Notes on Calabi-Yau Compactification

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The notes give an elementary introduction to Calabi-Yau compactification, conifold transition and other relevant topics.

Elementary facts of Calabi-Yau manifolds

Definition: *A (d-dimensional) Calabi-Yau manifold is a compact Kahler manifold which has $SU(d)$ holonomy.*

Sometimes we also refer to Kahler manifolds with Ricci-flat metric, which ensures that the holonomy lies in $SU(d)$ and vice versa. This is because the $U(1)$ part of the connection, i.e. the trace, give rise precisely to the Ricci tensor: $R_{i\bar{j}} = -\partial_{\bar{j}}\Gamma^k_{ik}$. Clearly the Ricci-flat condition requires vanishing first Chern class. The Calabi conjecture, proved by Yau, says if X is a Kahler manifold with $c_1(X) = 0$ and Kahler form J, then there is a unique Ricci-flat metric on X whose Kahler form $J' \sim J$ in $H^2(X)$. The following fact is very useful:

Proposition: *A compact, Kahler manifold M of $SU(d)$ holonomy has Hodge numbers $h^{p,0} = h^{0,p} = 0$ $(0 < p < d)$, $h^{0,d} = h^{d,0} = 1$. In other words, holomorphic p-form exists on M only for $p = d$, and is unique up to the cohomology class.*

Now we consider Calabi-Yau manifolds as projective varieties in $\mathbf{P}^n$. Let J be the Kahler form normalized as the generator of $H^1(\mathbf{P}^n)$, i.e. its integral over any $\mathbf{P}^1$ is 1. Also use J to denote its pullback on X. We want to compute the total Chern class $c(X)$.

Let ε^{n+1} be the $(n+1)$-plane trivial bundle over $\mathbf{P}^n$ induced from $C^{n+1} - \{0\}$, H be the canonical line bundle over $\mathbf{P}^n$, $H^\perp$ the orthogonal component of H in ε^{n+1}. The tangent bundle $T\mathbf{P}^n$ is naturally isomorphic to $\mathrm{Hom}(H, H^\perp)$.

$$\begin{aligned} T\mathbf{P}^n \oplus \varepsilon^1 &\cong \mathrm{Hom}(H, H^\perp) \oplus \mathrm{Hom}(H, H) \\ &\cong \mathrm{Hom}(H, \varepsilon^{n+1}) \cong H^{n+1} \end{aligned}$$

$c(H) = 1 + J$, therefore $c(\mathbf{P}^n) = (1 + J)^{n+1}$. Now let X be defined by k homogeneous polynomials of degree $d_1, \cdots, d_k$, $T\mathbf{P}^n|_X = TX \oplus NX$. In the case $k = 1$, X represents the homology cycle $d \cdot [\mathbf{P}^{n-1}]$. It can be shown that $c_1(NX) = c_1([X]) = [d \cdot J]$ where $[X]$ is the associated line bundle whose transition functions are given by the ratio of local defining meromorphic functions. In general the Chern class of the normal bundle NX is

$$c(NX) = \prod_{i=1}^{k} (1 + d_i J)$$

The Chern class of X is therefore given by

$$c(X) = \frac{c(\mathbf{P}^n)}{c(NX)} = \frac{(1+J)^{n+1}}{\prod(1+d_iJ)}$$

For the first Chern class to vanish we need $\sum d_i = n+1$, the Calabi-Yau manifold X constructed this way is usually denoted by $Y_{(n;d_1,\cdots,d_k)}$. All Calabi-Yau 3-folds arising in this form are $Y_{(4;5)}, Y_{(5;2,4)}, Y_{(5;3,3)}, Y_{(6;3,2,2)}, Y_{(7;2,2,2,2)}$, all of which are simply connected. There is another known example of Calabi-Yau 3-fold described as the following. The torus T of modulous $\tau = e^{\pi i/3}$ has $\mathbf{Z}_3$ symmetry of rotation with 3 fixed points. $(T \times T \times T)/\mathbf{Z}_3$ has vanishing Chern class with 27 conifold singularities. The singularities can be resolved by gluing in some suitable line bundle over $\mathbf{P}^2$ with vanishing first Chern class, the resulting manifold has $SU(3)$ holonomy.

