra}, and the L_0 entering here arises only the wrapped branes. This factor corresponds to the ground state energy of AdS_3.

5.3.1 Wrapped membranes

In this subsection we find the chiral primaries corresponding to membranes wrapped on holomorhpic curves in X.

Genus zero

For simplicity let's first consider the case of an isolated rational genus zero curve with degeneracy N_{q_A}, so that there is no internal (j_L, j_R) contribution. Summing over multiparticle states of this variety with weights and multiplicities given in (5.10)(5.11) gives

$$\begin{aligned} Z^0_{sugra} &= \prod_{q_A,l\geq 0,J_\psi\geq 0} (1-e^{2\pi i\tau(\frac{1}{2}q_Ap^A+l+J_\psi+1)}e^{2\pi i q_A y^A})^{N_{q_A}} \\ &\times \prod_{q_A,l\geq 0,J_\psi\geq 0} (1-e^{2\pi i\tau(\frac{1}{2}q_Ap^A+l+J_\psi+1)}e^{-2\pi i q_A y^A})^{N_{q_A}} \end{aligned} \tag{5.13}$$

The first factor comes from M2-branes while the second comes from anti-M2-branes. We can reorganize the product by defining $n = l + J_\psi + 1$

$$\begin{aligned} Z^0_{sugra}(\tau, y^A) = &\prod_{q_A,n>0} (1-e^{2\pi i\tau(\frac{1}{2}q_Ap^A+n)}e^{2\pi i q_A y^A})^{nN_{q_A}} \\ \times &\prod_{q_A,n>0} (1-e^{2\pi i\tau(\frac{1}{2}q_Ap^A+n)}e^{-2\pi i q_A y^A})^{nN_{q_A}} \end{aligned} \tag{5.14}$$

Higher genus

For general (j_L, j_R), instead of (5.13) we have

$$\begin{aligned} Z^{j_L,j_R}_{sugra} &= \prod_{q_A,l,J_\psi,m_L,m_R} (1-e^{2\pi i\tau(\frac{1}{2}q_Ap^A+l+J_\psi+1+2m_L)}e^{2\pi i q_A y^A})^{(-)^{2j_R+2j_L}N_{q_A,j_L,j_R}} \\ &\times \prod_{q_A,l,J_\psi,m_L,m_R} (1-e^{2\pi i\tau(\frac{1}{2}q_Ap^A+l+J_\psi+1+2m_L)}e^{-2\pi i q_A y^A})^{(-)^{2j_R+2j_L}N_{q_A,j_L,j_R}} \\ &= \prod_{q_A,n,m_L,} (1-e^{2\pi i\tau(\frac{1}{2}q_Ap^A+n+2m_L)}e^{2\pi i q_A y^A})^{(-)^{2j_R+2j_L}n(2j_R+1)N_{q_A,j_L,j_R}} \\ &\times \prod_{q_A,n,m_L} (1-e^{2\pi i\tau(\frac{1}{2}q_Ap^A+n+2m_L)}e^{-2\pi i q_A y^A})^{(-)^{2j_R+2j_L}n(2j_R+1)N_{q_A,j_L,j_R}} \end{aligned} \tag{5.15}$$

where J_ψ and l are non-negative integers, n is a positive integer and $-j_{L,R} \leq m_{L,R} \leq$

$j_{L,R}$. We will see below that these terms give all the loop contributions of the squared topological string partition function.

5.3.2 Supergravity modes

The massless spectrum of M-theory compactified on X consists of $n_H = 2(h^{2,1}(X)+1)$ hypermultiplets, $n_V = h^{1,1}(M) - 1$ vector multiplets, and a graviton multiplet. Their spectrum on $AdS_3 \times S^2$ organizes into short representations of $SL(2,\mathbf{R}) \times SU(1,1|2)$. The corresponding chiral primaries can be labelled by their $(L_0, \overline{L}_0 = J^3_R)$ quantum numbers, with the spectrum:

$$\begin{aligned} & n_H \bigoplus_{l\geq 0}(l+1, l+\tfrac{1}{2}) + n_V \bigoplus_{l\geq 0}\left[(l+1,l+1)+(l+1,l)\right] \\ & + \bigoplus_{l\geq 0}\left[(l+1,l+2)+(l+1,l+1)+(l+1,l)+(l+2,l)\right] \end{aligned} \tag{5.16}$$

The spectrum is obtained in [23, 36]. We have assumed here the range of allowed values of l is so as to include all possibilities with $L_0 > 0$. Whether or not singleton contributions with $\overline{L}_0 = 0$ should be included is a subtle issue which depends on the details of the asymptotic AdS_3 boundary conditions, and is beyond the scope of this thesis. The ambiguity leads to terms that depend on q but not any of the Calabi-Yau data.

