The idea of Morse Theory

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Abstract

Since Morse first presented his theory at the beginning of 20th century, there has been a great development of algebraic topology, differential topology and differential geometry based on his theory: h-cobordism by Smale, proof of the existence of exotic sphere by Milnor, and periodicity theorem by Bott to name a few. I do not, in the least, mean to exhaust the list of useful ideas coming out of Morse Theory in this short paper. This paper is meant to cover some interesting applications of Morse Theory within the reach of students who start learning differential topology. I hope to convey the excitement I felt when I learned about Morse Theory through this paper. Contained in this paper is some relation of Morse theory with physics.

1 Morse function and CW-complex

1.1 The relation between critical points and the structure of manifolds

Let M be a finite dimensional, differentiable manifold (from now on, manifold will automatially mean differentiable manifold). Also let f be a smooth function $f : M \to \mathbb{R}$. Morse Theory is a theory that connects the topology of M and the behavior of f on M. In other words, we get to know the topology (homotopy class) of manifold M by just studying a function defined on it. Although this connection between topology and analysis is fascinating, this should not be too much surprise for those who have studied de Rham cohomology. de Rham cohomology allows us to differentiate homotopy classes of manifolds by investigating forms. For example, Poincare Lemma tells us that the cohomology group $(H^*(M))$ is always zero whenever the manifold M is star-shaped. However, there is an important difference between cohomology and Morse Theory; whereas cohomology starts with smaller, local parts of manifolds and ascend to higher cohomology group by using tools such as Mayer-Vietris sequence, Morse Theory allows us to grasp the global structure at start. Let me define few terms and describe what the precise statement of the theory is.

A critical point p of a function $f : M^n \to \mathbb{R}$ on a n-dimensional manifold M^n is a point such that $Df|_p = 0$. In local coordinates, this is equivalent to $\frac{\partial f}{\partial x_1}|_p = \cdots = \frac{\partial f}{\partial x_n}|_p = 0$. Also, a point q is called non-degenerate whenever the Hessian of f is non-singular at q, i.e. $det([H_{ij}]) = det([\frac{\partial f}{\partial x_i \partial x_j}|_q]) \neq 0$. Now a function $f : M \to \mathbb{R}$ is called a Morse function if any critical point of f is non-degenerate. From now on, we only consider a Morse function f. An index of f at $p \in M$ is the dimension of vector space in which Hessian of f at p is negative definite bilinear form.

Now we can state an important result of Morse theory ([Milnor]).

Theorem 1 *Let $M^a = p \in M : f(p) \leq a$. If f is a C^∞ function on a manifold M with no degenerate critical points, and if each M^a is compact, then M has the homotopy type of a CW-complex, with one cell of dimension λ for each critical point of index λ.*

CW-complex X is intuitively a union of n-disk, $e^n = \{x \in R^n : |x| \leq 1\}$ quotiented by gluing (quotient) maps $\phi : \partial e^n \to X$, where ∂e^n is the boundary of n-disk, $n-1$-sphere S^{n-1}. A precise description of CW-complex can be found in [Hatcher].

I will not prove the theorem above, but give a general idea of how one can proceed to prove it. There are three main steps. First of all, one wants to prove

Theorem 2 (No critical point) *Let $a < b$ and supposed that the set $f^{-1}([a, b])$ has no critical point in M and is compact. Then M^a is a deformation retract of M^b.*

The deformation retract of M^b to M^a can be obtained by following the "flow lines" of f. That is, we have the vector field Df on $M^b - M^a$ and can find a curve $c : \mathbb{R} \to M$ such that $c_*(\frac{\partial}{\partial t}|_t)$ is precisely $Df|_{c(t)}$, where c_* is a push-forward map between tangent spaces, i.e. $c_* : T\mathbb{R}_t \to TM_{c(t)}$. Note that we can find such flow line from $M^a \to M^b$ precisely because Df does not terminate in the middle point, i.e. Df is non-zero in $M_b - M_a$.

Then, we want to investigate the critical point next.

Theorem 3 (one critical point) *Let p be a non-degenerate critical point with index λ, and $f(p) = c$. Then take ϵ small enough so that $f^{-1}([c-\epsilon, c+\epsilon])$ is compact and contains only one critical point, namely p. Then $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

To guarantee the existence of such ϵ, we need the following Lemma.

Lemma 1 (Lemma of Morse) . *Let p be a non-degenerate critical point for f. Then there is a local coordinate system $(x_1, \cdots, x_n)$ in a neighborhood U of p with $x_i(p) = 0$ for all i and such that the identity*

$$f = f(p) - (x_1)^2 - \cdots - (x_\lambda)^2 + (x_{\lambda+1})^2 + \cdots + (x_n)^2$$

holds throughout U, where λ is the index of f at p.