To compute the Euler characteristic of Calabi-Yau 3-fold Y defined by polynomials of degree $d_1, \cdots, d_k$ in $\mathbf{P}^{k+3}$, first we know that the top Chern class is the same as the Euler class of the underlying real manifold, i.e. $\chi(Y) = c_3([Y])$. The integral of J^3 over any $\mathbf{P}^3$ is 1. A generic k-plane intersects Y at $d = \prod_{i=1}^k d_i$ points, therefore the homology class $[Y] = d\,[\mathbf{P}^3]$, $\int_Y J^3 = d$. For example, $\chi(Y_{(4;5)}) = -200$.

All $Y_{(N;d_1,\cdots,d_k)}$ have exactly 1 harmonic 2-form: the Kahler form, therefore $h^{1,1} = b_2 = 2$. From $h^{3,0} = h^{0,3} = 0$, we can compute $h^{2,1} = h^{1,2} = \chi/2$.

Definition: *A 2-dimensional compact, Kahler manifold X satisfying*

$$c_1(X) = 0, \quad h^{1,0} = 0$$

is called a $K3$ surface.

It turns out that all $K3$ surfaces are diffeomorphic. For example, a smooth hypersurface in $\mathbf{P}^3$ defined by a quartic polynomial is a $K3$ surface, with total Chern class $c(X) = (1+J)^4/(1+4J) = 1 + 6J^2$. Hence $\chi(X) = 24$, $h^{1,1} = 20$.

Unbroken supersymmetry of $N = 1$ supergravity on $M_4 \times K$

The low energy effective theory of superstrings is $D = 10, N = 1$ supergravity coupled to super Yang-Mills fields. The supergravity multiplet contains the metric, spin $\frac{3}{2}$ gravitino, 2-form potential, spin $\frac{1}{2}$ fermion and a scalar field:

$$G_{MN},\ \psi_M,\ B_{MN},\ \lambda,\ \phi$$

and the super-Yang-Mills multiplet

$$F^a_{MN},\ \chi^a$$

We consider the compactification on $M_4 \times K$, where K is compact and M_4 is maximally symmetric. The supersymmetry transformation is written as

$$\begin{aligned}
\delta\psi_\mu &= \nabla_\mu\varepsilon + \frac{1}{32}\sqrt{2}e^{2\phi}(\gamma_\mu\gamma_5 \otimes H)\varepsilon \\
\delta\psi_m &= \nabla_m\varepsilon + \frac{1}{32}\sqrt{2}e^{2\phi}(\gamma_m H - 12H_m)\varepsilon \\
\delta\lambda &= \sqrt{2}\gamma^m\nabla_m\phi \cdot \varepsilon + \frac{1}{8}e^{2\phi}H\varepsilon \\
\delta\chi^a &= -\frac{1}{4}e^{\phi}F_{mn}\gamma^{mn}\varepsilon
\end{aligned} \tag{1}$$

where Greek and Latin indices refer to M_4 and K respectively, $H = H_{pqr}\gamma^{pqr}$, $H_m = H_{mqr}\gamma^{qr}$, ε is a 16-component $SO(9,1)$ Marjorana-Weyl spinor. The spin connection $\nabla_m = \partial_m + \frac{1}{2}\omega_m^{ab}\gamma_{ab}$, where ω_m is a *spin*(6) gauge field. The 3-form field strength

$$H = dB - \omega_{YM} + \omega_L$$

where ω_{YM} and ω_L are the Chern-Simons 3-form of the gauge field and spin connection respectively. Let tr denote the trace in the vector representation of $SO(3,1)$, Tr the trace in the adjoint representation of the gauge group,

$$dH = \mathrm{tr}R \wedge R - \frac{1}{30}\mathrm{Tr}F \wedge F$$

If there is unbroken supersymmetry, $Q_\alpha|0\rangle = 0$ for some supercharge Q_α. This implies for example, $\langle 0|\{Q_\alpha, \chi_\beta\}|0\rangle = 0$, i.e. classically the supersymmetric variation of fermions must vanish, otherwise the supersymmetry will be spontaneously broken. We are looking for conditions such that nonzero ε exists for (1) to vanish.