As before, one can act on them with L_{-1} and generate further chiral primary states with nonzero orbital angular momenta J_ψ in AdS_3. The contribution from (5.16) to the elliptic genus is

$$\begin{aligned} & \prod_{l,J_\psi\geq 0} \frac{(1-q^{l+J_\psi+1})^{n_H}}{(1-q^{l+J_\psi+1})^{2n_V+3}(1-q^{l+J_\psi+2})} \\ & = \prod_{n\geq 1}(1-q^n)M(q)^{-\chi(X)} \end{aligned} \tag{5.17}$$

where $M(q) = \prod_{n\geq 1}(1-q^n)^n$ is the Macmahon function and $\chi(X) = 2(h^{1,1} - h^{2,1})$ is the Euler characteristic of X. We will henceforth drop the η function prefactor which does not depend on Calabi-Yau data. The net contribution from massless neutral supergravity modes including singletons is then simply

$$Z^{massless}_{sugra} = \prod_{n\geq 1}(1-q^n)^{-n\chi}. \tag{5.18}$$

5.3.3 Putting it all together

Let us now summarize and compile the results of this section into a formula for $Z_{BH} = Z_{CFT}$ The elliptic genus of the $(0,4)$ CFT as a Ramond sector trace is [5]

$$Z_{CFT}(\tau, y^A) = \mathrm{Tr}_R(-)^F q^{L_0 - \frac{c_L}{24}} \overline{q}^{\overline{L}_0 - \frac{c_R}{24}} e^{2\pi i y^A q_A} \tag{5.19}$$

In the dilute gas expansion around $\Im\tau \to \infty$

$$\begin{aligned} Z_{CFT}(\tau, y^A) &= e^{-\pi i\tau c_L/12} Z_{sugra}(\tau, y^A) \\ &= e^{-\pi i\tau c_L/12} \prod (1 - e^{2\pi i\tau n})^{-n\chi} \\ &\quad \times \prod \left(1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + n + 2m_L)} e^{2\pi i q_A y^A}\right)^{(-)^{2j_R+2j_L} n(2j_R+1)N_{q_A,j_L,j_R}} \\ &\quad \times \prod \left(1 - e^{2\pi i\tau(\frac{1}{2}q_A p^A + n + 2m_L)} e^{-2\pi i q_A y^A}\right)^{(-)^{2j_R+2j_L} n(2j_R+1)N_{q_A,j_L,j_R}} \end{aligned} \tag{5.20}$$

where we take the products over the q_A charge lattice, positive integral p, n, integral or half integral j_L, j_R and $-j_{L,R} \leq m_{L,R} \leq j_{L,R}$.

[5] There are some subtleties here in the spectral flow related to the fact that the $U(1)$ current involves membrane charges and is in a supermultiplet with the center of mass degrees of freedom [56] which we shall not try to address.

5.4 Derivation of OSV conjecture

In the preceding section we computed the dilute gas approximation to the elliptic genus of the $(0,4)$ CFT, denoted Z_{BH}, as a product of terms coming from massless supergravity modes and wrapped membranes. In this section we wish to compare our result with the OSV formula [59] for the same object as the square of the topological string partition function Z_{top}. The OSV result begins with the Bekenstein-Hawking relation and then includes all orders perurbative corrections. This is the regime in which many BPS excitations are present and is the opposite of a dilute gas. However, modular invariance relates the dilute gas to the high-temperature regime needed for comparison to OSV as follows.

