On Eilenberg-MacLanes Spaces
(Term paper for Math 272a)

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Abstract

This paper discusses basic properties of Eilenberg-MacLane spaces $K(G, n)$, their cohomology groups and some classic applications. We construct $K(G, n)$ using both CW complexes and classifying spaces. Two proofs are given to the basic fact that cohomology of a CW complex X has 1-1 correspondence with homotopy classes of maps from X into Eilenberg-MacLane spaces: a barehanded proof (original) and a categorical proof using loop spaces. We use it to demonstrate the connection between cohomology operations and cohomology groups of $K(G, n)$'s. Finally we use the technique of spectral sequence to compute the cohomology of some classes of Eilenberg-MacLane spaces, and apply it to the calculation $\pi_5(S^3)$.

1. Introduction

A space X having only one nontrivial homotopy group $\pi_n(X) \cong G$ is called an Eilenberg-MacLane space $K(G, n)$. The simplest examples are $K(\mathbf{Z}, 1) \simeq S^1$, $K(\mathbf{Z}/2, 1) \simeq \mathbf{RP}^\infty$, and $K(\mathbf{Z}, 2) \simeq \mathbf{CP}^\infty$. In general they are more complicated objects. The Eilenberg-MacLane spaces play a fundamental role in the connection between homotopy and (co)homology. The first basic fact we will prove is that, given a CW complex X, there is a bijection between its cohomology group $H^n(X; G)$ and the homotopy classes of maps from X to $K(G, n)$. Using this, it is not hard to show that cohomology operations are completely classified by the cohomlogy groups of $K(G, n)$'s. The latter seems to be quite complicated to compute, since the construction of $K(G, n)$ as CW complexes involves attaching cells to cancel arbitrarily high dimensional homotopy groups. Fortunately they are computable via spectral sequences. For example, it has been proved by Serre that $H^*(K(\mathbf{Z}/2, n); \mathbf{Z}/2)$ is a polynomial ring over $\mathbf{Z}/2$, with generators obtained by all the different way (unrelated by Adem relations) of acting the Steenrod squares on a fundamental class $\alpha \in H^n(K(\mathbf{Z}/2, n); \mathbf{Z}/2)$.

The results in this paper have been known for 50 years. We are mostly following Hatcher's book *Algebratic Topology* and the book of Bott and Tu, *Differential Forms in Algebraic Topology.* Other sources are listed in the references. The bare-handed proof of theorem 2 is original.

The paper is organized as follows. In section 2 we construct the Eilenberg-MacLane spaces, using CW complexes and classifying spaces. Section 3 studies some basic properties of $K(G, n)$, their role in the connection between homotopy and cohomology, and classifying cohomology operations. As applications we show that the first Stiefel-Whitney class and the first Chern class classify the real and complex line bundles over CW complexes. In section 4 we employ the technique of spectral sequence, and use it to compute a number of example of cohomology groups of $K(G, n)$'s. Finally results of $H^i(K(\mathbf{Z}/2, 3))$ are used to compute $\pi_5(S^3)$.

2. Construction of Eilenberg-MacLane Spaces

Theorem 1 *There exists a CW complex $K(G, n)$ (unique up to homotopy type) for any group G at $n = 1$, and Abelian group G at $n > 1$.*

Proof: Step 1 - Construction of $K(G, 1)$.

Let EG be the Δ-complex whose n-simplices are given by $\{[g_0, \cdots, g_n] | g_i \in G\}$, glued along the faces $[g_0, \cdots, \hat{g}_i, \cdots, g_n]$ in the obvious way. Consider a homotopy $h_t : EG \to EG, t \in [0, 1]$, such that the restriction of h_t to the simplex $[g_0, \cdots, g_n]$ is given by the linear retraction of $[e, g_0, \cdots, g_n]$ onto e, with e being the identity element of G. It is clear that h_t is a homotopy between EG and the point e, so EG is contractible. Note that h is not a deformation retraction, since it moves e along the loop $[e, e]$.

There is a natural action of G on EG,

$$\begin{array}{cccc} g: & EG & \longrightarrow & EG \\ & [g_0, \cdots, g_n] & \longrightarrow & [gg_0, \cdots, gg_n] \end{array}$$

where g maps the simplex $[g_0, \cdots, g_n]$ linearly onto $[gg_0, \cdots, gg_n]$. It is not hard to see that this is a covering space action. Let $BG = EG/G$ be the quotient space, then $\pi_1(BG) = G$. Since $EG \to BG$ is a fiber bundle with fiber G having discrete topology, the long exact sequence of homotopy groups implies $\pi_i(BG) = 0$ for all $i > 1$. So BG is a $K(G, 1)$ space.

As a side remark, the space BG is also a Δ-complex, since the group action permutes the simplices. Explicitly, a simplex of BG has the form

$$[g_1|g_2|\cdots|g_n] := G[e, g_1, g_1g_2, \cdots, g_1\cdots g_n]$$

and the boundary of the simplex is

$$\partial[g_1|\cdots|g_n] = [g_2|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n]$$

Written in this form, it is clear that simplicial (co)homology groups of the Δ-complex BG is nothing but the group (co)homology of G. So we learned that the cohomology groups of $K(G,1)$ (if we have proved its uniqueness) is the group cohomology of G.