For simplicity (though not necessary), we are going to assume that M_4 is Minkowski space, $H_{pqr} = 0$, then (1) give constraints

$$\begin{gathered}
\nabla_m\phi = \nabla_m\varepsilon = 0, \\
R_{mnpq}\gamma^{pq}\varepsilon = 0, \;\; F^a_{mn}\gamma^{mn}\varepsilon = 0
\end{gathered} \tag{2}$$

where the Riemann tensor is related to the spin connection by $[\nabla_m, \nabla_n]\varepsilon = \frac{1}{4}R_{mnpq}\gamma^{pq}\varepsilon$. ε is in the $\mathbf{16}$, decomposes as $(\mathbf{2},\mathbf{4}) \oplus (\bar{\mathbf{2}},\bar{\mathbf{4}})$ under $SO(3,1) \times SO(6)$, we can write

$$\varepsilon(y) = \sum_{\Lambda=1}^{4} \xi_\Lambda \otimes \eta_\Lambda(y)$$

where ξ_Λ are constant real spinors of $SO(3,1)$, η_Λ are real spinors of $SO(6)$. It is easy to see that there are at least 2 independent η satisfying our conditions, with ε replaced by η. $\nabla_m\eta = 0$ implies that $\eta^\dagger\eta$ is a constant, normalized to be 1. Define

$$J_m^n = -i\eta^\dagger\gamma_m^n\eta$$

By Fierz rearrangement (e.g. $\eta\eta^\dagger = \frac{1}{6}(1-\eta^\dagger\gamma_{mn}\eta\gamma^{mn})(1+\gamma)$ for chiral η), we find $J^2 = -\mathbf{1}$ and $J_m^p J_n^q g_{pq} = g_{mn}$. Therefore J defines an almost complex structure on K and g_{mn} is hermitian with respect to J. J is covariantly constant, therefore it is a complex structure and K is Kahler with respect to J, and the holonomy is contained in $U(3) \subset O(6)$. Consider the $U(1)$ part of the connection

$$\Gamma = \Gamma^q_{mp} J^p_q dx^m$$

$R_{mnpq}\gamma^{pq}\varepsilon = 0$ implies that $d\Gamma = 0$, i.e. the $U(1)$ part of the holonomy vanishes, or K has vanishing first Chern class. The spin connection ω is a gauge field of $spin(6) \cong SU(4)$. The covariantly constant real spinor η transforms as $4 \oplus \bar{4}$, requires that ω only takes value in the little group $SU(3)$ of η. Under $SU(3) \subset SU(4)$, $4\oplus\bar{4} \to 3\oplus\bar{3}\oplus 1\oplus 1$, therefore a 6 dimensional manifold of $SU(3)$ holonomy always has 2 covariantly constant spinors η and $i\gamma\eta$.

Also from $R_{mnpq}\gamma^{pq}\eta = 0$, contracting with γ^n and use the relation $\gamma^n\gamma^{pq} = \gamma^{npq} - g^{np}\gamma^q + g^{nq}\gamma^p$, we find

$$R_{mn} = 0$$

In sum, K is a Ricci-flat Kahler manifold with vanishing first Chern class, with its holonomy contained in $SU(3)$. $H = 0$ implies

$$\frac{1}{30}\mathrm{Tr}F \wedge F = \mathrm{tr}R \wedge R$$

The simplest solution is to embed the spin connection in the gauge group $spin(32)/\mathbf{Z}_2$ or $E_8 \times E_8$, and set the gauge field equal to the spin connection. E_8 has maximal subgroup $SU(3) \times E_6$, under which the adjoint of E_8 decomposes as $(8,1) \oplus (1,78) \oplus (3,27) \oplus (\bar{3},\overline{27})$. The quadratic Casimir of $SU(3)$ in this representation is $3 + 27 \times \frac{1}{2} + 27 \times \frac{1}{2} = 30$, hence $\mathrm{tr}R \wedge R = \frac{1}{30}\mathrm{Tr}F \wedge F$ is satified when we set $A = \omega$.

