Notes on D-branes and Dualities

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This set of notes is meant to give a self-contained description of D-branes and some elementary applications to string dualities. Most part of the notes is following Polchinski's TASI LECTURES ON D-BRANES, with more details filled in.

T-duality in bosonic string theory

First consider the mode expansion of closed bosonic strings

$$X^\mu(z,\bar{z}) = X_L^\mu(z) + X_R^\mu(\bar{z}) = x^\mu - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\sigma^2 + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma^1 + (osc.)$$

The periodicity in σ^1 requires

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$$

Compactified on a circle, say $X^{25} \cong X^{25} + 2\pi R$, we have

$$p_L^{25} = \sqrt{\tfrac{2}{\alpha'}}\alpha_0^{25} = \left(\tfrac{n}{R} + \tfrac{wR}{\alpha'}\right)$$
$$p_R^{25} = \sqrt{\tfrac{2}{\alpha'}}\tilde{\alpha}_0^{25} = \left(\tfrac{n}{R} - \tfrac{wR}{\alpha'}\right)$$

The mass spectrum is

$$m^2 = -p_\mu p^\mu = \frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

together with level matching condition $L_0 = \tilde{L}_0$:

$$N - \tilde{N} + nw = 0$$

T-duality maps

$$X_L^{25} \to X_L^{25}, \quad X_R^{25} \to -X_R^{25}$$

Correspondingly

$$p_L^{25} \to p_L^{25}, \quad p_R^{25} \to -p_R^{25}$$

The mass spectrum remains invariant, while the momentum along the circle and the winding number interchanges,

$$R \to R' = \frac{\alpha'}{R}, \quad n \leftrightarrow w$$

At the self-dual radius $R = \sqrt{\alpha'}$, there is enhanced $SU(2)_L \times SU(2)_R$ gauge symmetry with currents

$$\begin{array}{ll} SU(2)_L: & \partial X^{25}(z), \;\; \exp(\pm 2iX_L^{25}(z)/\sqrt{\alpha'}) \\ SU(2)_R: & \bar{\partial} X^{25}(\bar{z}), \;\; \exp(\pm 2iX_R^{25}(\bar{z})/\sqrt{\alpha'}) \end{array}$$

The marginal operator $\partial X^{25}\bar{\partial}X^{25}$ corresponds to the change of radius, transforms as $(\mathbf{3},\mathbf{3})$ of the $SU(2)_L \times SU(2)_R$. T-duality corresponds to a rotation by π in $SU(2)_R$, which maps $\partial X^{25}\bar{\partial}X^{25}$ to minus itself.

The effective 25-dimensional coupling is $e^{\phi}R^{-1/2}$, as $e^{-2\phi}$ appears in the low energy effective action. T-duality requires

$$e^{\phi}R^{-1/2} = e^{\phi'}R'^{-1/2} \Rightarrow e^{\phi'} = e^{\phi}R^{-1}\alpha'^{1/2}$$

To compactify on d-torus T^d of periodicity $2\pi R$ in all compact directions, $G_{mn} = e_m^r e_n^r$, the left- and right-moving momenta are

$$k_{rL,R} = e_r^m \left(\frac{n_m}{R} - \frac{B_{mn}w^n R}{\alpha'} \pm \frac{w^m R}{\alpha'} \right)$$

The mass-shell condition is

$$\begin{array}{rcl} m^2 & = & \frac{1}{2}(k_{rL}k_{rL} + k_{rR}k_{rR}) + \frac{2}{\alpha'}(N + \tilde{N} - 2) \\ 0 & = & \alpha'(k_{rL}k_{rL} - k_{rR}k_{rR}) + 4(N - \tilde{N}) \end{array}$$

The vertex operators in the compact directions are

$$: \exp(ik_L \cdot X_L(z) + ik_R \cdot X_R(\bar{z})) :$$

where (k_{rL}, k_{rR}) lies in the Narain lattice Γ, namely

$$\frac{\alpha'}{2}(k_L \cdot k_L - k_R \cdot k_R) \in 2\mathbf{Z}$$

Γ is a self-dual even Lorentzian lattice of signature (d,d). The momentum in the compact direction (k_{rL}, k_{rR}) are the charges of the corresponding vertex operators under gauge fields

$$\partial X_L^r(z), \quad \bar{\partial} X_R^r(\bar{z})$$

The T-duality group is $O(d,d,\mathbf{Z})$. The space of inequivalent theories compactified on T^d is given by

$$(O(d) \times O(d)) \backslash O(d,d) / O(d,d,\mathbf{Z})$$

Now we turn to open strings. Under T-duality $X_L^{25} \to X_L^{25}, X_R^{25} \to -X_R^{25}$, the Neumann boundary condition becomes Dirichlet boundary condition,

$$\partial_n X^{25} = -i\partial_t X'^{25}$$

Moreover,

$$X'^{25}(\pi) - X'^{25}(0) = 2\pi\alpha' p^{25} = 2\pi n R'$$

where $p^{25} = n/R$. The endpoints are constrained to lie on the same hyperplane. Including $U(N)$ Chan-Paton factor, turn on Wilson line $A_{25} = \text{diag}\{\theta_1, \cdots, \theta_N\}/2\pi R$. Going around the circle, the state $|ij\rangle$ picks up phase $\exp i(\theta_j - \theta_i)$, therefore the momentum is $p^{25} = (2\pi n + \theta_j - \theta_i)/2\pi R$. The same analysis as above gives

$$X'^{25}(\pi) - X'^{25}(0) = (2\pi n + \theta_j - \theta_i)R'$$

So in the T-dual picture open string with Chan-Paton factor $|ij\rangle$ is stretched between hyperplanes with $X'^{25} = \theta_i R'$ and $X'^{25} = \theta_j R'$ respectively. The gauge symmetry is enhanced when the hyperplanes coincide. The mass is

$$\begin{aligned} m^2 &= (p^{25})^2 + \frac{1}{\alpha'}(N-1) \\ &= \left(\frac{2\pi n + \theta_i - \theta_j}{2\pi\alpha'} R'\right)^2 + \frac{1}{\alpha'}(N-1) \end{aligned}$$

Massless states have both ends lying on the same hyperplane, with vertex operators

$$\partial_t X^\mu, \quad \partial_t X^{25} = \partial_n X'^{25}$$

The latter corresponds to fluctuations of the branes. If we turn on general background gauge fields instead of a constant Wilson line, the shape of the branes are changed. Therefore we learned that D-branes are dynamical objects of the theory.

The generalization to several compact directions is straightforward, if we take $X^{p+1}, \cdots, X^{25}$ to be periodic, in the T-dual picture we have Dp-branes. A simple but important observation is, a Dp-brane becomes a D$(p-1)$-brane under T-duality in the tangent direction, or a D$(p+1)$-brane under T-duality in the orthogonal direction.

D-branes in bosonic string theory

Let $\xi^a,\ a = 0, \cdots, p$ be coordinates on the brane. The world-brane theory contains a $U(1)$ gauge field $A_a(\xi)$ and embedding $X^\mu(\xi)$. The action is

$$S_p = -T_p \int d^{p+1}\xi e^{-\phi} \det{}^{1/2}(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}) \tag{1}$$

where G_{ab} and B_{ab} are the pullbacks of spacetime fields onto the brane. Now we argue that this is the consistent Dp-brane action. The G_{ab} and dilaton are easy to understand. Consider a D-brane extended in X^1, X^2 directions. Suppose we turn on a constant $U(1)$ gauge field F_{12}, and choose the gauge such that $A_2 = X^1 F_{12}$. T-dual in the 2-direction,

$$X'^2 = \theta R' = 2\pi\alpha' X^1 F_{12}$$

This modifies the action from the world-volume to

$$\int dX^1 \sqrt{1+(\partial_1 X'^2)^2} = \int dX^1 \det{}^{1/2}(1+(2\pi\alpha' F_{12})^2)$$

which is a special case of (1). The open string world-sheet action containing B field and gauge field A is

$$\frac{1}{2\pi\alpha'}\int_M B + \int_{\partial M} A$$

To preserve the invariance under gauge transformation in B, A must transform accordingly such that $B + 2\pi\alpha' F$ is left gauge invariant. This combination should appear in the D-brane action. So we determined the full action (1).

The mass of a Dp-brane wrapped around T^d is

$$T_p e^{-\phi} \prod_{i=1}^{p} (2\pi R_i)$$

Take the T-dual in X^p direction, $e^{-\phi} = e^{-\phi'} R_p^{-1} \alpha'^{1/2}$, we have a D$(p-1)$-brane wrapping around T^{p-1} with tenson T_{p-1}. The mass is unchanged, therefore

$$T_p = T_{p-1}/(2\pi\sqrt{\alpha'})$$

In the presence of N D-branes, the gauge fields A_a and embedding X^μ become $n \times n$ matrices. A_a is understood as $U(N)$ gauge fields, while X^μ as matrices is of noncommutative geometry nature. The leading term in the scalar potential would be of the form $\mathrm{Tr}([X_a, X_b][X^a, X^b])$. The flat direction is when $[X_a, X_b] = 0$, X_a's can be diagonized simultaneously, giving rise to N independent D-branes. The action is of the form

$$S_p = -T_p \int d^{p+1}\xi e^{-\phi} \mathrm{Tr}\left\{\det{}^{1/2}(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}) + O([X_a, X_b]^2)\right\}$$

Now we compute the tension of D-branes. The idea is to consider the closed string exchange amplitude between two parallel separated Dp-branes. The amplitude can be thought as the 1-loop partition function of open strings stretched between the D-branes, obtained from integration over the moduli space; or as the exchange amplitude of graviton and dilaton between the D-branes, obtained from a field theory calculation using the string and brane effective action.

