# Physical interpretation of Hopf fibration

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### Abstract

Hopf fibration $S^3 \overset{S^1}{\to} S^2$ is one of the simplest example of non-trivial fibration. It is interesting to note that the description of a physical system called qubit is given by the hopf fibration, and the non-triviality of the fiber has a physical meaning. I present the hopf fibration from physics point of view and extend the hopf fibration to describe two qubits system, as was originally done by Mosseri *et.al*[1] .

## 1 $S^3 \overset{S^1}{\longrightarrow} S^2$

### 1.1 General construction

Let $S^3$ be described as $\{(a,b) \in \mathbb{C} \times \mathbb{C} : |a|^2 + |b|^2 = 1\}$. Then the hopf map is given by the composition of two maps $h_2 \circ h_1$:

$$\begin{array}{rccc} h_1 & : & S^3 & \to & \mathbb{C} \cup \{\infty\} \\ & & (a,b) & \mapsto & \overline{ab^{-1}} \\ h_2 & : & \mathbb{C} \cup \{\infty\} & \to & S^2 \\ & & x+yi & \mapsto & (x\cos\theta, y\cos\theta, \cos\theta), \tan\theta = \sqrt{x^2+y^2} \end{array}$$

$h_2$ is just the inverse of stereographic map. The complex conjugation of the map $h_1$ is not necessary, but is added there so that we can analogously construct the fibration $S^7 \overset{S^3}{\to} S^4$ It is easy to see that $e^{i\theta}(a,b) \in S^7$ is mapped to the same point in $S^2$ for all $\theta \in [0, 2\pi]$. We see that the lift is locally trivial (looks like the product $S^1 \times D^2(\epsilon)$ where $D^2(\epsilon)$ is the disk with radius $\epsilon$ ), therefore, this is a fiber bundle. Non-triviality of this fiber bundle implies that $S^3 \neq S^2 \times S^1$. It is well known that $\pi_3(S^2) = \mathbb{Z}$ and this hopf map is the generator of the group $\pi_3(S^2)$.

The idea is almost the same, but one way to define the hopf map without going to $\mathbb{C} \cup \{\infty\}$ is by sending

$$h : \begin{array}{ccc} S^3 & \to & \mathbb{CP}^1 \\ (a,b) & \mapsto & [a,b] \end{array}$$

where the equivalent class $[a, b]$ is given by the relation $(a, b)\ (za, zb)$ for any $z \in \mathbb{C}$. Now since $\mathbb{CP}^1 \simeq S^2$, $h$ is the hopf map.

Another way to get $S^3 \overset{S^1}{\to} S^2$ is by noting the homeomorphisms $S^1 \simeq U(1), S^3 \simeq SU(2)$. Consider the map $f : U(1) \to SU(2), e^{i\theta} \mapsto \left( \begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array} \right)$. Then $S^2$ can be obtained as a left coset of the subset $T = f(U(1))$, i.e. $T = \{\left( \begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{array} \right) : \theta \in [0, 2\pi)\}$. Then there is a homeomorphism $SU(2)/U(1) \simeq S^2$ . We do not use this description of hopf fibration, so I leave the details.

### 1.2 Physical interpretation

Now I'd like to describe the physical interpretation of hopf fibration. Physical system that we consider is spin-1/2 space, called a qubit. It's a two dimensional vector space with complex coefficient, so that the general vector is given by $\left( \begin{array}{c} a \\ b \end{array} \right)$, with $a, b \in \mathbb{C}$. Physicists use the notation $|\psi >$ to denote the vector with $|0 >$ and $|1 >$ the orthonormal basis of the space, so that the general vector in this space is given by $|\psi >= a|0 > +b|1 >$. The hermitian conjugate of the matrix $|\psi >$ is denoted by $< \psi|$ and naturally, the standard inner product of two vectors is denoted by $< \phi|\psi >$. Physically, $|\phi >$ is a spin state. Spin is measured as a magnetic moment ( a vector in $R^3$) in real space. Therefore, a state of a spin, $|\psi >$, essentially contains the information of which direction the spin is directed to in $R^3$. Now we only want to consider normalized vectors $|\psi >$ $(< \psi|\psi >= 1)$. This corresponds to setting the sum of probabilities of measurement equal to 1. (quantum mechanics gives only the probability of measurement. In our case, if we measure the z-component of the spin, we get a number $-\frac{\hbar}{2}$ with probability $|a|^2 = | < 0|\psi > |^2$ and $\frac{\hbar}{2}$ with probability $|b|^2 = | < 1|\psi > |^2$. The normalization condition gives $|a|^2 + |b|^2 = 1$. ) Therefore, this qubit space is $S^3 = \{(a, b) \in \mathbb{C} \times \mathbb{C} : |a|^2 + |b|^2 = 1\}$.