String theory viewpoint

The action of supersymmetric sigma model with curved target space can be written as

$$S = \int d\sigma d\tau \left(\frac{1}{2}g_{ij}\partial_\alpha X^i \partial^\alpha X^j + \frac{1}{2}ig_{ij}\bar{\psi}^i\rho^\alpha D_\alpha\psi^j - \frac{1}{6}R_{ijkl}\bar{\psi}^i\psi^j\bar{\psi}^k\psi^l\right) \tag{3}$$

where $D_\alpha\psi^i = \partial_\alpha\psi^i + \Gamma^i_{jk}\partial_\alpha X^j\psi^k$ is the pullback of the affine connection in the target space. In the usual flat background the spacetime supersymmetry is manifest in the Green-Schwarz formulation

$$S = \int d\tau d\sigma \left(\frac{1}{2}\partial_\alpha X^i \partial^\alpha X^i + \frac{1}{2}i\bar{S}\gamma^-\rho^\alpha\partial_\alpha S\right) \tag{4}$$

where S transforms as spacetime spinor. If the target space has holonomy $SU(3)$, under $SU(3) \subset SO(8)$ both the vector and spinor decomposes as $1\oplus 1\oplus 3\oplus\bar{3}$, there is no difference between S^α and ψ^i. The generalization (4) to curved background is analogous to (3):

$$S = \int d\sigma d\tau \left(\frac{1}{2} g_{ij} \partial_\alpha X^i \partial^\alpha X^j + \frac{1}{2} i \bar{S} \gamma^- \rho^\alpha D_\alpha S - \frac{1}{6} R_{ijkl} \bar{S} \gamma^{ij} S \, \bar{S} \gamma^{kl} S \right)$$

where $D_\alpha = \partial_\alpha + \partial_\alpha X^i \omega_i$ is the pullback of the spin connection. The spacetime supersymmetries include the ε supersymmetry

$$\delta X^i = (p^+)^{1/2} \bar{\varepsilon} \gamma^i S$$
$$\delta S = i (p^+)^{-1/2} \gamma_- \gamma_M \rho \cdot \partial X^M \varepsilon$$

which holds for Calabi-Yau target space as above, and the δ supersymmetry

$$\delta X^i = 0$$
$$\delta S^a = \delta^a$$

where δ^a is an $SO(8)$ real spinor. The invariance under δ symmetry demand $D_\alpha \delta^\alpha = 0$, i.e. δ^a is covariantly constant. This also agrees with the constraints from low energy supergravity. Furthermore, the world sheet β function vanishes on a Calabi-Yau background, which is another essential consistency condition for a string theory.

Massless spectrum of type II on Calabi-Yau

Type II theory compactified on Calabi-Yau have $d = 4, N = 2$ supersymmetry. The relevant massless representation of the supersymmetry algebra are

hypermultiplet	$(-\frac{1}{2}, 0^2, \frac{1}{2}) + (-\frac{1}{2}, 0^2, \frac{1}{2})$
vector multiplet	$(-1, -\frac{1}{2}^2, 0) + (0, \frac{1}{2}^2, 1)$
supergravity multiplet	$(-2, -\frac{3}{2}^2, -1) + (1, \frac{3}{2}^2, 2)$

For type IIA the bosonic fields in the low energy effective theory are NS-NS fields g_{MN}, b_{MN}, ϕ, and R-R fields F_2, F_4, where the fluctuations of the latter are given by C_M, C_{MNP}. $g_{\mu\nu}$ and c_μ form the bosonic part of the supergravity multiplet. $h^{3,0} = 1$ for Calabi-Yau 3-folds, the zero modes of $c_{ijk}, c_{\bar{i}\bar{j}\bar{k}}, \phi$ and $b_{\mu\nu} \sim a$ (axion) form the bosonic part of 1 hypermultiplet. $g_{\mu i}, b_{\mu i}$ are $(1,0)$ forms on K, since for Calabi-Yau manifolds $h^{1,0} = 0$, they have no zero modes. The metric being hermitian requires the variation of g_{ij} be made up by deformation of the complex structure. One can write

$$h_{ij} = \omega_{i\bar{k}\bar{l}} \varepsilon_j^{\bar{k}\bar{l}}$$

and show that the zero modes of h_{ij} correspond precisely to the harmonic $(1,2)$ forms $\omega_{i\bar{k}\bar{l}}$. Each $(2,1)$ harmonic form gives rise to the zero modes of $g_{ij}, g_{\bar{i}\bar{j}}, c_{ij\bar{k}}, c_{\bar{i}\bar{j}k}$, contained in $h^{2,1}$ hypermultiplets, correspond to the moduli of complex structures on K. Each $(1,1)$ harmonic form gives rise to zero modes of scalars $g_{i\bar{j}}, b_{i\bar{j}}$ and a vector $c_{\mu i\bar{j}}$, contained in $h^{1,1}$ vector multiplets, correspond to the Kahler moduli.