The world-sheet Hamiltonian for open strings strength between two D-branes separated by Y^μ is

$$L_0 = \alpha' k^2 + \frac{1}{4\pi^2\alpha'} Y^2 + \sum_{n=1}^{\infty} n\alpha_{-n}^i \alpha_n^i - 1$$

The amplitude is the same as cylinder partition function

$$\mathcal{A} = \int_0^\infty \frac{dt}{2t} \mathrm{Tr} e^{-2\pi t L_0}$$

$$
\begin{aligned}
&= iV_{p+1}\int \frac{d^{p+1}k}{(2\pi)^{p+1}}\int_0^\infty \frac{dt}{2t}\exp\left[-2\pi\alpha' t(k^2+Y^2/4\pi^2\alpha'^2)\right]\cdot\sum_{i\in\mathcal{H}_0^\perp}\exp[-2\pi t(h_i-1)] \\
&= iV_{p+1}\int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-(p+1)/2}\exp(-tY^2/2\pi\alpha')\eta(it)^{-24}
\end{aligned}
$$

Using the modular property

$$\eta(it) = t^{-1/2}\eta(i/t)$$

In the limit $t \to 0$ we can expand

$$
\begin{aligned}
\eta(i/t)^{-24} &= \exp(2\pi/t)\prod_{n=1}^{\infty}[1-\exp(-2\pi n/t)]^{-24} \\
&= \exp(2\pi/t) + 24 + O(\exp(-2\pi/t))
\end{aligned}
$$

The amplitude can be written as

$$\mathcal{A} = \frac{iV_{p+1}}{(8\pi^2\alpha')^{(p+1)/2}}\int_0^\infty dt\, t^{(21-p)/2}\exp(-tY^2/2\pi\alpha')\cdot[\exp(2\pi/t)+24+\cdots]$$

The first term comes from tachyon exchange, while the second term comes from exchanges of massless modes. The contribution from massless modes is computed as

$$\mathcal{A} \sim iV_{p+1}\frac{24}{2^{12}}(4\pi^2\alpha')^{11-p}\,\pi^{(p-23)/2}\,\Gamma(\frac{23-p}{2})\,|Y|^{p-23}$$

Go to momentum space (orthogonal directions),

$$\mathcal{A}(k) = \int d^{25-p}y\, e^{ik\cdot y}\mathcal{A}(y) \sim V_{p+1}\frac{24\pi}{2^{10}}(4\pi^2\alpha')^{11-p}\frac{i}{k^2}$$

Now we turn to field theory computation. The relevant spacetime effective action in the Einstein frame is

$$S = \frac{1}{2\kappa^2}\int d^{26}X(-\tilde{G})^{1/2}\left(\tilde{R} - \frac{1}{6}\tilde{\nabla}_\mu\tilde{\Phi}\tilde{\nabla}^\mu\tilde{\Phi}\right)$$

where $\kappa = \kappa_0 e^{\Phi_0}$ is gravitational coupling constant, the dilaton $\tilde{\Phi} = \Phi - \Phi_0$ has zero vacuum expectation value, $\tilde{G}_{ab} = \exp(-\tilde{\Phi}/6)G_{ab}$. In terms of these variables, the relevant term in D-brane action is

$$S_p = -\tau_p\int d^{p+1}\xi\exp\left(\frac{p-11}{12}\tilde{\Phi}\right)(-\det\tilde{G}_{ab})^{-1/2}$$

where $\tau_p = T_p e^{-\Phi_0}$. To compare with the previous result from stringy calculation, we need to consider tree level amplitude of exchanging a single graviton or dilaton between the D-branes. Let $\tilde{G}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and define

$$F_\nu = \eta^{\mu\rho}\left(\partial_\mu h_{\rho\nu} - \frac{1}{2}\partial_\nu h_{\mu\rho}\right)$$

Add the gauge fixing term $-F_\nu F^{\hat{\nu}}/4\kappa^2$, where the hat means we raise the index using the flat metric η instead of g. Expand the spacetime action to second order in $h_{\mu\nu}$ and $\tilde{\Phi}$, we get

$$S = -\frac{1}{8\kappa^2}\int d^{26}X \left(\partial_\mu h_{\nu\lambda}\partial^{\hat{\mu}} h^{\hat{\nu}\hat{\lambda}} - \frac{1}{2}\partial_\mu h^{\hat{\nu}}_{\nu}\partial^{\hat{\mu}} h^{\hat{\lambda}}_{\lambda} + \frac{2}{3}\partial_\mu \tilde{\Phi}\partial^{\hat{\mu}}\tilde{\Phi}\right)$$

One immediately obtains the propagators

$$\begin{aligned}
\langle \tilde{\Phi}\tilde{\Phi}\rangle &= -\frac{6i\kappa^2}{k^2} \\
\langle h_{\mu\nu} h_{\sigma\rho}\rangle &= -\frac{2i\kappa^2}{k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{12}\eta_{\mu\nu}\eta_{\sigma\rho}\right)
\end{aligned}$$

The D-brane action is expanded to the first order as

$$S_p = -\tau_p \int d^{p+1}\xi \left(\frac{p-11}{12}\tilde{\Phi} - \frac{1}{2}h_{aa}\right)$$

We've taken the world-brane metric to be δ_{ab} and the trace of $h_{\mu\nu}$ here is taken only over the tangent directions. The amplitude is simply

$$\mathcal{A} = \frac{i\kappa^2\tau_p^2}{k^2}V_{p+1}\left\{6\left(\frac{p-11}{12}\right)^2 + \frac{1}{2}\left[2(p+1) - \frac{1}{12}(p+1)^2\right]\right\} = \frac{6i\kappa^2\tau_p^2}{k^2}V_{p+1}$$

where V_{p+1} comes from the overall delta function of the diagram, and consequently k^2 contains only the orthogonal components. Compare with our previously result,

$$\tau_p^2 = \frac{\pi}{256\kappa^2}(4\pi^2\alpha')^{11-p}$$

As noted before, including $U(N)$ Chan-Paton factor will give an additional factor N^2 for the stringy calculation; the field theory calculation also has a factor N^2 because on each side there are N coincident D-branes. Also note that the D-brane mass (density), which is proportional to the tension, becomes small when strings are strongly coupled.

Now we turn to study the T-dual picture of unoriented bosonic string theories. First consider the closed strings. The world sheet parity acts as

$$\Omega: \quad X^M_L(z) \leftrightarrow X^M_R(\bar{z})$$

In the T-dual picture

$$\begin{aligned}
\Omega: \quad X'^\mu(z,\bar{z}) &\leftrightarrow X'^\mu(\bar{z},z) \\
X'^m(z,\bar{z}) &\leftrightarrow -X'^m(\bar{z},z)
\end{aligned}$$

We impose $\Omega = +1$ to obtain the unoriented theory. In the T-dual picture, if we separate the string wavefunction into its dependence on the center-of-mass and the internal part, and assume the latter is an eigenstate of Ω, then the string wavefunction at $-x^m$ is determined by the wavefunction at x^m up to a sign. The T-dual spacetime is T^{25-p} modulo reflection $\mathbf{Z}_2$, with 2^{25-p} fixed planes (orientifold plane). Unlike D-branes, these orientifold planes are rigid objects, there are no string modes representing the fluctuation of the planes.

The situation is slightly more interesting in the open string case. In general the world-sheet parity acts on Chan-Paton factors as

$$\Omega : \quad \lambda_{ij}|k, ij\rangle \to \lambda'_{ij}|k, ij\rangle, \quad \lambda' = M\lambda^T M^{-1}$$

$\Omega^2 = 1$ acting on the fields, therefore we demand $\lambda = (M^T M^{-1})^T \lambda (M^T M^{-1})$. In general the set of λ matrices is irreducible, because a splitting-joining interaction leads to the process $ij + kl \to ik + jl$, can always mix components in different diagonal blocks. By Schur's Lemma $M = \beta M^T$, $\beta = \pm 1$. Therefore λ must lie in the Lie algebra $SO(N)$ or $USp(N)$, giving the corresponding gauge groups.

Consider $SO(N)$ Chan-Paton factors and the presence of a single compact dimension. A general constant Wilson line can be brought to

$$W = \mathrm{diag}\left\{e^{i\theta_1}, e^{-i\theta_1}, \cdots, e^{i\theta_{N/2}}, e^{-i\theta_{N/2}}\right\}$$

In the dual picture there are $N/2$ D25-branes on $0 \le X'^{25} \le \pi R'$, and $N/2$ at their images under orientifold reflection.