Step 2 - Construction of $K(G,n)$ for Abelian group G.

To each generator g_α of G we associate an n-sphere S^n_α. So we have $\pi_n(\bigvee_\alpha S^n_\alpha) = \bigoplus_\alpha \mathbf{Z}$. A relation between the generator g_α's can be realized as the class $[\varphi_\beta]$, represented by the map $\varphi_\beta : S^n \to \bigvee_\alpha S^n_\alpha$. Consider the CW complex X obtained from $\bigvee_\alpha S^n_\alpha$ by attaching $(n+1)$-cells e^{n+1}_β via the maps φ_β.

Since any map $S^i \to X$ is homotopic to a cellular map, and X has no i-cells for $0 < i < n$, we see that $\pi_i(X) = 0$ for $i < n$. When $n > 1$, homotopy excision implies $\pi_{n+1}(X, X^n) = \pi_{n+1}(X/X^n) = \pi_{n+1}(\bigvee_\beta e^{n+1}_\beta/\partial e^{n+1}_\beta) = \bigoplus_\beta \mathbf{Z}$. The LES for the pair (X, X^n) gives the exact sequence

$$\pi_{n+1}(X, X^n) = \bigoplus_\beta \mathbf{Z} \longrightarrow \pi_n(X^n) = \bigoplus_\alpha \mathbf{Z} \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, X^n) = 0$$

So $\pi_n(X) = \bigoplus_\alpha \mathbf{Z}/\{[\varphi_\beta]\} = G$.

The space X in general has complicated higher homotopy groups. We use an inductive procedure to add higher dimensional cells to make π_i vanish, without affecting lower dimensional homotopy groups. Choose maps $\varphi_\alpha : S^{n+1} \to X$ representing the generators of $\pi_{n+1}(X)$, and attach $(n+2)$-cells e^{n+2}_α via φ_α's. Let's call the resulting space $X^{(n+1)}$. By cellular approximation, $\pi_{n+1}(X^{(n+1)}) = 0$. Now we attach $(n+3)$-cells in a similar way to get space $X^{(n+2)}$ with $\pi_{n+2}(X^{(n+2)}) = 0$, and so on. The direct limit under inclusion

$$Y = \varinjlim X^{(n+k)}$$

is an Eilenberg-MacLane space $K(G,n)$.

Step 3 - Uniqueness.

Given a CW complex $K(G,n)$ space Z, we want to show that there is a homotopy equivalence $f : Y \to Z$ where Y is the $K(G,n)$ constructed above. It suffices to construct a map f that induces isomorphisms on all homotopy groups.

First consider the n-skeleton $Y^n = \bigvee_\alpha S^n_\alpha$. We get a map $f : Y^n \to Z$ whose restriction to each S^n_α represents the homotopy class $[S^n_\alpha \hookrightarrow Y] \in \pi_n(Y) = G = \pi_n(Z)$. It is clear that f defined this way extends onto the $(n+1)$-skeleton of Y, since its composition with the attaching maps are null-homotopic. It further extends to cells of dimension$\geq n+2$, since $\pi_i(Z) = 0$ for $i > n$. This finishes the proof. **QED.**

Having constructed general Eilenberg-MacLane spaces using CW complexes, we now give an alternative construction using the classifying spaces, which is a direct generalization of the previous construction of $K(G,1)$.

Consider spaces $E_n(G) = G^{n+1}$, $B_n(G) = G^n$, regarded as collections of simplices. Define "face" and "degeneracy" operators on $E_*(G)$ by

$$\begin{aligned}
&d_0(g_1,\cdots,g_{n+1}) = (g_2,\cdots,g_{n+1}),\\
&d_i(g_1,\cdots,g_{n+1}) = (g_1,\cdots,g_{i-1},g_ig_{i+1},g_{i+2},\cdots,g_{n+1}), \quad 1 \leq i \leq n,\\
&s_i(g_1,\cdots,g_{n+1}) = (g_1,\cdots,g_{i-1},e,g_i,\cdots,g_{n+1}), \quad 0 \leq i \leq n.
\end{aligned}$$

And similarly on $B_*(G)$, where the only difference is that we define

$$d_n(g_1,\cdots,g_n) = (g_1,\cdots,g_{n-1}).$$

In a similar fashion, we can define face and degeneracy maps on the standard simplices as

$$\begin{aligned}
&\delta_i : \Delta_{n-1} \longrightarrow \Delta_n, \quad (t_0,\cdots,t_{n-1}) \mapsto (t_0,\cdots,t_{i-1},0,t_i,\cdots,t_n),\\
&\sigma_i : \Delta_{n+1} \longrightarrow \Delta_n, \quad (t_0,\cdots,t_{n+1}) \mapsto (t_0,\cdots,t_{i-1},t_i+t_{i+1},\cdots,t_{n+1}).
\end{aligned}$$