For type IIB the R-R fluctuations are c, c_{MN}, c_{MNPQ}. The supergravity multiplet contains $g_{\mu\nu}$ and $c_{\mu ijk}$. One hypermultiplet contains $\phi, b_{\mu\nu} \sim a, c, c_{\mu\nu} \sim a'$. Each harmonic $(1,1)$ form gives $g_{i\bar{j}}, b_{i\bar{j}}, c_{i\bar{j}}$ and axion $c_{\mu\nu i\bar{j}}$, which are the bosonic part of a hypermultiplet, instead of vector multiplet as in IIA. Finally each $(2,1)$ harmonic form gives $g_{ij}, g_{\bar{i},\bar{j}}$ and a vector $c_{\mu ij\bar{k}}$, which are the bosonic part of a vector multiplet.

The massless content of type II compactified on CY 3-fold is summarized as

IIA: b^2 vector multiplets, $h^{2,1}+1$ hypermultiplets

IIB: $h^{2,1}$ vector multiplets, b^2+1 hypermultiplets

They are related by mirror symmetry, which exchanges the complex structure with Kahler moduli.

Extremal black p-branes

Type IIB compactified on Calabi-Yau 3-fold X will contain black holes which becomes massless near conifold singularities. The starting point of the story is to look for extended charged black hole solution to the low energy supergravity of 10D superstrings. Consider the following action

$$S = \int d^{10}x\sqrt{-g}\left[e^{-2\phi}[R + 4(\nabla\phi)^2] - \frac{2e^{2\alpha\phi}}{(D-2)!}F^2\right] \tag{5}$$

where F is a $(D-2)$-form field strength satisfying $dF = 0$. We look for black $(10-D)$-brane solutions, i.e. require the symmetry of Euclidean group in $(10-D)$ dimensions. By a redefinition of the metric

$$ds^2 = e^A d\hat{s}^2 + e^B dx_i dx_i$$

this yields a D-dimensional effective action

$$S = \int d^D x\sqrt{-\hat{g}}\left[\hat{R} - \frac{1}{2}(\nabla\rho)^2 - \frac{1}{2}(\nabla\sigma)^2 - \frac{2e^{\beta\rho}}{(D-2)!}F^2\right]$$

The equations of motion are

$$d * e^{\beta\rho}F = 0, \qquad \nabla^2\sigma = 0, \qquad \nabla^2\rho = \tfrac{2\beta}{(D-2)!}e^{\beta\rho}F^2,$$

$$\hat{R}_{\mu\nu} = \tfrac{1}{2}\nabla_\mu\rho\nabla_\nu\rho + \tfrac{1}{2}\nabla_\mu\sigma\nabla_\nu\sigma + \tfrac{2e^{\beta\rho}}{(D-3)!}F_{\mu\alpha_1\cdots\alpha_{D-3}}F_\nu^{\alpha_1\cdots\alpha_{D-3}} - \hat{g}_{\mu\nu}\tfrac{2(D-3)e^{\beta\phi}}{(D-2)(D-2)!}F^2$$

Spherically symmetric solutions that are asymptotically flat takes the form

$$d\hat{s}^2 = -\lambda(\hat{r})^2 dt^2 + \lambda(\hat{r})^{-2}d\hat{r}^2 + R(\hat{r})^2 d\Omega_{D-2}^2$$