The orientifold plane also couples to the graviton and dilaton like D-branes, but with the cylinder replaced by Moebius strip and the Klein bottle. We can find the tension of the orientifold plane by computing the amplitude of exchanging a closed string with a parallel D-brane as follows. The field theory calculation is similar as before, except in the amplitude we replace a τ_p by the orientifold plane tension τ'_p, and in order to compare with the stringy calculation we

$$\mathcal{A} = \frac{6i\kappa^2 \tau_p \tau'_p}{k^2} V_{p+1}$$

To compute the unoriented string amplitude, because of the projection operator $\frac{1}{2}(1+\Omega)$, all amplitudes differ by a factor 1/2 from the oriented case. The Moebius strip amplitude is given by

$$\mathcal{A}_M = iV_{p+1} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \int_o^\infty \frac{dt}{2t} \sum_{i \in \mathcal{H}_0^\perp} \frac{\Omega_i}{2} \exp[-2\pi\alpha' t(p^2 + m_i^2)]$$

For $SO(N)$ open string the Chan-Paton factors have $\frac{1}{2}N(N+1)$ even states under parity Ω, and $\frac{1}{2}N(N-1)$ odd states. The net contribution has a factor N. For $USp(N)$ these numbers are reversed and the net contribution is $-N$ instead. For simplicity consider one D-brane at X^m and its image at $-X^m$ (i.e. gauge group $O(2)$), which gives an additional factor of 2. The open strings in our consideration do not end at the fixed planes but are stretched between the D-brane and its image. Effectively the separation is $Y^m = 2X^m$. The amplitude is

$$\mathcal{A}_M = \pm iNV_{p+1}\int_0^\infty \frac{dt}{2t}(8\pi^2\alpha' t)^{-(p+1)/2}e^{-2X^2t/\pi\alpha'}\cdot e^{2\pi t}\prod_{m=1}^{\infty}\left[1-(-1)^n e^{-2\pi nt}\right]^{-24}$$

where the $\pm$ sign depends on whether the gauge group is $SO(N)$ or $USp(N)$. The last product can be written as $\theta_{00}(0|2it)^{-12}\eta(2it)^{-12}$. We can use the modular property of θ_{00} and η under $2it \to i/2t$, and take the limit $t \to 0$ to separate the contribution from massless modes. The answer is

$$\mp 2^{p-13}V_{p+1}\frac{3\pi}{2^6}(4\pi^2\alpha')^{11-p}\frac{i}{k^2}$$

We should compare this with 2 times the field theory result, because there we didn't count in the image of the D-brane (i.e. the flux can go both directions). The tension of the fixed plane is related to the D-brane tension by

$$\tau'_p = \mp 2^{p-13}\tau_p$$

As we've seen before τ'_p is proportional to the dilaton coupling of orientifold planes. The total 2^{25-p} fixed planes give $\mp 2^{12}\tau_p$. The total coupling should vanish because the volume is finite, the dilaton flux has nowhere to go. This can be achieved if there are 2^{12} D-branes, which requires the gauge group to be $SO(2^{13})$. This seems to be an unrealistic number, but in the type I superstring theory the similar calculation gives a more realistic $SO(32)$. The same result can be reproduced by a purely stringy calculation requiring the tadpole from the disk and projective plane cancel.

T-duality and D-branes in superstring theory

We first review the massless spectrum of 10-dimensional superstring theory. The open string can be identified with one sector (left or right moving) of the closed string theory by the doubling trick, i.e. $\psi^\mu(2\pi-\sigma^1,\sigma^2) \cong \tilde{\psi}^\mu(\sigma^1,\sigma^2)$. In the cylinder coordinate $(\sigma^1,\sigma^2)\in S^1\times\mathbf{R}$, NS sector corresponds to antiperiodic world-sheet fermions and Ramond sector corresponds to periodic fermions. Mapping onto the disk, because ψ^μ and $\tilde{\psi}^\mu$ have conformal weight 1/2, fermion fields in NS sector become well-defined while those in Ramond sector have branch cuts.

The fermions are bosonized as

$$\begin{aligned}(\mp\psi^0+\psi^1)/\sqrt{2} &\cong e^{\pm iH^0}\\ (\psi^{2a}\pm i\psi^{2a+1})/\sqrt{2} &\cong e^{\pm iH^a}, \quad a=1,\cdots,4\end{aligned}$$

where H field has OPE

$$H(z)\cdot H(0)\sim -\ln z$$

By bosonizing superconformal ghosts, the vertex operator of ground states in NS sector is $e^{-\phi}$, while for R sector the ground states are

$$|s_0, \cdots, s_4\rangle \cong \exp\left(-\phi/2 + i\sum s_a H^a\right)$$

where $s_a = \pm 1/2$, $a = 0, \cdots, 4$. Physical states are annihilated by G_0, acting as $p_\mu \psi_0^\mu$ on ground states. Ground states in R sector are massless, in the light cone gauge $p^0 = p^1$, physical states are those of $s_0 = +1/2$. Either modular invariance of 1-loop partition function or the absence of branch cuts in vertex operator OPE demand the GSO projection. The R sector ground states are further projected onto $\mathbf{8}_s$ or $\mathbf{8}_c$ (chiral or antichiral spinor of $SO(8)$).

For open strings, the ground state spectrum coming from NS and R sector as representation of $SO(8)$ is

$$\mathbf{8}_v \oplus \mathbf{8}_s$$

form a vector multiplet of $d = 10, N = 1$ supersymmetry. For type II the massless spectra are

$$\begin{aligned} \text{Type IIA:} \quad & (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_c) \\ \text{Type IIB:} \quad & (\mathbf{8}_v \oplus \mathbf{8}_s) \otimes (\mathbf{8}_v \oplus \mathbf{8}_s) \end{aligned}$$

They form $d = 10, N = 2$ IIA and IIB supergravity multiplets respectively.

Type IIB theory is invariant under world-sheet parity which exchange left and right moving sectors. Projecting onto $\Omega = +1$ one can form an oriented string theory, with massless fields form $d = 10, N = 1$ supergravity multiplet. However this theory alone suffers from 1-loop divergence, in other words, the closed string tadpole doesn't vanish. To cancel the tadpole one can introduce open strings with $SO(32)$ Chan-Paton factors. The resulting theory is type I unoriented string theory. The massless spectrum of type I $SO(32)$ theory is

$$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \oplus \mathbf{8_c} \oplus \mathbf{56_s} \oplus (\mathbf{8}_v \oplus \mathbf{8}_s)_{SO(32)}$$

It is important to look at R-R ground state vertex operators:

$$e^{-\phi/2 - \tilde{\phi}/2} S_\alpha \tilde{S}_\beta$$

where S_α and $\tilde{S}_\beta$ are left and right moving spin fields as before. Decomposed into antisymmetric tensors of $SO(9,1)$, we have

$$\mathcal{V} = e^{-\phi/2 - \tilde{\phi}/2} S_\alpha \tilde{S}_\beta (\Gamma^{\mu_1 \cdots \mu_n} C)_{\alpha\beta} F_{\mu_1 \cdots \mu_n}(X)$$

with C being the charge conjugation matrix, i.e. flipping all the spins s_a. For spinors of $SO(9,1)$,

$$\begin{aligned} \text{IIA:} \quad & \mathbf{16} \otimes \mathbf{16'} = [0] \oplus [2] \oplus [4] \\ \text{IIB:} \quad & \mathbf{16} \otimes \mathbf{16} = [1] \oplus [3] \oplus [5]_+ \end{aligned}$$

In type IIA theory R-R fields are even forms while in IIB they are odd forms, with the 5-form being self-dual. Again we impose physical condition

$$G_0 \cdot \mathcal{V} = \tilde{G}_0 \cdot \mathcal{V} = 0$$

In a flat background, $G_0 \sim p_\mu \psi_0^\mu$ acting on ground states, where p_μ acts as derivative on $F_{\mu_1 \cdots \mu_n}(x)$, ψ_0^μ acts as Γ^μ on the Clifford module. From the identity

$$\Gamma^\nu \Gamma^{\mu_1 \cdots \mu_n} = \Gamma^{\nu \mu_1 \cdots \mu_n} + \eta^{\nu[\mu_1} \Gamma^{\mu_2 \cdots \mu_n]}$$

The physical condition is equivalent to

$$dF = 0, \quad d * F = 0$$

They are Bianchi identity and field equations for F, therefore F should be understood as field strength. Note that in type II theory R-R field strengths and R-R ground states (fundamental string excitation) have different degree as forms. If one computes the amplitude of string states coupled to R-R vertex operators, the result must be contracted with the external momentum. If we set the external momentum to be zero, the amplitude vanishes, which indicates that string states don't carry R-R charge. Indeed we'll later see that R-R charges are carried by D-branes as solitons of string theory.

When there is a nontrivial dilaton background, $G_0 \sim \sqrt{\alpha'/2}\psi_0^\mu (p_\mu + i\partial_\mu \Phi)$. The physical condition on F becomes

$$(d - d\Phi\wedge)F = 0, \quad (d - d\Phi\wedge) * F = 0$$

The correct field strength is $F' = e^{-\Phi}F$, decoupled from the dilaton as one can see from the low energy effective action.

Now we study T-duality of superstrings. For type II theory compactified on a circle in X^9 direction of radius R, we are looking for mapping between theories at $R \to 0$ and $R \to \infty$ limit in analogous to the bosonic string case. The action on bosonic fields is the same

$$X_R^9 \to -X_R^9$$

Superconformal invariance requires

$$\tilde{\psi}^9 \to -\tilde{\psi}^9$$

This operation flips the $\pm$ sign of $\tilde{\psi}_0^8 \pm i\tilde{\psi}_0^9$, hence reverse the chirality of right moving R-sector ground states. Therefore type IIA and IIB theory are interchange under T-duality:

$$\text{Type IIA at radius } R \longleftrightarrow \text{Type IIB at radius } \frac{\alpha'}{R}$$

If one T-dualizes more than one directions, type IIA (IIB) is mapped to IIA (IIB) or IIB (IIA) when the number of T-dualities is even or odd.