Under the modular transform $\tau \to -1/\tau$, we have

$$\begin{aligned} Z_{BH} &= Z_{CFT}(\tau, y^A) = Z_{CFT}(-1/\tau, y^A/\tau) e^{-\frac{2\pi i}{\tau} y^2} \\ &= \exp\left[\frac{2\pi i}{\tau}\left(\frac{c_L}{24} - y^2\right)\right] Z_{sugra}(-1/\tau, y^A/\tau) \end{aligned} \tag{5.21}$$

where $y^2 = D_{ABC} p^A y^B y^C$. In this form we can consider high temperatures $\Im\tau \to 0$ since the RHS will then involve Z_{sugra} at low temperatures. The $(0,4)$ CFT of [56] has $c_L = 6D + c_2 \cdot P = 6D_{ABC} p^A p^B p^C + c_{2A} p^A$, The prefactor in (5.21) is then

$$\exp\left[\frac{2\pi i}{\tau}\left(\frac{c_L}{24} - y^2\right)\right] = \exp\left\{\frac{\pi^2}{\phi^0}\left[D_{ABC} p^A \left(p^B p^C - \frac{\phi^B \phi^C}{\pi^2}\right) + \frac{1}{6} c_{2A} p^A\right]\right\} \tag{5.22}$$

This is precisely $|\exp(\mathcal{F}_0^{(0)} + \mathcal{F}_1^{(0)})|^2$, where $\mathcal{F}_{0,1}^{(0)}$ denote the part of topological string amplitude that is perturbative on the world sheet, at genus 0 and 1. Z_{sugra} then give the rest of $|Z_{top}|^2$. To see this we need to use the fundamental relation between the integral degeneracies N_{q_A, j_L, j_R} of BPS states and the coefficients of the topological

string expansion found from a Schwinger computation in [41]. Indeed comparing with [41][6] we find precisely that, for purely imaginary $\tau = i\phi^0/2\pi$ and $y^A = i\phi^A/2\pi$,

$$\begin{aligned} Z_{BH} &= \mathrm{Tr}_R[(-)^F \overline{q}^{\overline{L}_0 - \frac{c_R}{24}} e^{-\phi^A q_A - \phi^0 (L_0 - \frac{c_L}{24})}] \\ &= \left| Z_{top}(g_{top} = \frac{4\pi^2}{\phi^0}, t^A = \frac{\phi^A - \pi i p^A}{\phi^0}) \right|^2 . \end{aligned} \tag{5.23}$$

The first line is the OSV definition of the mixed partition function and the second is the OSV relation to the square of the topological string partition function.

In conclusion we have rederived the OSV relation in all detail from an M-theory partition function on an $AdS_3 \times S^2 \times X$. In this picture the factorization into holomorphic and antiholomorphic parts has a simple origin as the contributions from M2-branes and anti-M2-branes.

[6]To see this agreement one needs to use a well-known resummation of the formulae of [41] which is reproduced in the appendix.

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Appendix A

Appendix

A.1 The 10-dimensional Killing spinors

In order to write a ten-dimensional spinor as the tensor product of four-dimensional and internal (Calabi-Yau) spinors, it is necessary to work with a tensor product of Clifford algebras. Let Γ^M denote the ten-dimensional Clifford algebra matrices, with $M = 0, \ldots, 10$, $\mu = 0, \ldots, 3$, and $m = 4, \ldots, 9$. We can decompose the Γ^M into a tensor product of four and six-dimensional Clifford matrices, denoted by γ^μ and γ^m, as

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes 1, \\ \Gamma^m &= \gamma_{(4)} \otimes \gamma^m \,. \end{aligned} \tag{A.1}$$

Using a mostly-positive metric signature, the following matrices have the desired properties that they anticommute with the appropriate gamma matrices and square

to one:

$$\Gamma_{(10)} = -\Gamma^{0123456789},$$
$$\gamma_{(4)} = i\gamma^{0123},$$
$$\gamma_{(6)} = i\gamma^{456789}\,. \tag{A.2}$$

With these sign conventions, $\Gamma_{(10)}$ decomposes in the desired way as $\Gamma_{(10)} = \gamma_{(4)} \otimes \gamma_{(6)}$.

As an ansatz for the Killing spinors, we assume they take the form

$$\varepsilon_1 = \epsilon_1 \otimes \eta_+ + \epsilon^1 \otimes \eta_-\,, \qquad \varepsilon_2 = \epsilon^2 \otimes \eta_+ + \epsilon_2 \otimes \eta_-\,, \tag{A.3}$$

where the ε's are 10D Majorana-Weyl spinors, the η's are 6D covariantly-constant Weyl spinors on the Calabi-Yau, and the ϵ's are 4D Majorana spinors. We use chiral notation in which the chirality of the spinor is denoted by the position of the R-symmetry index. In particular, $\epsilon(A) = \epsilon^A + \epsilon_A$ where $\gamma_{(4)}\epsilon^A = \epsilon^A$ and $\gamma_{(4)}\epsilon_A = -\epsilon_A$. Of course, there are no Majorana-Weyl spinors in $3+1$ dimensions; the four-dimensional chiral projections are related by $\epsilon_A = \epsilon^{A*}$. For the six-dimensional Weyl spinors, we use the standard notation where $\gamma_{(6)}\eta_\pm = \pm\eta_\pm$. Since we will work with type IIA, the tensor products have been chosen such that the ten-dimensional spinors are of opposite chirality. In doublet notation,