In addition we require the scalar fields σ, ρ vanish asymptotically and the field strength

$$F = Q\epsilon_{D-2}$$

where Q is the charge of the black hole and ϵ_{D-2} the standard volume form of S^{D-2}. Solutions of the metric are

$$\begin{aligned} ds^2 = & -\frac{\left[1-(\frac{r_+}{r})^{D-3}\right]}{\left[1-(\frac{r_-}{r})^{D-3}\right]^{\gamma_x-1}}dt^2 + \frac{\left[1-(\frac{r_-}{r})^{D-3}\right]^{\gamma_r}}{\left[1-(\frac{r_+}{r})^{D-3}\right]}dr^2 \\ & + r^2\left[1-\left(\frac{r_-}{r}\right)^{D-3}\right]^{\gamma_r+1} d\Omega_{D-2}^2 + \left[1-\left(\frac{r_-}{r}\right)^{D-3}\right]^{\gamma_x} dx^i dx_i \end{aligned}$$

where

$$\begin{aligned} \gamma_r &= \delta(\alpha-1) - \frac{D-5}{D-3}, \\ \gamma_x &= \delta(\alpha+1), \\ \gamma_\phi &= -\delta(4\alpha+7-D), \\ \delta &= (2\alpha^2+(7-D)\alpha+2)^{-1} \end{aligned}$$

The event horizon is at $r = r_+$ where a curvature singularity occurs at $r = r_-$. They are related with the charge and mass of the black hole by

$$\begin{aligned} Q &= \left[\frac{\gamma(D-3)^3(r_+r_-)^{D-3}}{2\beta^2}\right]^{1/2} \\ M &= [1-(D-3)\gamma]r_-^{D-3} + r_+^{D-3} \end{aligned}$$

The expression is so horrible that I don't even bother to write down explicitly β and γ, which can be found in the paper of Horowitz & Strominger.

A particular interesting case is in type IIB theory, black 3-brane charged under the self-dual 5-form F. The extended black hole carries both electric and magnetic charge. The equation of motion is

$$R_{\mu\nu} = F_{\mu\alpha_1\cdots\alpha_4}F_n u^{\alpha_1\cdots\alpha_4}$$

The solution is

$$\begin{aligned} ds^2 = & -(1-r_+^4/r^4)(1-r_-^4/r^4)^{-1/2}dt^2 + \frac{dr^2}{(1-r_+^4/r^4)(1-r_-^4/r^4)} \\ & + r^2 d\Omega_5^2 + (1-r_-^4/r^4)^{1/2}dx_i dx^i \\ F = & \quad Q(\epsilon_5 + *\epsilon_5), \quad \phi = \phi_0 \end{aligned}$$

The charge is related with the inner and outer horizon by

$$Q = 2r_+^2 r_-^2$$

Due to Hawking radiation they will decay at the quantum level till the extremal limit $r_+ = r_-$, and the metric becomes

$$ds^2 = -(1-r_+^4/r^4)^{1/2}dt^2 + \frac{dr^2}{(1-r_+^4/r^4)^2} + r^2 d\Omega_5^2 + (1-r_+^4/r^4)^{1/2}dx_i dx^i$$

It is shown that the solution preserves half supersymmetries. The mass per 3-volume $\int d^6y\, T_{00}$ saturates the BPS bound. Roughly speaking,

$$M = Q \cdot V_3$$

Non-extremal black holes would have M/Q value greater than the BPS bound, which corresponds precisely to $r_+ > r_-$, the curvature singularity is protected by the event horizon.

Monopoles in $N = 2$ SYM

Recall that $N = 2$, $d = 4$ super-Yang-Mills theory is characterized by a holomorphic function $\mathcal{F}(\mathcal{A})$. Expressed in $N = 1$ superspace, the Lagrangian is

$$\frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A^I}\bar{A}^I + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(A)}{\partial A^I \partial A^J} W_\alpha^I W^{\alpha J}\right]$$

where A is the $N = 1$ chiral multiplet, together with W_α form the $N = 2$ vector multiplet. The Kahler potential is

$$\mathcal{K} = \mathrm{Im}\left(\frac{\partial \mathcal{F}(A)}{\partial A^I}\bar{A}^I \right)$$