Still consider T-duality in X^9 direction. To preserve the OPE with $\tilde{\psi}^\mu$, the action on spin fields must be

$$S_\alpha(z) \to S_\alpha(z), \quad \tilde{S}_\alpha(\bar{z}) \to (P_9)_{\alpha\beta}\tilde{S}_\beta(\bar{z})$$

where $P_9 = \Gamma^9\Gamma$ anticommutes with Lorentz generator $\tilde{M}^{\mu 9}$ and commutes with other $\tilde{M}^{\mu\nu}$'s, it is the X^9-parity transformation on the spinors. Acting on the R-R vertex operators

$$\bar{S}\Gamma^{\mu_1\cdots\mu_n}\tilde{S}$$

T-duality interchanges R-R fields $C_{\mu_1,\cdots,\mu_p}$ with $C_{\mu_1,\cdots,\mu_p,9}$ up to signs. This is consistent with the fact that IIA and IIB theory have odd and even R-R fields. (even and odd R-R field strengths, resp.)

The spin fields of R-R sector $S_\alpha, \tilde{S}_\alpha$ are the world-sheet currents associated with spacetime supersymmetry. The supercharges are

$$Q_\alpha = \oint \frac{dz}{2\pi i} e^{-\phi/2} S_\alpha, \quad \tilde{Q}_\alpha = \oint \frac{d\bar{z}}{2\pi i} e^{-\tilde{\phi}/2} \tilde{S}_\alpha$$

In the open string case, the left and right moving currents flow out of the boundary, only the total supercharge $Q_\alpha + \tilde{Q}_\alpha$ is conserved. The conserved supercharge for the D-brane is obtained by T-dualizing the orthogonal directions,

$$Q_\alpha + \left(\prod P_m\right) \tilde{Q}_\alpha$$

Therefore a D-brane break half spacetime supersymmetry, is a BPS state. They carry R-R charge, as we will compute later. A Dp-brane naturally couples to $(p+1)$-form potential C_{p+1}, or $(p+2)$-form field strength F_{p+2}. For example, the coupling can be obtained by integrating C_{p+1} over the brane; the electric charge is the total flux of the dual field strength through a sphere enclosing the D-brane in the transverse direction. By T-dualizing in different directions, we see that type IIA theory has $p = 0, 2, 4, 6, 8$-branes and type IIB theory has $p = -1, 1, 3, 5, 7, 9$-branes.

Now we determine the D-brane effective action and R-R charge. The convention is following Tasi lectures instead of Polchinski's book. The coupling of a D-brane to NS-NS fields takes the same form as in bosonic string theory. The coupling to R-R field should include

$$S = \frac{1}{2}\int F_{p+2} \wedge *F_{p+2} + i\mu_p \int_{p-brane} C_{p+1}$$

The R-R field strength F is the $F' = e^{-\Phi}F$ before, decoupled from the dilaton. The full coupling to R-R fields takes the form

$$i\mu_p \int_{p-brane} \mathrm{Tr}\left[\exp(2\pi\alpha' F + B) \wedge \sum_q C_{q+1}\right] \tag{2}$$

To justify this, we consider the example of a 1-brane in the 1-2 plane (at constant angle), without any gauge fields turning on. The R-R coupling is

$$\int C_1 = \int dx^1 \left(C_1 + \partial_1 X^2 C_2\right)$$

T-dualize the 2-direction, the action becomes

$$\int dx^1 \left(C_{12} + 2\pi\alpha' F_{12} C\right)$$

Agrees with (2) up to a constant, which is counted in the R-R charge. In fact, suppose the 1-brane is wrapped around a circle in 2-direction of radius R, after T-duality $e^{-\phi} = e^{-\phi'} R^{-1} \alpha'^{1/2}$. In general we find

$$\mu_p = 2\pi\alpha'^{1/2} \mu_{p+1}$$

This will be confirmed by a direct calculation of μ_p. The leading coupling to fermions $\lambda(\xi)$ takes the form

$$-i \int d^{p+1}\xi \mathrm{Tr}\left(\bar{\lambda}\Gamma^a D_a \lambda\right)$$

To obtain the D-brane tension and R-R charge we need to compute the graviton/dilaton and R-R field exchange amplitude. One would expect the force of NS-NS and R-R exchange cancel each other because the system of two parallel D-branes is still supersymmetric, the total energy is determined by the BPS mass, independent of the separation. The open string 1-loop amplitude is the bosonic result combined with fermionic trace:

$$\mathcal{A} = iV_{p+1} \int_0^\infty \frac{dt}{t} (8\pi^2\alpha' t)^{-(p+1)/2} e^{-tY^2/2\pi\alpha'} \eta(it)^{-8} \cdot \frac{1}{2} \left[Z_0^0(it)^4 - Z_1^0(it)^4 - Z_0^1(it)^4\right]$$

where

$$\begin{aligned}
Z_0^0(\tau) &= \mathrm{Tr}_{NS}\left[e^{2\pi i\tau H}\right] \\
Z_1^0(\tau) &= \mathrm{Tr}_{NS}\left[(-1)^F e^{2\pi i\tau H}\right] \\
Z_0^1(\tau) &= \mathrm{Tr}_{R}\left[e^{2\pi i\tau H}\right] \\
Z_1^1(\tau) &= \mathrm{Tr}_{R}\left[(-1)^F e^{2\pi i\tau H}\right]
\end{aligned}$$

$Z_1^1(\tau)$ is identically zero because the trace of $(-1)^F$ over R-sector ground states cancels out. The three terms appeared in the amplitude sum to zero by the Jacobi abstruse identity, which indicates that there is no force between parallel D-branes. If we look at only one sector, use the modular property

$$Z_\beta^\alpha(\tau) = Z_{-\alpha}^\beta(-1/\tau)$$

take the limit $t \to 0$ and separate out the amplitude of exchanging massless modes,

$$\begin{aligned}\mathcal{A}_{NS} &= -\mathcal{A}_R \sim \frac{i}{2} V_{p+1} \int_0^\infty \frac{dt}{t} (2\pi t)^{-(p+1)/2} (t/8\pi^2\alpha')^4 e^{-tY^2/2\pi\alpha'} \\ &\to V_{p+1} 2\pi (4\pi^2\alpha')^{3-p} \frac{i}{k^2}\end{aligned}$$

The last expression is written in the $(9-p)$-momentum space.

The field theory calculation of graviton and dilaton exchange amplitude is the same as the bosonic string case. The result is proportional to $(D-2)$, therefore 1/3 of the bosonic case

$$\mathcal{A}_{NS} = \frac{2i\kappa^2\tau_p^2}{k^2}$$

where κ is the 10-dimensional gravitation coupling. Compare with the stringy computation, we obtain the tension

$$\tau_p^2 = \frac{\pi}{\kappa^2} (4\pi^2\alpha')^{3-p}$$

The R-R exchange amplitude is easily obtained from the effective action as

$$\mathcal{A}_R = -i\mu_p^2/k^2$$

hence the R-R charge is

$$\mu_p^2 = 2\kappa^2\tau_p^2 = 2\pi(4\pi^2\alpha')^{3-p} \tag{3}$$

In Polchinski's book a different convention is used for R-R field kinetic term, consequently the charge is $\mu_p^2 = T_p^2$.

In the T-dual picture of type I theory there are also 2^{9-p} orientifold planes. The orientation projection preserves the supercharge $Q_\alpha + \tilde{Q}_\alpha$, which becomes $Q_\alpha + P_\perp \tilde{Q}_\alpha$ in the T-dual picture just as D-branes (though the origin of the supersymmetry breaking is slightly different). They are also NS-NS and R-R sources. A parallel calculation as in the bosonic case relates their tension and charge to D-branes by

$$\mu_p' = \mp 2^{p-5}\mu_p, \quad \tau_p' = \mp 2^{p-5}\tau_p$$

where $\pm$ sign corresponds to $SO(N)$ or $USp(N)$ Chan-Paton factors. The total R-R charge of orientifold planes is $\mp 16\mu_p$. In the compact directions, R-R flux has nowhere to go, the total source must be zero. When the open string is coupled to $SO(32)$ gauge fields, there are 16 D-branes, which precisely cancel the R-R charge of orientifold planes. Again, in the original picture without T-duality the $SO(32)$ gauge group is required for the cancellation of closed string tadpole, or the ultraviolet divergence of open string 1-loop amplitude.

Dp-brane and $(6-p)$-brane are coupled to field strength F_{p+2} and F_{8-p}, dual to each other. In other words, under F_{p+2}, Dp-brane is electrically charged while

D$(6-p)$-brane is magnetically charged. Pick a $(8-p)$-sphere enclosing a Dp-brane, the flux of $F_{8-p} = *F_{p+2}$ is given by the charge

$$\mu_p = \Phi = \int_{S^{8-p}} *F_{p+2}$$

We can pick a "Dirac string" (branch cut) in the transverse direction to the p-brane, and write $F_{8-p} = dC_{7-p}$ everywhere except at the Dirac string. The flux can be expressed as

$$\Phi = \int_{S^{7-p}} C_{7-p}$$

where the integral is over a small sphere on S^{8-p} surrounding the Dirac string. A $(6-p)$-brane wrapped around the small sphere picks up a phase $\exp(i\mu_{6-p}\Phi)$. The "string" should be invisible, requiring

$$\mu_{6-p}\mu_p = \mu_{6-p}\Phi = 2\pi n$$

From (3) we see that D-brane charges satisfy $n = 1$.