Consider the spaces

$$E(G) = \coprod_{n=0}^{\infty} E_n(G) \times \Delta_n / \sim,$$

$$B(G) = \coprod_{n=0}^{\infty} B_n(G) \times \Delta_n / \sim .$$

where the equivalence relation $\sim$ is defined by

$$(f, \delta_i u) \sim (d_i(f), u), \quad (f, \sigma_i u) \sim (s_i(f), u).$$

The map $p_* : E_*(G) \to B_*(G)$ that projects $E_{n+1}(G)$ onto its first n coordinates induces a projection map

$$p : E(G) \longrightarrow B(G)$$

This is a fiber bundle with fiber being G. Its group action on the simplices is given by

$$g : (g_1, \cdots, g_{n+1}) \longrightarrow (g_1, \cdots, g_{n+1} g)$$

As before $E(G)$ is contractible. So by the LES of the fibration $G \to E(G) \to B(G)$, we have

$$\pi_{i+1}(BG) \cong \pi_i(G), \quad \forall\ i \geq 0.$$

The group multiplication $G \times G \to G$ gives a group structure on $B(G)$:

$$B(G) \times B(G) \cong B(G \times G) \longrightarrow B(G)$$

We can iterate this procedure and get $B^n(G)$. In our previous construction of $K(G, 1)$, we were treating G as a discrete group. If we start out with a discrete Abelian group G, the Eilenberg-MacLane spaces can be constructed as

$$K(G, n) = B^n(G).$$

3. Basic Properties of $K(G, n)$ and some applications

Next we will give two proofs to the follwing fundamental result relating Eilenberg-MacLane spaces to singular cohomology groups.

Theorem 2 *Let G be an Abelian group, X be a CW complex. There is a natural bijection*

$$\begin{array}{rcl} T : \langle X, K(G, n) \rangle & \longrightarrow & H^n(X; G) \\ [f] & \longmapsto & f^* \alpha \end{array}$$

where $\alpha \in H^n(K(G, n); G) \cong Hom(H_n(K; \mathbf{Z}), G)$ is given by the inverse of Hurewitz isomorphism $G = \pi_n(K) \to H_n(K; \mathbf{Z})$.

Bare-handed Proof: Consider a homotopy class $[f] \in \langle X, K(G,n)\rangle$, with base-point preserving map $f : X \to K(G,n)$. Since $\pi_i(K) = 0$ for $i < n$, the restriction of f onto the $(n-1)$-skeleton X^{n-1} is null-homotopic. Without loss of generality, we can assume f maps X^{n-1} to the basepoint in $K(G,n)$. So we get

$$\overline{f} : X/X^{n-1} \longrightarrow K(G,n)$$

It induces the map on homotopy groups

$$\overline{f}_* : \pi_n(X/X^{n-1}) \longrightarrow G$$

Via Hurewitz isomorphism (rather trivial in this case), we can regard $\overline{f}_*$ as an element of $\mathrm{Hom}(H_n(X/X^{n-1}), G) \cong H^n(X/X^{n-1}; G)$. The two groups are isomorphic since H_{n-1} vanishes. It is not hard to see that the image of $\overline{f}_*$ under $q^* : H^n(X/X^{n-1}; G) \to H^n(X; G)$ is just $f^*\alpha$.

We first prove the surjectivity of T. From the LES

$$\cdots \longrightarrow H^{n-1}(X^{n-1}; G) \xrightarrow{\delta} H^n(X/X^{n-1}; G) \xrightarrow{q^*} H^n(X; G) \longrightarrow H^n(X^{n-1}; G) = 0$$

q^* is surjective. So given $\gamma \in H^n(X; G)$, we have $\psi : \pi_n(X/X^{n-1}) \to G$, $q^*\psi = \gamma$. This in particular gives a map $\widetilde{\psi} : \pi_n(X^n/X^{n-1}) \cong \bigoplus_\alpha \mathbf{Z} \to G$. It can be realized from the mapping between spaces

$$g : \ X^n/X^{n-1} \longrightarrow K(G,n)$$

We need to extend g to X. Since $\pi_i(K(G,n)) = 0$ for $i > n$, it suffices to extend g to X^{n+1}. Pick any $(n+1)$-cell e_β^{n+1} with attaching map $\varphi_\beta : \partial e_\beta^{n+1} \to X^n$. The map

$$g \circ \varphi_\beta : S^n \to X^n/X^{n-1} \to K(G,n)$$

induces $(g \circ \varphi_\beta)_* : H_n(S^n) = \mathbf{Z} \to H_n(K(G,n)) = G$. φ_β represents an n-dimensional homology class $\overline{[\varphi_\beta]}$ in the image of $H_{n+1}(X^{n+1}, X^n) \to H_n(X^n)$, or a class $[\varphi_\beta] \in H_n(X^n, X^{n-1})$. It follows that $(g \circ \varphi_\beta)_* = \widetilde{\psi}([\varphi_\beta]) = \gamma(\overline{[\varphi_\beta]}) = 0$. Hence $g \circ \varphi_\beta$ is null-homotopic, extends onto e_β^{n+1}. This proves the surjectivity of T.