$$\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \tag{A.4}$$

$\Gamma_{(10)}\varepsilon$ can be written as $-\sigma^3\varepsilon$. In addition, the following identities for the spinors $\eta_\pm$ will be useful:

$$\gamma_{\bar{i}}\eta_+ = 0\,, \quad \gamma_{ijk}\eta_+ = \Omega_{ijk}\eta_-\,, \quad \gamma_{ij}\eta_+ = \frac{1}{2}\Omega_{ijk}\gamma^k\eta_-\,, \quad \gamma_{\overline{ij}kl}\eta_+ = \left(g_{k\bar{j}}g_{l\bar{i}} - g_{k\bar{i}}g_{l\bar{j}}\right)\eta_+\,,$$

$$\gamma_i \eta_- = 0\,, \quad \gamma_{\overline{ijk}}\eta_- = \Omega_{\overline{ijk}}\eta_+\,, \quad \gamma_{\overline{ij}}\eta_- = \frac{1}{2}\Omega_{\overline{ijk}}\gamma^{\overline{k}}\eta_+\,, \quad \gamma_{ij\overline{kl}}\eta_- = \left(g_{\bar{k}j}g_{\bar{l}i} - g_{\bar{k}i}g_{\bar{l}j}\right)\eta_- \tag{A.5}$$

Given these ansätze, we want to check that the supersymmetry variations of the background vanish modulo conditions on the four-dimensional Majorana components of the Killing spinors. Since we work only with bosonic backgrounds, we need only check the variations of dilatino and gravitino.

The supersymmetry variation of the dilatino is [10]

$$\delta\lambda = \frac{1}{2}\left(3\not{F}_{(2)} i\sigma^2 + \not{F}_{(4)}\sigma^1\right)\varepsilon\,, \tag{A.6}$$

where $F_{(2)} = \frac{1}{R}\omega_{AdS_2}$ and $F_{(4)} = \frac{1}{R}\omega_{S^2}\wedge J$. Taking note of the fact that $g^{\bar{i}j}\gamma_{\bar{i}j}\eta_\pm = 3\gamma_{(6)}\eta_\pm$ and $\not{\omega}_{S^2} = -i\not{\omega}_{AdS_2}\gamma_{(4)}$, we find that

$$\not{F}_{(4)}\varepsilon = -3i\not{\omega}_{AdS_2}\gamma_{(4)}\gamma_{(6)}\varepsilon = -3\not{F}_{(2)}\sigma^3\varepsilon\,. \tag{A.7}$$

As a result, the dilatono variation vanishes automatically.

The gravitino variation is

$$\delta\psi_M = \nabla_M\varepsilon + \frac{1}{8}\left(\not{F}_{(2)}\Gamma_M i\sigma^2 + \not{F}_{(4)}\Gamma_M\sigma^1\right)\varepsilon = 0\,. \tag{A.8}$$

When the free index is holomorphic in the Calabi-Yau, this reduces to the following condition:

$$\left(\not{F}_{(2)}\gamma_m i\sigma^2 + \not{F}_{(4)}\gamma_m\sigma^1\right)\varepsilon = 0\,. \tag{A.9}$$

Using the fact that $g^{i\bar{j}}\gamma_{i\bar{j}}\gamma_m\eta_\pm = \gamma_m\gamma_{(6)}\eta_\pm$, we find that $\not{F}_{(4)}\gamma_m\varepsilon = -\not{F}_{(2)}\gamma_m\sigma^3\varepsilon$. This works similarly for an antiholomorphic index, so the gravitino variation is identically zero when the free index is in the Calabi-Yau.