D-brane dynamics

Now we study the $p - p'$ system. Given a p-brane and p'-brane parallel to the axes, there are different boundary conditions in each spatial direction for $p - p'$ strings: DD, NN, ND, DN. Let ν be the number of ND and DN directions. The first D-brane leaves unbroken the supersymmetries

$$Q_\alpha + P\tilde{Q}_\alpha$$

while the second only preserves $Q_\alpha + P'\tilde{Q}_\alpha$. The unbroken supersymmetry left corresponds to $\tilde{Q}_\alpha$ with $+1$ eigenvalue under $P^{-1}P'$. $P^{-1}P'$ is trivial (identity up to a sign) in the DD and NN directions, a parity transformation in the ND and DN directions. In type II or type I theory only even or odd dimensional branes exist, ν is always an even number. We can arrange them in pairs, correspondingly $P^{-1}P'$ can be written as

$$e^{i\pi(J_1+\cdots+J_{\nu/2})/2}$$

Acting on spinors $e^{i\pi J}$ has eigenvalues $\pm i$, there will be unbroken supersymmetry only if $\nu = 0(\mathrm{mod}4)$. In the case $\nu = 0$ half supersymmetries are preserved, otherwise only $1/4$ supersymmetries are preserved.

We can see this directly from the mass spectrum. The mode expansions of X^μ under different boundary conditions are

$$\text{NN:} \qquad X^\mu(z,\bar{z}) = x^\mu - i\alpha' p^\mu \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^\mu}{m}(z^{-m} + \bar{z}^{-m})$$

$$\text{DN, ND:} \qquad X^\mu(z,\bar z) = i\sqrt{\frac{\alpha'}{2}} \sum_{r\in \mathbf{Z}+1/2} \frac{\alpha_r^\mu}{r}(z^{-r} \pm \bar z^{-r})$$

$$\text{DD:} \qquad X^\mu(z,\bar z) = -i\frac{\delta X^\mu}{2\pi}\ln(z/\bar z) + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha_m^\mu}{m}(z^{-m} - \bar z^{-m})$$

In the DD case, δX^μ is the separation of two D-branes in X^μ-direction. We demand T_F has a branch in the R sector and no branch cut in the NS sector. Therefore the fermions in Ramond sector have the same periodicity as X^μ while those in NS sector have opposite periodicity. The zero energy of the integer moding bosons is $-1/24$. For half-integer moding bosons, the zero energy is $(\zeta(-1) - 2\zeta(-1))/2 = 1/48$. Corresponding fermions have opposite zero energy. In sum, the zero energy in Ramond sector is zero, while in NS sector the result is

$$(8-\nu)(-\frac{1}{24} - \frac{1}{48}) + \nu(+\frac{1}{48} + \frac{1}{24}) = -\frac{1}{2} + \frac{\nu}{8}$$

Only if $\nu = 0(\text{mod} 4)$ could the left and right energy levels match. This is consistent with the fact that in type I theory the orientation projection projects out R-R field strengths F_1 and F_5, only D$1,5,9$-branes exist.

Consider $\nu = 4$ system. Both NS and R sector have zero ground state energy. In NS sector 4 ND world-sheet fermions ψ^i generate $2^{4/2} = 4$ ground states, of which 2 survives the GSO projection. In R sector there are 4 NN and DD world-sheet fermions (light cone gauge), after GSO projection again give rise to 2 states. There is a another copy coming from orientation-reversed strings because the two endpoints are attached on different D-branes. The full massless content is an $N = 2$ hypermultiplet.

Restricting to $(p,p') = (9,5)$ system, the bosonic fields χ^A in the hypermultiplet is a doublet under the global $SU(2)$ of $N = 2$ supersymmetry. The string has endpoints on two branes, therefore χ^A carries charge $(+1,-1)$ under the $U(1) \times U(1)$ gauge fields on the brane. The minimally coupled action is of the form

$$\int d^6\xi \left[D_a\chi^\dagger D^a\chi + \left(\frac{1}{4g_p^2} + \frac{1}{4g_{p'}^2}\right) \sum_{I=1}^{3} (\chi^\dagger \tau^I \chi)^2 \right]$$

where $D_a = \partial_a + iA_a - iA'_a$. By T-dualizing r NN directions

$$A_a \to X_a/2\pi\alpha', \quad A'_a \to X'_a/2\pi\alpha'$$

We obtain the action for $(p,p') = (9-r, 5-r)$ system

$$\int d^{6-r}\xi \left[D_a\chi^\dagger D^a\chi + \left(\frac{X_a - X'_a}{2\pi\alpha'}\right)^2 \chi^\dagger\chi + \left(\frac{1}{4g_p^2} + \frac{1}{4g_{p'}^2}\right) \sum_{I=1}^{3} (\chi^\dagger \tau^I \chi)^2 \right]$$

The second term indicates that strings stretched between the two branes are massive.

As noted before, the world-sheet current for spacetime supersymmetry is the spin field $S_\alpha(z)$ and $\tilde{S}_\alpha(\bar{z})$. The supersymmetry algebra is

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= 2(\Gamma^0\Gamma^\mu)_{\alpha\beta}(P_\mu + Q_\mu^{NS}/2\pi\alpha') \\
\{\bar{Q}_\alpha, \bar{Q}_\beta\} &= 2(\Gamma^0\Gamma^\mu)_{\alpha\beta}(P_\mu - Q_\mu^{NS}/2\pi\alpha') \\
\{Q_\alpha, \bar{Q}_\beta\} &= 2\sum_p \tau_p(\Gamma^0\Gamma^{\mu_1\cdots\mu_p})_{\alpha\beta} Q^R_{\mu_1\cdots\mu_p}
\end{aligned}$$

where Q_μ^{NS} is the charge under NS-NS 2-form B, defined as

$$Q^\mu_{NS} = \int d^9x j^{\mu 0}, \qquad \int_M B = \frac{1}{2}\int d^{10}x\, j^{\mu\nu}(x)B_{\mu\nu}(x)$$

$Q^R_{\mu_1\cdots\mu_p}$ is the charge under R-R field C_{p+1}, normalized to 1 per world-volume (measured by $dx_{\mu_1}\wedge\cdots\wedge dx_{\mu_p}$).

So far we are free to choose the normalization of string coupling $g = e^\phi$. It can be fixed as the ratio of F-string tension to D-string tension:

$$g = \frac{\tau_{F1}}{\tau_{D1}} = \frac{1/2\pi\alpha'}{4\pi^{5/2}\alpha'\kappa^{-1}} = \frac{\kappa}{8\pi^{7/2}\alpha'^2}$$

and

$$\kappa_0 = \kappa e^{-\phi} = 8\pi^{7/2}\alpha'^2$$

Consider in type IIB theory a system of parallel q_1 F-strings and q_2 D-strings at rest, the supersymmetry algebra gives

$$\frac{1}{2}\left\{\begin{bmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{bmatrix}, [Q_\beta, \bar{Q}_\beta]\right\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} M + \begin{pmatrix} q_1 & q_2/g \\ q_2/g & -q_1 \end{pmatrix} \frac{v_1\Gamma^0\Gamma^1}{2\pi\alpha'}$$

where v_1 is the length of system. The BPS bound follows from that the LHS is positive definite:

$$\frac{M}{v_1} \geq \frac{\sqrt{q_1^2 + q_2^2/g^2}}{2\pi\alpha'}$$

This is saturated by F-string and D-string, as they preserve half supersymmetries. The formula is easily generalized to $p - p'$ system. For $\nu = 0(\mathrm{mod}4)$ the answer is

$$M \geq \tau_p v_p + \tau_{p'} v_{p'}$$

and for $\nu = 2(\mathrm{mod}4)$

$$M \geq \sqrt{\tau_p^2 v_p^2 + \tau_{p'}^2 v_{p'}^2}$$

This agrees with our previous result: $p - p'$ system with $\nu = 0(\mathrm{mod}4)$ perserves some supersymmetry, and the mass indeed saturate the BPS bound; when $\nu = 2(\mathrm{mod}4)$ all the supersymmetries are broken, and the system doesn't saturate the bound.

Consider one F-string and D-string parallel in X^1-direction. The total energy per unit length is

$$\tau_{D1}+\tau_{F1}=\frac{g^{-1}+1}{2\pi\alpha'}$$

certainly exceeds the BPS bound, therefore the system is not supersymmetric. In fact, the supercharges preserved by the F-string satisfies

$$\Gamma^0\Gamma^1 Q=Q, \quad \Gamma^0\Gamma^1\tilde{Q}=-\tilde{Q}$$

while the supercharges preserved by the D-string are $Q+P_\perp\tilde{Q}$, $\Gamma^0\Gamma^1 P_\perp=P_\perp\Gamma^0\Gamma^1$. There is no supersymmetry preserved by both of them.