From the above construction, it is clear that given $\psi \in H^n(X^n/X^{n-1}; G)$, the associated map $X/X^{n-1} \to K(G,n)$ is unique up to homotopy, therefore the preimage $[f] \in \langle X, K(G,n)\rangle$ is unique. However given $\gamma \in H^n(X; G)$, we have different choices of $\psi' = \psi + \delta\eta$, $\eta \in H^{n-1}(X^{n-1}; G)$.

They give rise to maps $f, f' : X \to K(G, n)$ that restrict to the constant map on X^{n-1}. It remains to prove that f and f' are homotopic, although not necessarily rel X^{n-1}. Again it suffices to prove that the restriction of f and f' on X^n are homotopic. Without loss of generality, we can

1. consider simply $f, f' : X^n/X^{n-2} \to K(G, n)$; (because a homotopy is a map $X \times I \to K$, X^{n-2} only affects the $(n-1)$-skeleton of $X \times I$)
2. assume η is represented by a generator of $H^{n-1}(X^{n-1}, X^{n-2}; G)$.

Pick any n-cell e^n_α with attaching map $\varphi_\alpha : \partial e^n_\alpha \to X^{n-1}$. Composed with the cellular boundary map, we get

$$H_n(e^n_\alpha/\partial e^n_\alpha) \xrightarrow{d_\alpha} H_{n-1}(S^{n-1}_\beta)$$

where S^{n-1}_β is the sphere in X^{n-1}/X^{n-2} dual to η, d_α is the degree of $p_\beta \circ \varphi_\alpha : \partial e^n_\alpha \to S^{n-1}_\beta$. We'll first prove that the map $\tilde{g}$ representing the preimage of $\delta\eta$ is null-homotopic. As before we can assume that $\tilde{g}$ is constant on X^{n-1}. Consider the map on the quotient space $g : X^n/X^{n-1} = \bigvee_\alpha S_\alpha \to K(G, n)$. Its restriction to each S^n_α represents the element $d_\alpha \eta \in G = \pi_n(K(G, n))$.

We claim that $\tilde{g}$ is null-homotopic. It suffices to show that the restriction of $\tilde{g}$ at

$$\tilde{g} : e^n_\alpha \cup p_\beta \circ \varphi_\alpha(\partial e^n_\alpha) \longrightarrow K(G, n)$$

is null-homotopic, for all e^n_α. Since $p_\beta \circ \varphi_\alpha$ is of degree d_α, and the map on the quotient space represents $d_\alpha \eta$, the above map factorize as

$$(e^n_\alpha \cup p_\beta \circ \varphi_\alpha(\partial e^n_\alpha), p_\beta \circ \varphi_\alpha(\partial e^n_\alpha)) \longrightarrow (B^n, S^{n-1}) \xrightarrow{\eta} K(G, n)$$

where the first map is constructed as follows. $p_\beta \circ \varphi_\alpha$ is homotopy equivalent to suspensions of $S^1 \to S^1, e^{i\theta} \mapsto e^{id_\alpha\theta}$. It can be trivially extended onto the ball e^n_α, which induces the map on the quotient space $e^n_\alpha \cup p_\beta \circ \varphi_\alpha(\partial e^n_\alpha) \to B^n$. Now $\tilde{g}$ factorizes through $B^n \to K(G, n)$, it is certainly null-homotopic.

We note that the homotopy classes of maps

$$e^n_\alpha \cup p_\beta \circ \varphi_\alpha(\partial e^n_\alpha) \longrightarrow K(G, n)$$

has a group structure, constructed in the same way as π_n (if $d_\alpha = 0$ it reduces to $\pi_n(K)$). Now $[f'] = [f] + [\tilde{g}] = [f]$, f and f' are homotopic. This finishes proving the injectivity of T. **QED.**

The above bare handed proof is straightforward but not very illuminating. A more elegant proof follows directly from two lemmas:

Lemma 1 *The functor $X \to h^n(X) = \langle X, K(G,n)\rangle$ defines a reduces cohomology theory on the category of base-pointed CW complexes.*

Remark: the definition for $h^n(X)$ extends to $n \le 0$ by $\langle X, K(G,n)\rangle = \langle X, \Omega^k K(G, n+k)\rangle$, where Ω is the loop space functor.

Lemma 2 *If a reduced cohomology theory h^* defined on CW complexes has $h^n(S^0) = 0$ for $n \neq 0$, then there are natural isomorphisms*

$$h^n(X) \cong \widetilde{H}^n(X; h^0(S^0))$$

for all CW complexes X and integer n.

The proof of lemma 2 is standard, which we will omit. We will sketch the proof of lemma 1.

Proof of Lemma 1: First of all, if we take $K(G,n)$ to be a CW complex, there is a weak homotopy equivalence

$$K(G,n) \to \Omega K(G, n+1)$$

This follows from the fact that for a space X, the loop space ΩX has homotopy groups $\pi_n(\Omega X) \cong \pi_{n+1}(X)$.

Secondly, there is an adjoint relation between the functor Ω and reduced suspension Σ,

$$\langle \Sigma X, K\rangle = \langle X, \Omega K\rangle$$

There is a natural group structure on $\langle X, \Omega K\rangle$. In particular, there is a group structure on $\langle X, K(G,n)\rangle$ since $K(G,n)$ are weak homotopy equivalent to loop spaces (the "weak" condition can be dropped, but that's unnecessary for our purpose).