Note that this lemma tells us that non-degenerate critical point is isolated, therefore, on compact manifold, there are only finitely many non-degenerate critical points.

The proof of Thereom 3 is similar to that of Theorem 2. We follow the flow lines as much as possible. Then the best one can do is to leave the handle with dimension λ. One should see the picture on [Milnor] p15.

Now that we know M is a disk with bunch of handles, we only need to make sure they have the same homotopy class as CW-complex. This is the third step. To see what this means, let's say we wan to glue e^n to X with the quotient map ϕ. Then what we want to show is that the composition of the homotopy equivalent target space ($X \cong Y$) and homotopic gluing map ($\phi \cong \psi$) will lead to the same homotopy class, so that $X \bigcup_\phi e^n \cong Y \bigcup_\psi e^n$. See [Milnor] for details.

1.2 Manifolds homeomorphic to spheres

As one of my favorite application of the idea above, I would like to present the following statement stated in [Matsumoto].

Theorem 4 *Suppose M^n is a compact n-dimensional manifold and that a Morse function f on M has only two critical points on M. Then M^n is homeomorphic to a sphere S^n. In fact, when $1 \leq n \leq 6$, M^n is diffeomorphic to S^n.*

Using the Theorem 1, the first statement of this theorem can be obtained almost for free.

Proof Since M is compact, f attains its maximum and minimum on M. Therefore, the two critical points correspond to maximum point p and minimum point q so that index of p is n and index of q is zero. Let $f(p) = b$ and $f(q) = a$. Then with ϵ small enough, we know by Theorem 1that $f^{-1}([a, a+\epsilon])$ is a n-disk, which is contractible. Using the theorem 1, we also know that $apoint \bigcup e^n$ is deformation retract of M. $e^0 \bigcup e^n$ is nothing but one point compactification of $\mathbb{R}^n$, which is homeomorphic to S^n. Therefore $M \cong S^n$.

In order to describe the diffeomorphism, let's describe the decomposition into disks in a different way. By symmetry (or by using the function $-f$), we can have $e^n \bigcup e^n$ with gluing map $\phi : S^{n-1} \to S^{n-1}$ as deformation retract of M. The gluing map is in fact diffeomorphism. Therefore, notice that everything up to the identification that two disks are homeomorphic to a sphere, we have diffeomorphisms in the proof above. Indeed, the diffulty to turn this homeomorphism into diffeomorphism prevents us from proving the statement above when $n > 6$. In order to have the diffeomorphism between $e^n \bigcup_\phi e^n$ and S^n, it suffices to prove the following lemma [Matsumoto].

Lemma 2 *Let $k : \partial D_0 \to \partial D_1$ be a diffeomorphism between the respective boundaries of two n-dimensional disks D_0 and D_1. Then we can extend k to a diffeomorphism $K : D_0 \to D_1$ of disks.*

Once we have the lemma above, then we can imagine the map S^n to $D_0 \bigcup D_1$ in the following way. Identify the upper hemisphere of S^n with D_0. Then the possible difficulty of diffeomorphism occurs when we cross "equator" into lower hemisphere of S^n, D_1. Consider the curve $c : \mathbb{R} \to D_0$ that goes into boundary of D_0. Then we can walk into D_1 smoothly by extending the curve c through the composition $K \circ c(-t)$, using the smooth map K above. Now the proof of the lemma above for $n \leq 6$ is done, but each dimension seems to need its own unique method. I'll omit the proof. See [Matsumoto] for its beautiful proof of this Lemma for $n = 2$ case.

This extension of map turns out to be possible only up to $n \leq 6$. Milnor showed that there exists more than one differentiable structure on S^7 (an object homeomorphic to S^7 is not necessarily diffeomorphic to S^7), thereby eliminating the possibility of extending the theorem 4 into bigger dimensions. This is one very interesting, surprising result.

1.3 example: SO_n

To illustrate how one can find the index of critical point, I'll find the critical points and indexes of Morse function on SO_n, following closely [Matsumoto]. The CW complex of SU_n can be found with almost analogous Morse function and method as described below. For details of SU_n, please refer to [Matsumoto]. As a Corollary, we obtain that $SU_2 \cong S^3$ and that SU_2 is simply connected.

First of all, we need to decide what Morse function we want to use. It turns out that that the nice function,

$$f(A) = c_1 x_{11} + \cdots + c_n x_{nn} \quad \text{where}$$
$$A = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

$$0 \le c_1 < c_2 < \cdots < c_n$$

will do the job.