If there is $SL(2,\mathbf{Z})$ duality in type IIB theory one would expect dyonic BPS bound states of 1 F-string and 1 D-string. Indeed, the above F-string can break with its endpoints attached on the D-string. Eventually let the endpoints go to infinity, the F-string is dissolved in the D-string. Both ends of the F-string are charged under D-string gauge field F_{ab}, leaving a flux on the D-string. We can compute this flux explicitly. Choose the gauge $B_{ab}=0$, the D-string world-sheet action contains the term

$$-\frac{1}{2}\tau_p(2\pi\alpha')^2\int d^2\xi\, F_{ab}F^{ab}$$

The conjugate momentum is $\Pi=2\pi\alpha' g^{-1}F$. The gauge group is $U(1)$, hence Π is quantized. One would expect $F_{ab}=g(2\pi\alpha')^{-1}\epsilon_{ab}$ for 1 F-string. The mass per unit world-volume is

$$\tau_p \det{}^{1/2}(G_{ab}+2\pi\alpha' F_{ab})=\frac{\sqrt{g^{-2}+1}}{2\pi\alpha'}$$

which is precisely the BPS bound. We can also see this from a T-duality argument. T-dualize the X^1-direction, the D-string with flux becomes a D0-brane moving with velocity

$$\dot{X}^1=2\pi\alpha'\dot{A}^1$$

It preserves the same number of supersymmetries as a D0-brane.

Now we study bound states of p-brane and p'-brane in general. By T-duality we can assume $p'=0$, $p=0,2,4,6,8$.

$0-0$ bound states: Compactify X^9-direction on a circle of radius R. Consider the system of n D0-branes with total momentum $p^9=m/R$, where $(n,m)=1$. The momentum can not be shared evenly by subsystems, therefore the energy of the bound state is lower than the continuum of separated states. One can T-dualize X^9-direction, D0-branes become D1-branes, and the momentum becomes winding numbers, corresponding to m F-strings. Therefore this is just the $(q_1,q_2)=(m,n)$ system considered before.

$0-2$ bound states: For incident $0-2$ brane system, the NS $0-2$ string has zero energy $-1/4$, is tachyonic. Hence one would expect a $0-2$ bound state whose mass saturate the BPS bound, below the continuum of separated states. The 0-brane dissolves in the 2-brane, leaving the flux of 2-brane gauge field F. The R-R action of 2-brane takes the form

$$i\mu_2 \int_{2-brane} \mathrm{Tr}(C_3 + 2\pi\alpha' F \wedge C_1)$$

So the flux is also a source for C_1, the 2-brane with flux carries 0-brane R-R charge. The picture is even more clear after T-dualizing one direction tangent to the 2-brane. We have two D-strings wrapped in transverse directions. The obvious BPS "bound" state with correct charges is a D-string at $\pi/4$ angle wrapping both directions once.

$0-4$ bound states: This is similar to the $0-0$ case. The bound state preserves 1/4 supersymmetry. It lies in supermultiplets generated by the broken supersymmetries.

When there is 1 D0-brane and 1 D4-brane, there are 16 supersymmetries broken by the 4-brane, giving rise to 2^8 states localized at the brane; 8 supersymmetries preserved by the 4-brane but broken by the 0-brane, giving rise to 2^4 states localized at the 0-brane. The bound state lies in a multiplet of 2^{12} states.

When there are 2 D0-branes and 1 D4-brane, the 8 supersymmetries preserved by the 4-brane but broken by the 0-branes generated 8 bosonic and 8 fermionic states. the unbounded 0-brane system lies in a multiplet of $8 + \frac{1}{2}16 \times 15 = 2^7$ states. The bound system of 0-branes has 2^4 ground states as before. The whole system lies in a multiplet of 9×2^{12} states.

In general the degenracy D_n of n D0-brane and 1 D4-brane bound state is given by

$$\sum_{n=0}^{\infty} q^n D_n = 256 \prod_{k=1}^{\infty} \left(\frac{1+q^k}{1-q^k} \right)^8$$

where the k term comes from the bound states of k D0-branes then bound to the 4-brane.

The $0-6$ system has no supersymmetric bound states, as the zero energy for NS $0-6$ strings is positive, the long distance force is repulsive. The $0-8$ system is more complicated because the R-R field of the 8-brane doesn't fall off as distance increases, the total energy in infinite. This indicates that D8-branes can not exists independently.

S, U Dualities and M-theory

First we study the $SL(2,\mathbf{Z})$ duality of type IIB theory. Consider a D-string stretched in X^1-direction. The gauge field has no dynamics in two dimensions.

Bosonic excitations are transverse fluctuations, the same as for F-string. The physical condition on R-sector massless states, namely the Dirac equation, projects the left-moving fermions into $\mathbf{8}_s$ of $SO(8)$ and right-moving fermions into $\mathbf{8}_c$. More explicitly,

$$(\Gamma^0\partial_0 + \Gamma^1\partial_1)u = 0$$

Left and right moving fermions have $s_0 = +1/2$ and $s_0 = -1/2$ respectively. Under $SO(1,1) \times SO(8) \subset SO(9,1)$,

$$\mathbf{16} \rightarrow (1/2, \mathbf{8}_s) + (-1/2, \mathbf{8}_c)$$

For F-strings, the massless fermionic excitations are generated by broken supersymmetries. Q_α and $\tilde{Q}_\alpha$ have the same chirality as $SO(9,1)$ spinors. Imposed the physical conditions, they have opposite chirality as $SO(8)$ spinors as D-strings. Therefore we see D-string and F-string have the same massless excitations.

Under the Einstein metric, $\tilde{G}_{\mu\nu} = g^{-1/2}G_{\mu\nu}$, the tensions are

$$\tau_{F1} = g^{1/2}/2\pi\alpha', \quad \tau_{D1} = g^{-1/2}/2\pi\alpha'$$

We expect the duality relating strongly coupled type IIB theory with weakly coupled theory: $g \leftrightarrow 1/g$, interchanging F-string and D-string, NS-NS and R-R 2-form potentials. The full duality group is expected to be $SL(2,\mathbf{Z})$: F-string is mapped to (q_1, q_2) FD bound states for q_1, q_2 coprime. The transformation on massless fields are

$$\begin{pmatrix} F_3' \\ H_3' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix},$$
$$F_5' = F_5, \quad \tilde{G}'_{\mu\nu} = \tilde{G}_{\mu\nu}, \quad \tau' = \frac{a\tau + b}{c\tau + d}$$

where

$$\tau = C_0 + ie^{-\phi}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$$

In type IIB theory D5-brane is magnetically charged under F_3, under the weak-strong duality it should be mapped to a magnetic source for NS-NS field strength H_3. This is the NS5-brane, or an extremal black 5-brane. The relevant supergravity action is

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\,(-G)^{1/2}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right)$$

Pick a 3-sphere around the NS5-brane, the magnetic charge is

$$Q = \int_{S^3} H_3$$

The extremal 5-brane solution is

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad G_{mn} = e^{2\Phi}\delta_{mn}$$
$$H = - *_4 d\Phi, \quad e^{2\Phi} = e^{2\Phi(\infty)} + \frac{Q}{2\pi^2 r^2}$$

where Greek and Latin indices denote the directions tangent and orthogonal to the 5-brane, respectively. $r^2 = x^m x^m$, $*_4$ is the Hodge dual in the 4 transverse directions. The solution preserves half (?) supersymmetries, is a BPS object. BPS bound together with Dirac quantization condition require:

$$\tau_{F1}\tau_{NS5} = \tau_{D1}\tau_{D5} \quad \Rightarrow \quad \tau_{NS5} = \frac{2\pi\alpha'}{\kappa^2} = \frac{1}{(2\pi)^5 g^2 \alpha'^3}$$

There are other black p-brane solutions charged under R-R fields. We will later see that these black branes and D-branes are good descriptions in different limits of the theory.

For a system of n D3-brane, the dynamics on the branes is $d = 4, N = 4$ $U(n)$ gauge theory. Expanding the D-brane effective action, the gauge coupling is related to string coupling by

$$g_{D3}^2 = \frac{1}{(2\pi\alpha')^2 \tau_{D3}} = 2\pi g$$

The S-duality of type IIB theory takes D3-brane to itself. Under strong-weakly duality, the gauge coupling transforms as

$$g_{D3}^2 \to \frac{4\pi^2}{g_{D3}^2}$$

The full duality implies the $SL(2, \mathbf{Z})$ duality of $d = 4, N = 4$ SYM theory.

Now we turn to study the duality between $SO(32)$ type I and heterotic string theory. For 1-1 strings (i.e. with both ends on the D1-brane) stretched in X^1 direction, the world-sheet parity Ω acts on $X^2, \cdots, X^9$ with an extra minus sign compared to the usual NN strings, as one can easily see from T-duality. This extends to ψ^μ by superconformal invariance. On type IIB fermionic ground states, Ω acts as

$$- \exp\left[i\pi(s_1 + s_2 + s_3 + s_4)\right]$$

The orientation projection removes the left-moving $\mathbf{8}_s$ and keeps the right-moving $\mathbf{8}_c$. For 1-9 strings, NS ground states are massive; R gound states are generated by the only periodic world-sheet fermions $\psi^{0,1}$:

$$|s_0; i\rangle, \qquad s_0 = \pm 1/2$$

where i is the $SO(32)$ Chan-Paton factor of the end attached on the D9-brane. The GSO projection removes the right-movers with $s_0 = -1/2$, keeps the left-mover with $s_0 = +1/2$. The orientation projection determines 9-1 states in terms of 1-9 states.