Now we will show that $h^n(X) = \langle X, K(G,n)\rangle$ defines a reduced cohomology theory. To check the homotopy axiom and the wedge sum axiom is trivial. We will do the more interesting one, the LES associated to CW pair (X, A). First note the sequence of inclusion of spaces

$$A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA) \tag{1}$$

where CX is the cone over X. There are homotopy equivalences

$$\begin{aligned} &X \cup CA \simeq X/A, \\ &(X \cup CA) \cup CX \simeq SA, \\ &((X \cup CA) \cup CX) \cup C(X \cup CA) \simeq SX. \end{aligned}$$

By repeating the above sequence with A, X replaced by SA, SX, and so on, we get a sequence of mappings

$$A \to X \to X \cup CA \to SA \to SX \to S(X \cup CA) \to S^2 A \to \cdots$$

It is also homotopy equivalent to the sequence of reduced suspensions

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \to \cdots$$

For an arbitrary space K, this gives a sequence on the homotopy classes of mappings

$$\langle A, K\rangle \leftarrow \langle X, K\rangle \leftarrow \langle X/A, K\rangle \leftarrow \langle \Sigma A, K\rangle \leftarrow \langle \Sigma X, K\rangle \leftarrow \cdots$$

It is interesting to note that in (1) each two consecutive maps are obtained from the previous ones in the same fashion. So to check that the above sequence is exact, it suffices to show that

$$\langle X/A, K\rangle = \langle X \cup CA, K\rangle \to \langle X, K\rangle \to \langle A, K\rangle$$

is exact. This is equivalent to say that, a map $f : X \to K$ whose restriction to A is null-homotopic extends to $X \cup CA \to K$, which is clear.

Now take $K = K(G, n)$ and use the adjoint relation $\langle \Sigma X, K\rangle = \langle X, \Omega K\rangle$, we get the desired LES for the pair (X, A). **QED.**

For later application we prove two very useful lemmas.

Lemma 3 (Postnikov Tower) *For any CW complex X, there is a sequence of fibrations*

$$Y_n \to Y_{n-1} \to \cdots \to Y_2 \to Y_1 = K(\pi_1(X), 1)$$

with the fiber of $Y_q \to Y_{q-1}$ being $K(\pi_q(X), q)$; and commuting maps $f_q : X \to Y_q$ such that the induced map $(f_q)_*$ on π_i are isomorphisms for $i \leq q$.

Proof: We first construct Y_n by attaching cells of dimension$\geq n+2$ to kill $\pi_i(X)$, for $i \geq n+1$, as in our construction of $K(G, n)$. Then by attaching cells of dimension $\geq n+1$ to Y_n we can kill $\pi_n(Y_n)$ too, and call the resulting space $\widetilde{Y}_{n-1}$, and so on. So we have a sequence of inclusion

$$Y_n \subset \widetilde{Y}_{n-1} \subset \cdots \subset \widetilde{Y}_1 = K(\pi_1(X), 1)$$

This is not quite a sequence of fibration. But up to homotopy we can regard the inclusion $\widetilde{Y}_q \hookrightarrow \widetilde{Y}_{q-1}$ as a fibration of the space of paths in $\widetilde{Y}_{q-1}$ that starts in $\widetilde{Y}_q$, since the latter deformation retracts onto $\widetilde{Y}_q$. So the above sequence of inclusion is up to homotopy a sequence of fibration

$$Y_n \to Y_{n-1} \to \cdots \to Y_1 = K(\pi_1(X), 1)$$

From the LES of the fiber bundle $p : Y_q \to Y_{q-1}$ and the fact that p induces isomorphisms on π_i for $i < q$, we see that the fiber is $K(\pi_q(X), q)$. **QED.**

Lemma 4 (Whitehead Tower) *Let X be a CW complex. There is a sequence of fibration*

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

where the fiber of $X_n \to X_{n-1}$ is $K(\pi_n(X), n-1)$, in particular X_1 is the universal cover of X. They satisfy $\pi_i(X_n) = 0$ for all $i \leq n$, and the map $X_n \to X$ induces isomorphisms on π_i for $i > n$.

Proof: This time we go backwards. Suppose we have X_{n-1} with $\pi_i(X_{n-1}) = 0$ for $i \leq n-1$, let's attach cells of dimension$\geq n+2$ to kill $\pi_i(X_{n-1})$ for $i \geq n+1$. The resulting space is $K(\pi_n(X), n) \supset X_{n-1}$. Let X_n be the space of paths in $K(\pi_n(X), n)$ that start from a basepoint and end in X_{n-1}. This is a fiber bundle over X_{n-1}, with fiber homeomorphic to the loop space $\Omega K(\pi_n(X), n) \simeq K(\pi_n(X), n-1)$. So we get a fibration

$$K(\pi_n(X), n-1) \to X_n \to X_{n-1}$$

It follows from the LES that $\pi_i(X_n) = \pi_i(X_{n-1})$ for all $i \geq n+1$, and $\pi_i(X_n) = \pi_i(X_{n-1}) = 0$ for all $i \leq n-2$. The rest of the LES looks like

$$0 \to \pi_n(X_n) \to \pi_n(X_{n-1}) \xrightarrow{\partial} \pi_n(X) \to \pi_{n-1}(X_n) \to 0$$