Claim 1 *The Morse function f defined above has critical points at*

$$A = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \pm 1 \end{pmatrix}$$

The sings of 1 are such that the determinant of A is 1

proof

$\Rightarrow$ Suppose A is critical point. Then the derivative of f should be always zero. I consider a curve given by a rotation of first and second coordinate $B_{12}(\theta)$ defined by

$$B_{12}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & \cdots & \cdots & 0 \\ \sin\theta & \cos\theta & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}$$

$B_{12}(\theta) \in SO_n$ and the multiplication by B_{12} on A gives us the curve in the space of SO_n. Therefore, we require that $\frac{d}{d\theta} AB_{12}(\theta) = \frac{d}{d\theta} B_{12}(\theta)A = 0$. This give us

$$\begin{aligned} c_1 x_{12} - c_2 x_{21} &= 0 \\ -c_1 x_{21} + c_2 x_{12} &= 0 \end{aligned}$$

Solving for x_{12}, x_{21} gives $x_{12} = x_{21} = 0$. We can carry out the similar calculation for $B_{ij}(\theta)$ with $i < j$ (B_{ij} is a matrix with entry $(i,i) = \cos\theta, (i,j) = \sin\theta, (j,i) = -\sin\theta, (j,j) = \cos\theta.$) and conclude that $x_{ij} = 0$ whenever $i \neq j$. So we conclude any critical point of f is a diagonal matrix. But since $A \in SO_n$, we should have $A^t A = I$ where I is the identity matrix. So each entry has to be ± 1.

On the other hand suppose that A takes the form above. In order to check it is indeed the critical point, we need to compute the derivative of f. The dimension of SO_n is $(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$ (consider choosing an orthonormal basis in $\mathbb{R}^n$, one vector at a time). If we could find $\frac{n(n-1)}{2}$ curves C_i that goes through A with velocity vecotr at A linearly independent from each other, we only need to check that the derivative of $f(C_i)$ vanishes to see $Df = 0$. This is because the velocity vector of C_i at

A plays a role of a local coordinate for A. Now, I claim $AB_{ij}(\theta)$s are in fact C_is. The derivative of $AB_{ij}(\theta)$ at A is (I do it for B_{12} case, but it's all the same for others)

$$\frac{d}{d\theta}AB_{12}(\theta)|_{\theta=0} = \begin{pmatrix} 0 & -\epsilon_1 & \cdots & \cdots & 0 \\ \epsilon_2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

where $\epsilon_1 = A_{11}$ and $\epsilon_2 = A_{22}$. It's quite easy to check these matrix for B_{ij} are all linearly independent.

Therefore, we check $\frac{d}{d\theta}AB_{ij}(\theta) = 0$. but this is exactly what we checked before. So, in fact A of the form above are critical points. $\square$

Now that we know the coordinate system of SO_n and the critical points, it is straightforward to calculate the Hessian of f at A. As before, suppose A is a diagonal matrix and $A_{ii} = \epsilon_i$, where $\epsilon_i = \pm 1$. Then we want to compute $\frac{\partial^2}{\partial\theta\partial\phi}f(AB_{\alpha\beta}(\theta)B_{\gamma\delta}(\phi))|_{\theta=0,\phi=0}$. But notice that $AB_{\alpha\beta}(\theta)B_{\gamma\delta}(\phi)$ is linear in θ and ϕ, and f is a linear function. Therefore, we can take the derivative inside.

$$\begin{aligned} \frac{\partial^2}{\partial\theta\partial\phi}f(AB_{\alpha\beta}(\theta)B_{\gamma\delta}(\phi))|_{\theta=0,\phi=0} &= f\left(A\frac{\partial}{\partial\theta}B_{\alpha\beta}(\theta)\frac{\partial}{\partial\phi}|_{\theta=0}B_{\gamma\delta}(\phi))|_{\phi=0}\right) \\ &= \begin{cases} -c_\alpha\epsilon_\alpha - c_\beta\epsilon_\beta & \text{if } \alpha=\gamma,\ \beta=\delta \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The calculation above becomes easier if we use the matrix multiplication $c_{ij} = \sum_k a_{ik}b_{kj}$. Note that whenever I write B_{ij} I assume that $i \leq j$.

The calculation above shows that the Hessian is diagonal. Since $c_\alpha \neq c_\beta$ for $\alpha \neq \beta$, the entries are non-zero. Therefore, f is non-degenerate. Let I_j be the subscript for which $\epsilon_{I_j} = -1$. Then the index of f at A is given by $(I_1 - 1) + \cdots + (I_k - 1)$ as one can check.