To summarize, the massless bosonic excitations of the D-string are the usual fluctuations; the massless fermionic excitations are right-movers transforming as $\mathbf{8}_c$ under the transverse $SO(8)$ and left-movers transforming as vector under gauge group $SO(32)$ and invariant under $SO(8)$. This is precisely the massless excitations of a long $SO(32)$ heterotic string. Again, the D-string tension is $\tau_{D1} = (2\pi\alpha' g)^{-1}$, D-strings become light at strong coupling. We expect SO(32) type I theory and heterotic theory related by strong-weak duality.

As a consistency check, the low energy effective action of $SO(32)$ type I and heterotic theory are related by the following identification of fields:

$$\begin{aligned} &G_{I\mu\nu} = e^{-\Phi_h} G_{h\mu\nu}, \quad \Phi_I = -\Phi_h \\ &\tilde{F}_{I3} = \tilde{H}_{h3}, \quad A_{I1} = A_{h1} \end{aligned}$$

The fundamental heterotic string can carry arbitrary representation of the gauge group. Under strong-weak duality it is mapped to D-string in type I theory. It is interesting to see that D-string can indeed carry, say, the spinor representation of $SO(32)$. The 1-9 strings have massless fermionic excitations Ψ^i. Their zero modes satisfy the Clifford algebra

$$\{\Psi_0^i, \Psi_0^j\} = \delta^{ij}, \qquad i, j = 1, \cdots, 32$$

generate the spinors $\mathbf{2^{15}} + \mathbf{2'^{15}}$ of $SO(32)$.

The gauge coupling and gravitational coupling in heterotic supergravity theory are related by

$$\frac{g_{YM}^2}{8\pi\kappa^2} = \frac{1}{2\pi\alpha'} \tag{4}$$

The RHS is heterotic string tension. The tension of D-branes in type I theory differs from type II by a factor of $2^{-1/2}$ because the cylinder amplitude has an additional factor of $1/2$ due to orientation projection. Recall that the gravitational coupling is related to string coupling by

$$\kappa = 8\pi^{7/2}\alpha'^2 g$$

And the gauge coupling of D9-brane is

$$g_{YM}^2 = g_{D9}^2 = \frac{1}{(2\pi\alpha')^2\tau_{D9}} = 2^{1/2}(2\pi)^7\alpha'^3 g$$

The D-string tension is therefore

$$\tau_{D1} = \frac{\pi^{1/2}}{2^{1/2}\kappa} \cdot 4\pi^2\alpha' = \frac{g_{YM}^2}{8\pi\kappa^2}$$

Notice that the expression of g_{YM} in two theories are different, in terms of g_{YM} the type I D-string tension agrees with the heterotic fundamental string tension (4).

Finally we sketch a nontrivial check between scattering amplitudes of two theories. We want to study the $F^4_{\mu\nu}$ interaction in the effective action. This is beyond the low energy effective supergravity theory. We first describe the computation of four gauge boson disk amplitude in type I theory. Vertex operators for gauge bosons in -1 picture and 0 picture are given by

$$\begin{aligned}\mathcal{V}^{-1} &= g_o t^a e^{-\phi} e_\mu \psi^\mu e^{ik\cdot X}\\ \mathcal{V}^0 &= g_o (2\alpha')^{-1/2} t^a e_\mu (i\dot{X}^\mu + 2\alpha' k\cdot\psi\,\psi^\mu) e^{ik\cdot X}\end{aligned}$$

The amplitude is

$$\frac{1}{\alpha' g_o^2}\sum_{cyclic\ orders}\int dx_4 \left\langle c\mathcal{V}_1^{-1}(x_1)\cdot c\mathcal{V}_2^{-1}(x_2)\cdot c\mathcal{V}_3^0(x_3)\cdot \mathcal{V}_4^0(x_4)\right\rangle$$

By carrying out the OPE's and perform the integral, the answer is

$$-16ig_{YM}^2\alpha'^2(2\pi)^{10}\delta^{10}(\sum k_i)K(e_1,\cdots,e_4)\left[\mathrm{Tr}_v(t^{a_1}\cdots t^{a_4})\frac{\Gamma(-\alpha's)\Gamma(-\alpha'u)}{\Gamma(1-\alpha's-\alpha'u)} + 2\ \mathrm{perm.}\right]$$

where

$$K(e_1,\cdots,e_4) = t^{\mu_1\nu_1\cdots\mu_4\nu_4} k_{1\mu_1}e_{1\nu_1}\cdots k_{4\mu_4}e_{4\nu_4}$$

The tensor t is antisymmetric within $\mu_i\nu_i$ and symmetric with respect to pairs of $\mu_i\nu_i$. In the low energy limit, i.e. effectively $\alpha'\to 0$,

$$\frac{\Gamma(-\alpha's)\Gamma(-\alpha'u)}{\Gamma(1-\alpha's-\alpha'u)} = \frac{1}{\alpha'^2 su} - \frac{\pi^2}{6} + O(\alpha')$$

The leading term gives the amplitude of the low energy theory, can be written as a sum of contributions from single poles. The second term gives the $O(\alpha'^2)$ order correction. To restore the usual convention of Yang-Mills kinetic term,

$$k_{[\mu}e_{\nu]} \simeq -iF_{\mu\nu}/2g_{YM}$$

The correction to effective Lagrangian is

$$\frac{\pi^2\alpha'^2}{2\cdot 4! g_{YM}^2}\, t^{\mu_1\nu_1\cdots\mu_4\nu_4}\mathrm{Tr}_v(F_{\mu_1\nu_1}\cdots F_{\mu_4\nu_4})$$

For type I theory, $g_{YM}^2 = 2(2\pi)^{7/2}\alpha'\kappa$, this can be rewritten as

$$\frac{g_{YM}^2}{2^{10}\pi^5 4!\kappa^2}(tF^4) \tag{5}$$

The contraction of indices in the last term is understood.
Now we compute the four gauge boson 1-loop amplitude in heterotic theory. The vertex operators are

$$\begin{aligned}\mathcal{V}^{-1} &= g_c \hat{k}^{-1/2} j^a e^{-\tilde{\phi}} e_\mu \tilde{\psi}^\mu e^{ik\cdot X}\\ \mathcal{V}^0 &= g_c (2/\alpha')^{1/2}\hat{k}^{-1/2} j^a e_\mu (i\bar{\partial}X^\mu + \frac{1}{2}\alpha' k\cdot\tilde{\psi}\,\tilde{\psi}^\mu) e^{ik\cdot X}\end{aligned}$$

where $\hat{k}$ is the constant appeared in the current algebra

$$j^a(z)\, j^b(0) \sim \frac{\hat{k}\delta^{ab}}{z^2} + \frac{if^{ab}_c}{z} j^c(0)$$

The current j^a in $SO(32)$ heterotic theory can be represented in the fermionic form

$$j^a = 2^{-1/2} it^a_{AB} \lambda^A \lambda^B$$

The 1-loop amplitude is

$$\sum_{spin\ structure} \int_F \frac{d\tau d\bar{\tau}}{8\tau_2} \prod_{i=1}^{4} \int d^2 w_i \left\langle b(0)\tilde{b}(0)\tilde{c}(0)c(0) \cdot \mathcal{V}^0_1(w_1, \bar{w_1}) \cdots \mathcal{V}^0_4(w_4, \bar{w_4}) \right\rangle_\tau$$

The key issue is to compute

$$\langle j^{a_1}(w_1) \cdots j^{a_4}(w_4) \rangle_{T^2}$$

In the limit $k_i \to 0$, all other correlation functions are independent of w_i's. We can replace j^a by its average over Rew_i, i.e. the charge. More precisely,

$$j^a(w) j^b(0) \to T[Q^a(w) Q^b(0)] - \pi\delta^2(w, \bar{w})\delta^{ab}$$

The singular piece comes from jj OPE. The correlation function of j^a's can be obtained from the generating function

$$\langle \exp(z \cdot \bar{j}) \rangle = \exp\left(-\frac{z \cdot z}{16\pi\tau_2}\right) \mathrm{Tr}\left[\exp(2\pi i \tau H) \exp(z \cdot Q)\right]$$

The trace includes a sum over the $SO(32)$ lattice, where the charge is

$$Q = 2^{-1/2} l, \quad l \in \Gamma$$

The result of effective Lagrangian is

$$\frac{1}{2^8 \pi^5 4! \alpha'} (tF^4)$$

For $SO(32)$ heterotic theory $g^2_{YM} = 4\kappa^2/\alpha'$. In term of gauge coupling, the answer is precisely the same as (5). This indicates that strong-weak duality can relate amplitudes of different orders in two theories.