The map $\partial : \pi_n(X) \to \pi_{n-1}(K(\pi_n(X), n-1)) = \pi_n(X)$ is an isomorphism, so $\pi_{n-1}(X_n) = \pi_n(X_n) = 0$. This shows that the X_n as constructed satisfy the desired properties. **QED.**

We now prove the basic fact that the cohomology groups of Eilenberg-MacLane spaces classify all cohomology operations. Let us recall that a cohomology operation is a natural transformation $H^m(X; G) \to H^n(X; H)$ defined for all spaces X and fixed G, H, m, n.

Theorem 3 *For fixed G, H, m, n, there is a bijection between all cohomology operations $\Theta : H^m(X;G) \to H^n(X;H)$ and $H^n(K(G,m);H)$, given by $\Theta \mapsto \Theta(\alpha)$, where $\alpha = T(\mathbf{1}) \in H^m(K(G,m);G)$ as in theorem 2.*

Proof: First, by CW approximation we can assume X is a CW complex. Given $\beta \in H^m(X;G)$, it corresponds to a map $\varphi : X \to K(G,m)$ with $\varphi^*\alpha = \beta$. So

$$\Theta(\beta) = \Theta(\varphi^*\alpha) = \varphi^*\Theta(\alpha)$$

This shows that $\Theta \to \Theta(\alpha)$ is injective. Now take any $\gamma \in H^n(K(G,m);H)$. It corresponds to a map $\theta : K(G,m) \to K(H,n)$, with $\theta^*\widetilde{\alpha} = \gamma$. θ induces the map

$$\Theta : \langle X, K(G,m)\rangle = H^m(X;G) \longrightarrow \langle X, K(H,n)\rangle = H^n(X;H)$$

Then

$$\Theta(\alpha) = \theta^*(\widetilde{\alpha}) = \gamma$$

This proves the surjectivity. The naturality of Θ is clear. **QED.**

Application 1: By Hurewitz theorem $H^i(K(G,n);H) = 0$ for all $i < n$, and $H^n(K(G,n);H) = \mathrm{Hom}(G,H)$. It follows that cohomology operations do not decrease in the dimension of the cohomology group, and the only operations that preserve the dimensions are induced by homomorphisms between the coefficient groups. A first nontrivial example is the Steenrod square operation

$$Sq^i : H^n(X;\mathbf{Z}/2) \longrightarrow H^{n+i}(X;\mathbf{Z}/2)$$

From theorem 3, we get nonzero cohomology classes of $K(\mathbf{Z}/2,n)$ by repeatedly acting Sq^i on the fundamental class $\alpha \in H^n(K(\mathbf{Z}/2,n);\mathbf{Z}/2)$. In fact, the whole cohomology ring $H^*(K(\mathbf{Z}/2,n);\mathbf{Z}/2)$ can be obtained in this way. This means that one cannot essentially make new mod 2 cohomology operations other than the Steenrod squares. We will not attempt to prove this important fact.

Application 2: Expressed as the bases of fibrations of S^∞, it is easy to see that $\mathbf{RP}^\infty$ is $K(\mathbf{Z}/2,1)$, $\mathbf{CP}^\infty$ is $K(\mathbf{Z},2)$. For CW complex X it follows that there are isomorphisms

$$\begin{aligned} w_1 : \mathrm{Vect}^1_{\mathbf{R}}(X) = \langle X, \mathbf{RP}^\infty\rangle &\longrightarrow H^1(X;\mathbf{Z}/2) \\ c_1 : \mathrm{Vect}^1_{\mathbf{C}}(X) = \langle X, \mathbf{CP}^\infty\rangle &\longrightarrow H^2(X;\mathbf{Z}) \end{aligned}$$

where $\mathrm{Vect}^1_{\mathbf{R},\mathbf{C}}(X)$ are the collections of homotopy classes of real and complex line bundles over X. w_1 is known as the first Stiefel-Whitney class, and c_1 is the first Chern class. We see that they completely classify real and complex line bundles over CW complexes.

4. Cohomlogy of Eilenberg-MacLane Spaces

We first compute the rational cohomology rings $H^*(K(\mathbf{Z}_p, n), \mathbf{Q})$ and $H^*(K(\mathbf{Z}, n), \mathbf{Q})$. To do this we need to use the technique of spectral sequence.

Theorem 4 (Leray) *For a fibration $F \to X \to B$ with B simply connected, there is a spectral sequence $\{E_r^{p,q}, d_r\}$ that converges to $H^*(X; G)$ with*

$$E_2^{p,q} = H^p(B; H^q(F; G)).$$

Namely we have coboundary maps $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, and $E_{r+1} = H(E_r, d_r)$. There is a filtration

$$H^n(X; G) = F_0^n \supset F_1^n \supset \cdots \supset F_n^n \supset 0$$

such that each quotient $F_p^n / F_{p+1}^n \cong E_\infty^{p,n-p}$, where $E_\infty^{p,n-p}$ are the stable groups of the sequence.