When the gravitino equation has its free index in the $AdS_2\times S^2$ space, the variation becomes

$$\delta\psi_\mu = \left[\nabla_\mu \pm \frac{1}{8}\gamma_\mu\left(\not{F}_{(2)} i\sigma^2 - \sigma^1\not{F}_{(4)}\right)\right]\varepsilon = 0\,, \tag{A.10}$$

where the $\pm$ is $+$ if μ is in the S^2 and $-$ if μ is in the AdS_2. Using the same identity used for the dilatino equation, we get

$$\delta\psi_\mu = \left[\nabla_\mu \pm \frac{i}{2}\gamma_\mu \not{F}_{(2)}\sigma^2\right]\varepsilon = \left[\nabla_\mu + \frac{i}{2}\not{F}_{(2)}\gamma_\mu\sigma^2\right]\varepsilon\,. \tag{A.11}$$

Demanding that the terms linear in η_+ and linear in η_- must vanish separately, we get the 4D equations

$$\left[\nabla_\mu + \frac{i}{2}\not{F}_{(2)}\gamma_\mu\sigma^2\right]\epsilon = 0\,, \tag{A.12}$$

where $\epsilon = (\epsilon_1, \epsilon^2)^T$.

It is useful to derive the action of $\Gamma_{(0)} = \frac{1}{(p+1)!\sqrt{\det G}}\epsilon^{\widehat{\mu}_0\cdots\widehat{\mu}_p}\Gamma_{\widehat{\mu}_0\cdots\widehat{\mu}_p}$ on the $\eta_\pm$ which live on the world-volume of holomorphically wrapped D-branes (see (4.25)). For D0-branes we have simply $\Gamma_{(0)} = \gamma^0$. For D2-branes, we have

$$\Gamma_{(0)}\eta_\pm = \gamma^0\epsilon^{i\bar{j}}\gamma_{i\bar{j}}\eta_\pm = i\gamma^0\gamma_{(6)}\eta_\pm \tag{A.13}$$

For D4-branes, we have

$$\Gamma_{(0)}\eta_\pm = \gamma^0\frac{1}{4}\epsilon^{i\bar{j}k\bar{l}}\gamma_{i\bar{j}k\bar{l}}\eta_\pm = -\gamma^0\eta_\pm \tag{A.14}$$

where we used the last column of (A.5). Finally for D6-branes, we have $\Gamma_{(0)} = -i\gamma^0\gamma_{(6)}$ using (A.2). These formulae can be summarized as $\Gamma_{(0)}\varepsilon = \gamma^0(i\gamma_{(6)})^{p/2}\varepsilon$.

A.2 Resummation of the GV formula

In this appendix we rearrange the expression for the topological string partition function to express it in terms of the Gopakumar-Vafa invariants rather than the degneracies N_{q_A,j_L,j_R} of the irreducible representations. Taking minus the log of the

first product in (5.15) and summing over j_L, j_R, resumming and setting $g_{top} = -2\pi i\tau$, $t^A = y^A + \frac{\tau}{2}p^A$ gives

$$\begin{aligned} F &= \sum_{q_A,n,m_L,j_Lj_R} (-)^{2j_R+2j_L} n(2j_R+1)N_{q_A,j_L,j_R} \ln(1-e^{-g_{top}(n+2m_L)}e^{2\pi i q_A t^A}) \\ &= -\sum_{q_A,n,m_L,j_L,j_R,k} (-)^{2j_R+2j_L} \frac{n}{k}(2j_R+1)N_{q_A,j_L,j_R} e^{-kg_{top}(n+2m_L)}e^{2\pi i k q_A t^A} \\ &= -\sum_{q_A,j_L,j_R,k} (-)^{2j_R+2j_L} \frac{1}{k}(2j_R+1)N_{q_A,j_L,j_R} \frac{\sinh[(2j_L+1)kg_{top}]}{\sinh^2[\frac{1}{2}kg_{top}]\sinh[kg_{top}]} e^{2\pi i k q_A t^A} \end{aligned}$$

$(A.15)$

In [42] the Gopakumar-Vafa invariants α_{r,q_A} are defined by

$$\sum_r \alpha_{r,q_A}(-)^r(2\sinh\frac{\theta}{2})^{2r} = \sum_{j_L,j_R} (-)^{2j_R+2j_L}(2j_R+1)N_{q_A,j_L,j_R}\frac{\sinh[(2j_L+1)\theta]}{\sinh[\theta]}, \quad \text{(A.16)}$$

so that

$$F = \sum_{q_A,r,k} \frac{(-)^{r-1}}{k}\alpha_{r,q_A}(2\sinh\frac{kg_{top}}{2})^{2r-2}e^{2\pi i k q_A t^A}. \quad \text{(A.17)}$$

This agrees precisely with the expression for the topological string partition function given in [42].