As I mentioned above, spin state contains the information of which direction this spin is pointing at in $R^3$. Explicitly, the projection map of this vector onto each coordinate is given by the following

$$X =< \psi|\sigma_x|\psi >, \qquad \text{where} \quad \sigma_x = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$$

$$Y =< \psi|\sigma_y|\psi >, \qquad \text{where} \quad \sigma_y = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right)$$

$$Z =< \psi|\sigma_z|\psi >, \qquad \text{where} \quad \sigma_z = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$

Since the interpretation of these numbers are the projection of a unit vector in $R^3$ onto $x, y, z$ coordinates, we expect $X^2 + Y^2 + Z^2 = 1$. Indeed, this is the case. What's more, this map $(a, b) \to (X, Y, Z)$ is exactly the hopf map described above. So hopf map is the map from a spin state into a unit vector in $R^3$ that the spin state represents in real space. Non-triviality of the fibration indicates the peculiarity of spin space. Let me note to the readers with physics background that the fact that this hopf fibration has no global bundle section helps us to understand the lack of topological point defect in the spin-1/2 ferromagnetic BEC.

This relation between hopf map and qubit goes further than this. In the next section, I'd like to analogously construct the hopf map $S^7 \xrightarrow{S^3} S^4$, using quarternions[1]. This map gives us the description of two-qubits system.

# 2 $S^7 \xrightarrow{S^3} S^4$

## 2.1 Quarternions

Before constructing the hopf map $S^7 \xrightarrow{S^3} S^4$, let me briefly describe the quarternions, an extension of an imaginary number. We consider the three imaginary numbers $i, j, k$ with the relation $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. Then the general element of the quarternion $Q$ is given by $q = a_1 + a_2 i + a_3 j + a_4 k$, where $a_i \in \mathbb{R}, i = 1, 2, 3, 4$. In particular, quarternions is non-commutative. The conjugate of $q$ is given by $\overline{q} = a_1 - a_2 i - a_3 j - a_4 k$. As w can check, $\overline{q}q = |q|^2 = \sum_{i=1}^4 a_i^2$. We can describe the quarternion through the usual imaginary number as $q = c_1 + c_2 j$ such that $c_1 = a_1 + a_2 i$, $c_2 = a_3 + a_4 i$. Now we are ready to construct $S^7 \xrightarrow{S^3} S^4$.

## 2.2 Construction

Consider $S^7 = \{(a, b, c, d) \in \mathbb{C}^4 : |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\}$, and let $q_1 = a + bj, q_2 = c + dj$ (this choice of $q_1$ and $q_2$ is rather arbitrary[1]). Then the hopf map is given by the composition of two maps $h_2 \circ h_1$as for $S^3 \xrightarrow{S^1} S^2$, where the $h_2$ is just inverse of stereographic map, and $h_1$ is given by

$$h_1 \; : \; \begin{array}{ccc} S^7 & \to & \overline{\mathbb{C}^2 \cup \{\infty\}} \\ (q_1, q_2) & \mapsto & q_1 q_2^{-1} \end{array}$$

## 2.3 Physical system

The physical system that corresponds to this hopf map is two-qubits. It is a tensor product of two identical qubit space given above, $H_1 \otimes H_2$. The basis of this vector space is $\{|00>, |01>, |10>, |11>\}$ where $|00>=|0>_1$ $\otimes|0>_2$, etc. The general vector in $H_1 \otimes H_2$ is given by $|\psi>=a|00>$ $+b|01>+c|10>+d|11>$, with $a, b, c, d \in \mathbb{C}$. Define the elements of quartenions from imaginary numbers as before, $q_1 = a + bj$, $q_2 = c + dj$. Then we can consider a vector in $H_1 \otimes H_2$ as two tuples. Then the hopf map given above is the following map[5].

$$
\begin{aligned}
x_0 &= <\psi|\sigma_z|\psi>, && \text{where} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
x_1 &= <\psi|\sigma_x|\psi>, && \text{where} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
x_2 &= <\psi|\sigma_{y_i}|\psi>, && \text{where} \quad \sigma_{y_i} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
x_3 &= <\psi|\sigma_{y_j}|\psi>, && \text{where} \quad \sigma_{y_j} = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} \\
x_4 &= <\psi|\sigma_{y_k}|\psi>, && \text{where} \quad \sigma_{y_k} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}
\end{aligned}
$$

So the analogy with $S^3 \xrightarrow{S^1} S^2$ is clear and clean. However, this way of writing the map does not give us the physical interpretation immediately. We'll see that $x_3$ and $x_4$ of this map gives the quantity called "entanglement." Entanglement is a non-local relation between the firts qubit $H_1$ and the second qubit $H_2$. As a motivation, I will explain what entanglement is in the next section.

## 2.4 Entanglement

Entanglement attracts physicists' attention for its utility in quantum computation, quantum teleportation, quantum cryptography, etc [3]. It is indeed this property of quantum system that makes quantum information different from classical information. First of all, let me give you the definition of "entangled state."