Classically $\mathcal{F}$ takes the form

$$\mathcal{F}_0(A) = \frac{2\pi i}{g^2}A^2$$

In the effective action $\mathcal{F}(A)$ receives one-loop correction and instanton corrections

$$\mathcal{F} = \mathcal{F}_0 + \frac{i}{2\pi}A^2 \ln \frac{A^2}{\Lambda^2} + \sum_{k=1}^{\infty} F_k \left(\frac{\Lambda}{A}\right)^{4k} A^2$$

which is exactly solved by Seiberg and Witten using electromagnetic duality. Let Z^I denote the A^I moduli (vacuum expectation value) , $F_I = \partial\mathcal{F}/\partial Z^I$. Dyonic BPS objects with electric charge n_I and magnetic charge m^I have mass

$$M = |n_I Z^I + m^I F_I|$$

Resolving conifold singularities

The extremal black 3-branes are very similar to the situation above. On Calabi-Yau space X one can choose a basis of holomology 3-cycles A_I, B^J, $I, J = 1, \cdots, b_3/2$ satisfying

$$A_I \cap B^J = -B^J \cap A_I = \delta_I^J, \quad A_I \cap A_J = B^I \cap B^J = 0$$

The intersection form is preserved under $Sp(b_3, \mathbf{Z})$. Up to an overall scale there is a unique holomorphic 3-form Ω on X, define

$$F_I = \int_{A_I} \Omega, \quad Z^J = \int_{B^J} \Omega$$

We can think of Z^I's (or F_I's) as projective coordinates on the moduli space $\mathcal{M}$ of complex structures on X. $\{F_I, Z^J\}$ form a section of the projectivized $Sp(b_3, \mathbf{Z})$ bundle over $\mathcal{M}$. The metric on the projective variety $\mathcal{M}$ from the Fubini-Study metric is given by $g_{I\bar{J}} = \partial_I \partial_{\bar{J}} \mathcal{K}$, with the Kahler potential being

$$\mathcal{K} = -\ln\left(iF_I \bar{Z}^I - iZ^I \bar{F}_I\right)$$

The extremal 3-branes can wrap around nontrivial homology 3-cycles in X, which appear as black holes in 4 dimensions. The R-R charges are given by

$$\int_{A_I \times S^2} F = n_I g_5, \quad \int_{B^J \times S^2} F = m^J g_5$$

where S^2 is in the noncompact direction surrounding the black hole. We can write $F = (1 + *)G_I \alpha^I$, with α^I being the harmonic 3-form dual to A_I, then n_I is the charge under $U(1)$ field strength G_I. Up to an overall constant the Kahler and $Sp(b_3, \mathbf{Z})$ invariant BPS mass is

$$M = g_5 e^{\mathcal{K}/2} |m^I F_I - n_I Z^I|$$

When X has conifold singularities one of the homology cycle degenerates, i.e. say, $Z^1 = 0$. The 3-brane with $n_1 = 1$ and all other charges zero would have mass $M = g_5 e^{\mathcal{K}/2} |Z^1|$, vanishes at the conifold. This agrees with the fact that the mass is proportional to the volume of the 3-brane, which approaches zero as B^1 shrinks to zero size.

The hypersurface in $\mathcal{M}$ defined by $Z^1 = 0$ can be encircle by a loop. Transporting around the loop, there is a monodromy in the $Sp(b_3, \mathbf{Z})$ bundle over $\mathcal{M}$

$$Z^1 \to Z^1, \quad F_1 \to F_1 + Z^1$$

Therefore near $Z^1 = 0$,

$$F_1(Z^1) \sim \frac{1}{2\pi i} Z^1 \ln Z^1 + (\text{regular terms}) \tag{6}$$

and the metric diverges as

$$g_{I\bar{J}} \sim \ln(Z^1 \bar{Z}^1)$$

The distance to $Z^1 = 0$ is finite and the curvature blows up. The BPS state with $n_1 = 1$ and other charges zero becomes massless near $Z^1 = 0$ and can be excited in a hypermultiplet H^1. Hence the low energy effective theory involving only the moduli fields breaks down, instead we should consider the effective theory containing the black hole hypermultiplet H^1 and the kinetic term of Z^1 modulous doesn't diverge. Integrating out H^1 gives one-loop correction $F_1(Z^1) \sim \frac{1}{2\pi i} Z^1 \ln Z^1 + (\text{regular terms})$, which agrees with (6).