Let's start with $K(\mathbf{Z}_p, 1)$. It follows from the LES of homotopy groups for the fiber bundle $\mathbf{Z}_p \to S^\infty \to L(\infty, p)$ that the infinite dimensional lens space $L(\infty, p)$ is $K(\mathbf{Z}_p, 1)$. It has nonzero cohomlogy groups $H^0 = \mathbf{Z}$ and $H^{2i} = \mathbf{Z}_p$ for $i \geq 1$. So in particular $H^*(K(\mathbf{Z}_p, 1); \mathbf{Q})$ is $\mathbf{Q}$ at H^0 and zero in other dimensions. We are going to prove inductively that this is true for $H^*(K(\mathbf{Z}_p, n); \mathbf{Q})$ with arbitrary n. Let us abbreviate $K_n = K(\mathbf{Z}_p, n)$.

Suppose $H^*(K_{n-1}, \mathbf{Q})$ is $\mathbf{Q}$ at dimension 0 and vanishes otherwise. Since K_{n-1} is homotopic to the loop space of K_n, we get a fiber bundle

$$K_{n-1} \to PK_n \to K_n$$

where PX is the path space of X, always contractible. Then

$$E_2^{p,q} = H^p(K_n; H^q(K_{n-1})) = \begin{cases} H^p(K_n; \mathbf{Q}), & q = 0, \\ 0, & q > 0. \end{cases}$$

We see that $E_2^{p,q}$ are already stable. Since PK_n is contractible, $E_\infty^{p,q}$ is $\mathbf{Q}$ at $E_\infty^{0,0}$ and vanishes for all other p, q's. It follows that

$$H^*(K(\mathbf{Z}_p, n); \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{at dimension } 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now let's turn to $K(\mathbf{Z}, n)$. We know that $K(\mathbf{Z}, 1)$ is S^1 and $K(\mathbf{Z}, 2)$ is $\mathbf{CP}^\infty$. Their cohomology groups are free algebras on one generator of dimension n (polynomial algebra for n even and exterior algebra for n odd). We will prove inductively that this is true for all n. Again we abbreviate $K(\mathbf{Z}, n)$ by K_n.

Suppose $H^*(K_{n-1}, \mathbf{Q})$ is the free algebra on one generator of dimension $n-1$. If n is even, then the only non-vanishing cohomology group of K_{n-1} are $H^0, H^{n-1} \cong \mathbf{Q}$. Let a be the generator for H^{n-1}, then $H^*(K_{n-1}; \mathbf{Q}) = \mathbf{Q}[a]/a^2$. So from the fibration $K_{n-1} \to PK_n \to K_n$ we get the spectral sequence $\{E_r^{p,q}, d_r\}$ with

$$E_2^{p,q} = H^p(K_n; H^q(K_{n-1})) = \begin{cases} H^p(K_n; \mathbf{Q}), & q = 0, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

We immediately see that $E_2^{p,q} = \cdots = E_n^{p,q}$, and $E_{n+1}^{p,q} = E_\infty^{p,q}$. Since PK_n is contractible, $E_\infty^{p,q}$ is $\mathbf{Q}$ at $E_\infty^{0,0}$ and vanishes for other p, q's. In particular, it follows that $E_n^{i,0} = 0$ for $0 < i < n$, which we already know from Hurewitz theorem. Further, $d_n : E_n^{0,n-1} \to E_n^{n,0}$ is nonzero, and $d_n(a)$ generates $E_n^{n,0} = H^n(K_n; \mathbf{Q}) \cong \mathbf{Q}$. Now at the n-th column we have $E_n^{n,n-1}$ generated by $d_n(a) \otimes a$. It is mapped by d_n to $d_n(a)^2$, which generates $E_n^{2n,0} = H^{2n}(K_n; \mathbf{Q})$, and so on. So we conclude that $H^*(K_n; \mathbf{Q}) = \mathbf{Q}[d_n(a)]$ for n even, where $d_n(a)$ is a free generator.

If n is odd, by assumption $H^*(K_{n-1}; \mathbf{Q}) = \mathbf{Q}[a]$ where a is an $(n-1)$-dimensional generator. Compared with the above discussion for n even, now $d_n(a)$ still generates $E_n^{n,0}$, but it follows that the map $d_n : E_n^{0,(i+1)(n-1)} \to E_n^{n,i(n-1)}$ is nonzero, in fact, an isomorphism for all i. So $d_n : E_n^{n,(i+1)(n-1)} \to E_n^{2n,i(n-1)}$ is zero, which further implies that $E_{n+1}^{p,q}$'s already stabilize, and $E_n^{kn,0} = 0$ for $k \geq 2$. We conclude that $H^*(K_n; \mathbf{Q}) = \mathbf{Q}[\epsilon]/\epsilon^2$ where $\epsilon = d_n(a)$ is an n-dimensional generator.