**Theorem 1 ( Schmidt decomposition )** *Suppose $|\psi>$is a pure state of a composite system, $H_1 \otimes H_2$, where $H_1$ and $H_2$ are n dimensional Hilbert space with complex coefficients. Then there exist orthonormal states $|i>_{H_1}$ for system $H_1$, and orthonormal states $|i>_{H_2}$ of system $H_2$ such that*

$$|\psi>=\sum_{i=1}^{k} \lambda_i |i>_{H_1} \otimes |i>_{H_2}$$

*where $\lambda_i$ are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt co-efficients. The number of Schmidt coefficient in the decomposition (in the case above, $k$) is called Schmidt number.*

**Definition 1** *A composite state in $H_1 \otimes H_2$ is called entangled state iff Schmidt number of the state is bigger than 1. When a state $|\psi >$ is not entangled, we call the state separable, since $|\psi >= |\phi >_1 \, |\phi >_2$.*

One example of entangled state (in fact, this is a maximally entangled state) in $H_1 \otimes H_2$ where $H_1$ and $H_2$ are qubits, is $|\phi^+ >_{H_1 H_2} = \frac{1}{\sqrt{2}}(|00> + |11>)$.

The entangled state contains its information about the state in non-local relation between two spaces $H_1$ and $H_2$. In other words, one cannot re-construct an entangled state, given the information that can be obtained by the local measurements in $H_1$ or $H_2$.

Another property of entangled states is that the local operation (operation on $H_1$) changes the state in the other space $H_2$. Therefore, the two spaces are "entangled."

It is not obvious how one can quantify this notion of entanglement. However, research has been done recently and physicists show that the measure of entanglement is closely related to the notion of entropy that one frequently encounter in the study of thermodynamics. Entropy is a measure of randomness. Intuitively, the connection between entropy and entanglement is reasonable.

Explicitly, " the entanglement in the state $|\psi >$" is given by $|C(\psi)|$ where $C(\psi)$ is given by [4]

$$C(\psi) = < \psi|\sigma_y \otimes \sigma_y|\psi^* > \quad \text{where} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$|\psi^* >$ is complex conjugation of $|\psi >$. In particular, we have $C(\psi) = 0$ for separable state.

Now we can re-interpret the hopf map$S^7 \overset{S^3}{\rightarrow} S^4$. The map is actually equivalent to

$$\begin{aligned} x_0 &= < \psi|\sigma_z \otimes Id|\psi > \\ x_1 &= < \psi|\sigma_x \otimes Id|\psi > \\ x_2 &= < \psi|\sigma_y \otimes Id|\psi > \\ x_3 &= Re(C(\psi)) \\ x_4 &= Re(C(\psi)) \end{aligned}$$

so that $x_3$ and $x_4$ (in fact, $\sqrt{x_3^2 + x_4^2}$) gives "the amount of entanglement" of $|\psi >$. Hopf map is entanglement sensitive.

### 2.5 Separable State

There is really an intimate relation between two-qubits space and hopf fibration. The expression of hopf map above gives us the idea that the first 3-tuple $(x_0, x_1, x_2)$ is simply the information of the first qubit space, $H_1$, and the last two-tuple $(x_3, x_4)$ is the information about entanglement, that is, the global relation between two qubits. Then we can expect that the map kills (or integrate) the information about the second qubit space $H_2$ during the process. We can see that the fibre $S^3$ contains the information of $H_2$ for separable case in the following way. In the separable state case, we have $x_3 = x_4 = 0$. Since we have $S^3 \overset{S^1}{\rightarrow} S^2$ for one qubit system, we expect that the hopf map to give us $S^7 \rightarrow S^2 \otimes S^2$. The first $S^2$ is obviously given by $(x_0, x_1, x_2)$ above. It is interesting to note that if we make an appropriate choice of basis for $S^3$(the fiber) in $S^7$ , and use the hopf map $S^3 \overset{S^1}{\rightarrow} S^2$ to project onto three-tuples, we have [1] (three-tuples are denoted by $(y_0, y_1, y_2)$ )

$$\begin{aligned} y_0 &=< \psi|Id \otimes \sigma_z|\psi > \\ y_1 &=< \psi|Id \otimes \sigma_x|\psi > \\ y_2 &=< \psi|Id \otimes \sigma_y|\psi > \end{aligned}$$

Therefore, we expect that understanding the topology of a hopf map $S^7 \overset{S^3}{\rightarrow} S^4$ gives us a deep understanding of two qubit system. As far as I know, this has not been done by physicists yet, so maybe some mathematicians can give us an interesting insight.

## 3 Conclusion

Mathematics has been more than a tool for physics( General relativity of Einstein (differential geometry) and Quantum Mechanics (Linear algebra) are two examples that come up to my mind ). The relation depicted here between entanglement and hopf maps could be one of such examples. Believing that this intimate relation goes further in nature, physicists try to generalize the hopf map further and try to understand qubits system (see the efforts to extend the idea above to three qubits, using octernion in [5]).It was to mathematicians' surprise that, conversely, physics could be "a tool" for mathematicians. One example is the proof and extension of Morse Inequality by a great physicist and mathematician, Edward Witten[6]. It is when this happens that a mathematician gets excited about physics. I hope that I have conveyed the intimate relation between physics and mathematics through this topic, and get some students in mathematics interested in physics.

References

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