Conifold transition

The conifold singularities can be locally described as the zero locus

$$\sum_{i=1}^{4} x_i^2 = 0 \tag{7}$$

in $B^8 \subset \mathbf{C}^4$. It is a cone over its intersection with the S^7: $\sum |x_i|^2 = r^2$. Write $x = u + iv$, where $u, v \in \mathbf{R}^4$, the equations of the intersection are

$$u \cdot u = r^2/2, \quad v \cdot v = r^2/2, \quad u \cdot v = 0$$

Obviously u lies on an S^3 and v on the equator, the intersection is S^2 fibered over S^3. The fibration is trivial because S^3 is parallelizable, therefore (7) is a cone over $S^2 \times S^3$. The singularity can be resolved in two ways: blowing up S^3, one gets

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

which is the same as $SL(2, \mathbf{C}) \cong T^*S^3$; or blowing up S^2, one gets

$$\begin{aligned} xu + yv &= 0 \\ zu + wv &= 0 \end{aligned}$$

The resulting spaces are $S^3 \times B^3$ and $S^2 \times B^4$ respectively.

As an example we consider the Calabi-Yau conifold X in $\mathbf{P}^4$ with all singular points on a $\mathbf{P}^2 : x_3 = x_4 = 0$, defined by a quintic homogeneous polynomial

$$f(x) = x_3 g(x) + x_4 h(x)$$

The singular locus is $\mathbf{P}^2 \cap \{g(x) = h(x) = 0\}$, has generically 16 points. By gluing in $S^3 \times B^3$, we get a Calabi-Yau manifold with 16 homology 3-cycles γ^a satisfying

$$\sum_{a=1}^{16} \gamma^a = 0$$

Alternatively, we can blow up the $\mathbf{P}^2$, the resulting Calabi-Yau manifold is defined by

$$\begin{aligned} y_0 x_4 - y_1 x_3 &= 0 \\ y_0 g(x) + y_1 h(x) &= 0 \end{aligned}$$

This corresponds to gluing in copies of $S^2 \times B^4$ with all S^2's homologous to each other (of the same area under the Ricci-flat metric), giving rise to an additional $(1,1)$ harmonic form dual to the S^2. The first Calabi-Yau manifold has Hodge numbers $(h^{2,1}, h^{1,1}) = (101, 1)$, the second one has $(h^{2,1}, h^{1,1}) = (86, 2)$.

Black hole condensation

Back to type IIB compactified on Calabi-Yau space X, extremal black 3-branes can wrap around any of the 16 degenerating cycles $\gamma^a = n^a_I B^I$, with charge $q^a_I = n^a_I$ under the $U(1)^{15}$ field strengths G_I's. We choose a basis for B^I's such that $n^a_I = \delta^a_I, a = 1, \cdots, 15$ and $n^{16}_I = -1$. The mass of the black hole is determined by the BPS bound, $m^a = |n^a_I Z^I|$, up to a constant. As $Z^I \to 0$, the 16 charged hypermultiplets H^a become massless. Denote the two charged complex scalars of H^a by $h^{a\alpha}(\alpha = 1, 2)$, transforming as $\mathbf{2}$ of the global $SU(2)$. The most general potential for these scalar fields restricted by supersymmetry takes the form

$$V = \sum_{I,J=1}^{15} M^{IJ} D^{\alpha\beta}_I D_{J\alpha\beta}$$

where M^{IJ} is positive definite, and

$$D^{\alpha\beta}_I = \sum_{a=1}^{16} n^a_I (h^{*a\alpha} h^{a\beta} + h^{*a\beta} h^{a\alpha})$$

The flat directions are $D^{\alpha\beta}_I = 0$, which precisely correspond to that the hypermultiplets pick up vacuum expectation value $h^{a\alpha} = v^\alpha$ for all a, where v is a complex constant vector. The VEV's break all $U(1)^{15}$, hence the corresponding vector multiplets pick up mass. There is still one massless hypermultiplet remains. Moving along the flat direction of V, we have $101 - 15 = 86$ massless vector multiplets, and $2 + 1 = 3$ massless hypermultiplets. This is precisely the massless spectrum of the $(86, 2)$ model. We come to the crucial observation: passing from one Calabi-Yau moduli space to another through the conifold point, massless vector multiplets of type IIB fundamental string modes becomes massive, while one of the black hole hypermultiplets (a linear combination) becomes massless.

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