To summarize, we found that

$$H^*(K(\mathbf{Z}, n); \mathbf{Q}) = \begin{cases} \mathbf{Q}[a], & a \in H^n, \quad n \text{ even}, \\ \mathbf{Q}[a]/a^2, & a \in H^n, \quad n \text{ odd}. \end{cases}$$

As a beautiful application of the cohomology of $K(G,n)$'s, we will prove

Theorem 5

$$\pi_4(S^3) \cong \pi_5(S^3) \cong \mathbf{Z}/2.$$

To do this, we'll make use of the Whitehead tower for S^3:

$$\begin{array}{ccc} K(\pi_4(S^3),3) & \to & X_4 \\ & & \downarrow \\ K(\mathbf{Z},2) & \to & X_3 \\ & & \downarrow \\ & & S^3 \end{array}$$

We have $H_4(X_3) \cong \pi_4(X_3) \cong \pi_4(S^3)$ and $H_5(X_4) \cong \pi_5(X_4) \cong \pi_5(S^3)$ by the construction of Whitehead tower. To compute the cohomology of X_3, we consider the spectral sequence for the fibration $\mathbf{CP}^\infty \simeq K(\mathbf{Z},2) \to X_3 \to S^3$. The table for $E_2^{p,q} = E_3^{p,q}$ looks like

4	$\mathbf{Z}a^2$	0	0	$\mathbf{Z}a^2x$	0
3	0	0	0	0	0
2	$\mathbf{Z}a$	0	0	$\mathbf{Z}ax$	0
1	0	0	0	0	0
0	$\mathbf{Z}$	0	0	$\mathbf{Z}x$	0
	0	1	2	3	4

Since $H^3(X_3) = 0$, $d_3(a) = x$ is the generator for $E_3^{3,0}$. So $d_3 : a^2 \mapsto 2ax, a^3 \mapsto 3a^2x$. We see that the non-vanishing cohomology groups of X_3 are

$$H^0(X_3) = \mathbf{Z}, \quad H^{2k+1}(X_3) = \mathbf{Z}/k \quad \text{for all } k > 1.$$

In particular we learned that $H_4(X_3) = \mathbf{Z}/2$, $H_6(X_3) = \mathbf{Z}/3$. It follows that $\pi_4(S^3) = \mathbf{Z}/2$.

Next we look at the homology spectral sequence for $K(\pi_4(S^3),3) \to X_4 \to X_3$. To compute $H_5(X_4)$ we first need to know the (co)homology of $K(\pi_4(S^3),3) = K(\mathbf{Z}/2,3)$. For the fibration $\mathbf{RP}^\infty \simeq K(\mathbf{Z}/2,1) \to PK(\mathbf{Z}/2,2) \to K(\mathbf{Z}/2,2)$, it is not hard to determine the first few terms of $E_2^{p,q} = E_3^{p,q}$ to be

4	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$		
3	0	0	0	0	0		
2	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$		
1	0	0	0	0	0	0	0
0	$\mathbf{Z}$	0	0	$\mathbf{Z}/2$	0	$\mathbf{Z}/4$	$\mathbf{Z}/2$
	0	1	2	3	4	5	6

and read off the cohomology groups of $K(\mathbf{Z}/2, 2)$:

$$H^q(K(\mathbf{Z}/2, 2); \mathbf{Z}) = \mathbf{Z},\ 0,\ 0,\ \mathbf{Z}/2,\ 0,\ \mathbf{Z}/4,\ \mathbf{Z}/2 \qquad \text{for } q = 0, \cdots, 6.$$

Now one step further, for the fibration $K(\mathbf{Z}/2, 2) \to PK(\mathbf{Z}/2, 3) \to K(\mathbf{Z}/2, 3)$, the table for $E_2^{p,q}$ is

5	$\mathbf{Z}/4$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	
4	0	0	0	0	0	0	
3	$\mathbf{Z}/2$	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	
2	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$
	0	1	2	3	4	5	6

from which we read off the cohomology of $K(\mathbf{Z}/2, 3)$:

$$H^q(K(\mathbf{Z}/2, 3); \mathbf{Z}) = \mathbf{Z},\ 0,\ 0,\ 0,\ \mathbf{Z}/2,\ 0,\ \mathbf{Z}/2 \qquad \text{for } q = 0, \cdots, 6.$$

In particular we learned that $H_4(K(\mathbf{Z}/2, 3)) = 0$ and $H_5(K(\mathbf{Z}/2, 3)) = \mathbf{Z}/2$. This enables us to deduce the table for $E^2_{p,q}$ of the homology spectral sequence of $K(\mathbf{Z}/2, 3) \to X_4 \to X_3$:

5	$\mathbf{Z}/2$	0	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0
4	0	0	0	0	0	0	0
3	$\mathbf{Z}/2$	0	0	0	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0
2	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}/2$	0	$\mathbf{Z}/3$
	0	1	2	3	4	5	6

Since $H_3(X_4) = 0$, $d^4 : \mathbf{Z}/2 \to \mathbf{Z}/2$ is an isomorphism, which verifies $\pi_4(S^3) = \mathbf{Z}/2$. On the other hand, $d^6 : \mathbf{Z}/3 \to \mathbf{Z}/2$ must be zero, therefore $\pi_5(S^3) \cong H_5(X_4) = \mathbf{Z}/2$. This finishes the calculation.

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