Under toroidal compactification the duality group can be enhanced, referred to as U-duality. We study the example of type IIB string compactified on T^5. The gauge fields are: 10 from NS-NS sector $G_{\mu n}$, $B_{\mu n}$; 16 from R-R sector $C_{\mu n}$, $C_{\mu npq}$, $C_{\mu npqrs}$; and the 2-form field strength $*_5 H_3$ with the dual taken in noncompact directions. Under the T-duality group $O(5, 5, \mathbf{Z})$ they transform as $SO(10)$ representation $\mathbf{10} + \mathbf{16} + \mathbf{1}$. Correspondingly type IIA theory will have $\mathbf{16}'$. The charges of **10** are carried by winding strings, charges of **16** are carried by wrapped D-branes, and the charge of **1** is carried by NS5-brane wrapping T^5 once. The low energy supergravity theory of T^5 compactification has symmetry $E_{6(6)}$. The U-duality group $E_{6(6)}(\mathbf{Z})$ is conjectured as the full duality group of the theory. It has representation **27** induced from E_6, which decomposes under $SO(10)$ subgroup as $\mathbf{27} \to \mathbf{10} + \mathbf{16} + \mathbf{1}$.

Denote by $D_{mn\cdots}$ a D-brane extended in the indicated directions, F_m for a fundamental string in X^m-direction, and p_m a BPS state carrying the corresponding momentum. As a simple example consider duality chain

$$(D_9, F_9) \xrightarrow{T_{78}} (D_{789}, F_9) \xrightarrow{S} (D_{789}, D_9) \xrightarrow{T_9} (D_{78}, D_\emptyset)$$

Therefore FD bound state in type IIB theory is U-dual to type IIA D0-D2 bound state. Now consider

$$(D_{6789}, D_\emptyset) \xrightarrow{T_6} (D_{789}, D_6) \xrightarrow{S} (D_{789}, F_6) \xrightarrow{T_{6789}} (D_6, p_6) \xrightarrow{S} (F_6, p_6)$$

This says (n, m) D0-D4 bound states are U-dual to fundamental strings with momentum n and winding number m in one direction. We want to count the BPS state degeneracy in the latter case. The supersymmetry algebra of fundamental string excitations in the compactified theory can be written as

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= 2(\Gamma^0\Gamma^\mu)_{\alpha\beta}P_\mu + 2(\Gamma^0\Gamma^m)_{\alpha\beta}P_{Lm} \\
\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} &= 2(\Gamma^0\Gamma^\mu)_{\alpha\beta}P_\mu + 2(\Gamma^0\Gamma^m)_{\alpha\beta}P_{Rm} \\
\{Q_\alpha, \tilde{Q}_\beta\} &= 0
\end{aligned}$$

For strings with momentum $k_{L,R}$ in the compact direction, the RHS has eigenvalues

$$2(M \pm |k_{L,R}|)$$

Therefore a BPS state has $M^2 = k_L^2$ or $M^2 = k_R^2$, corresponding to $N(\tilde{N}) = 1/2$ in NS-sector or $N(\tilde{N}) = 0$ in R-sector, i.e. the "ordinary ground states". The restriction on left and right moving sectors can not be simultaneously satisfied. The possible BPS states have

$$(N, \tilde{N}) = \left\{ \begin{array}{ll} (nm, 0), & nm > 0 \\ (0, -nm), & nm < 0 \end{array} \right.$$

The degeneracy of BPS states is given by the generating function

$$\mathrm{Tr} q^N = 2^8 \prod_{k=1}^{\infty} \left(\frac{1+q^k}{1-q^k} \right)^8$$

and the same for $\tilde{N}$. In the case $m = 1$ this agree with our previous result on BPS states of D0-D4 system.

Now we turn to study the strong coupling limit of type IIA theory. D0-branes have mass

$$\tau_0 = \alpha'^{-1/2} g^{-1}$$

This becomes the lightest scale in the limit $g \to \infty$. There is BPS bound state of n D0-branes with mass $n\tau_0 = n\alpha'^{-1/2} g^{-1}$, which is a type IIB fundamental string carrying momentum n in the T-dual picture. In the strong coupling limit this indicates an eleventh dimension compactified on a circle of radius

$$R_{10} = g\alpha'^{1/2}$$

The 11-dimensional gravitational coupling is

$$\kappa_{11}^2 = 2\pi R_{10}\kappa^2 = \frac{1}{2}(2\pi)^8 g^3 \alpha'^{9/2}$$

Introducing 11-dimensional Planck mass

$$M_{11} = g^{-1/3}\alpha'^{-1/2}$$

We have

$$g = (M_{11}R_{10})^{3/2}, \quad \alpha' = M_{11}^{-3}R_{10}^{-1}$$

For type II theory compactified on a circle, the symmetry of low energy theory is $SL(2,\mathbf{R}) \times SO(1,1)$, the U-duality group is $SL(2,\mathbf{Z})$. For type IIB theory this is just the S-duality group, while for type IIA this can be thought as the modular group the T^2 on which M-theory is compactified.

For type II compactified on T^2, the symmetry of low energy theory is $SL(3,\mathbf{R}) \times SL(2,\mathbf{R})$, the U-duality group is $SL(3,\mathbf{Z}) \times SL(2,\mathbf{Z})$. The T-duality group is $SL(2,\mathbf{Z}) \times SL(2,\mathbf{Z})$, with one factor from the discrete shift of B field, another factor from the modular group of T^2. As M-theory compactified on T^3, the geometric factor is enlarged to the modular group $SL(3,\mathbf{Z})$.

Type IIB theory has $SL(2,\mathbf{Z})$ duality. This is interpreted by Vafa as the modular group of toroidal compactification of a 12-dimensional F-theory, with the complex structure of the torus characterized by $\tau = C_0 + ie^{-\Phi}$.

The low energy effective theory of M-theory is 11-dimensional supergravity. There is one 3-form gauge field B_3, $F_4 = dB_3$. The bosonic part of the action is given by

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\,(-G)^{1/2}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa_{11}^2}\int B_3 \wedge F_4 \wedge F_4$$

Under F_4, the electrically and magnetically charged objects are M2- and M5-brane, given by black p-brane solutions, respectively. Now we study the correspondence between objects in IIA theory and M-theory compactified on a circle.

D0-branes: These corresponds to BPS states of nonzero p_{10}. In type IIA theory they have 2^8 massless excitations generated by 8 broken supersymmetries. In

M-theory they correspond to states of an ultrshort massless graviton multiplet.

F-strings: One would natural expect them to be M2-branes wrapped on the circle. Such membranes are coupled to $B_{\mu\nu 10}$, which reduces to $B_{\mu\nu}$ fields coupled to IIA strings. The $d = 11$ supergravity action reduces to $d = 10$ type IIA supergravity.

D2-branes: These are naturally expected to be M2-branes transverse to the $X^1$0-direction. The IIA D2-brane are coupled to R-R 3-form potential $B_{\mu\nu\rho}$, which are the transverse components of the 11-dimensional field B_3. As a consistency check, since F-strings are D2-branes wrapped on the X^{10}-direction, their tension should be related in the obvious way. Indeed,

$$\tau_{M2} = \tau_{D2} = \frac{1}{(2\pi)^2 g \alpha'^{3/2}} = \frac{M_{11}^3}{(2\pi)^2}$$
$$\tau_{F1} = \frac{1}{2\pi\alpha'} = 2\pi R_{10} \tau_{D2}$$

The bosonic action of D2-brane in flat spacetime can be written as

$$S = -\tau_2 \int d^3x \left\{ \left[-\det\left(\eta_{\mu\nu} + \partial_\mu X^m \partial_\nu X^m + 2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2} + \frac{1}{2}\epsilon^{\mu\nu\rho}\lambda \partial_\mu F_{\nu\rho} \right\}$$

An auxiliary field λ is introduced and now we regard $F_{\mu\nu}$ as an independent field. In 3 dimensions we have the relation

$$\det(G_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}) = \det G_{\mu\nu} \left[1 + \frac{1}{2}(2\pi\alpha')^2 F_{\alpha\beta} F^{\alpha\beta}\right]$$

The F equation of motion gives

$$d\lambda = 2(2\pi\alpha')^2 \left[1 + (2\pi\alpha')^2 F_{\mu\nu} F^{\mu\nu}\right]^{-1/2} * F$$

Integrating out $F_{\mu\nu}$, and after some algebra we can write the action as

$$S = -\tau_2 \int d^3x \left[-\det\left(\eta_{\mu\nu} + \partial_\mu X^m \partial_\nu X^m + (2\pi\alpha')^{-2} \partial_\mu \lambda \partial_\nu \lambda\right)\right]^{1/2}$$

Identify

$$\lambda = 2\pi\alpha' X^{10}$$

This is precisely the action of 11-dimensional membranes. The 11-dimensional Lorentz invariance is also manifest. Extension to fermionic fields is worked out in (P. Townsend, hep-th/9512062).

D4-branes: This is naturally identified with wrapped M5-branes.

NS5-branes: They are M5-branes transverse to X^{10}-direction. The tension is the same as type IIB NS5-branes:

$$\tau_{M5} = \tau_{NS5} = \frac{1}{(2\pi)^5 g^2 \alpha'^3} = \frac{M_{11}^6}{(2\pi)^5}$$

The relation to D4-brane tension is

$$\tau_{D4} = \frac{1}{(2\pi)^4 g \alpha'^{5/2}} = 2\pi R_{10} \tau_{M5}$$

The collective coordinate for the eleventh dimension is given by a scalar λ: roughly speaking, $*F_5 = d\lambda$ on the brane.

D6-branes: These are dual to D0-branes. They are Kaluza-Klein magnetic monopoles.

D8-branes: They are dissolved as gauge dynamics on the boundary of M-theory.

References

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