2 Morse Inequality

2.1 Theorem

So far, I have presented how one can obtain the structure of CW-complex, using a Morse function. But often, what we are interested is quantity that is invariant under homotopy equivalence, namely homology group. Indeed, Morse's original presentation of theory contains a resul that relates the number of critical points on M and homology group.

Let $R_\lambda(M)$ be $dim(H_\lambda(M))$, rank of λth homology group. Also let C_λ be the number of critical points with index λ. Then *Morse inequality* states that ([Milnor])

Theorem 5 (Morse's inequalities)

$$R_\lambda(M) - R_{\lambda-1}(M) + \cdots \pm R_0(M) \leq C_\lambda - C_{\lambda-1} + \cdots \pm C_0$$

for all λ*.*

An interesting Corollary follows immediately

Corollary 1 *If* $C_{\lambda+1} = C_{\lambda-1} = 0$ *then* $R_\lambda = C_\lambda$ *and* $R_{\lambda+1} = R_{\lambda-1} = 0$.

proof Let the equation in the theorem 5 be $E(\lambda)$. Then $E(\lambda+1) + E(\lambda)$ gives $R_\lambda \leq C_\lambda$. So if $C_{\lambda+1} = 0$, then $R_{\lambda+1} = 0$. However, this implies from $E(\lambda+1)$ and $E(\lambda)$ that

$$R_\lambda(M) - R_{\lambda-1}(M) + \cdots \pm R_0(M) = C_\lambda - C_{\lambda-1} + \cdots \pm C_0$$

Let this equation be $E'(\lambda)$. Then if $C_{\lambda-1} = 0$, then we obtain $E'(\lambda-2)$ and $R_{\lambda-1} = 0$. It easily follows from $E'(\lambda) - E'(\lambda-2)$ that $R_\lambda = C_\lambda$. $\square$

2.2 Homology group of $\mathbb{CP}^n$

I would like to use the results from previous sections to compute the homology group of $\mathbb{CP}^n$. I certainly remember computing the cohomology group of $\mathbb{CP}^n$ in my mid-term for differential topology class.

We can find the CW-complex structure of $\mathbb{CP}^n$ by thinking about it, suggested by [Hatcher]. Consider $\mathbb{CP}^n$ as a complex n-sphere with antipodal points identified, or a real $2n$-sphere with antipodal points identified. Then each point of $\mathbb{CP}^n$ can be written as $[z_1, z_2, \cdots, z_{n+1}]$ with $|z_1|^2 + \cdots + |z_{n+1}|^2 = 1$. $[,]$ denotes an equivalent class. The equivalent relation is given by $(z_1, z_2, \cdots, z_n) \sim \lambda(z_1, z_2, \cdots, z_{n+1})$ with $\lambda = e^{i\theta}$ for some $\theta \in R$. Let's divide this object into two. One is when $z_1 = 0$. Then we obtain $\mathbb{CP}^{n-1}$. Suppose $z_1 \neq 0$. Then we can identify an element $[z_1, z_2, \cdots, z_{n+1}]$ of $\mathbb{CP}^n$ with $\lambda(z_1, z_2, \cdots, z_{n+1}) = (x_1, z_2, \cdots, z_{n+1}) = (\sqrt{1-K^2}, z_2, \cdots, z_{n+1})$ where $\lambda = \frac{|z_1|}{z_1}$ is chosen to make x_1 real and positive, and $K = \sqrt{|z_2|^2 + \cdots + |z_{n+1}|^2}$. Now each$(z_2, \cdots, z_{n+1})$ gives us a complex $n-1$ sphere with radius K. Conveniently, $K = 0$ corresponds to a point $(1, 0, \cdots, 0)$ so that in fact, $\mathbb{CP}^n$ with $z_1 \neq 0$ is the same as complex open n disk. Therefore $\mathbb{CP}^n = \mathbb{CP}^{n-1} \bigcup e^{2n}$. By induction, we obtain $\mathbb{CP}^n$ is the same homotopy class as $e^0 \bigcup e^2 \cdots \bigcup e^{2(n-1)} \bigcup e^{2n}$. If $\mathbb{CP}^n$ admits Morse function at all, then we already know the number of critical points and the indexes, using Theorem 1!! This is an interesting change of point of view. In a sense, this is a feedback from topology to analysis.

Now that we understand the structure of $\mathbb{CP}^n$ through thinking, it's relatively easy to reprove the statement $\mathbb{CP}^n \cong e^0 \bigcup e^2 \cdots \bigcup e^{2(n-1)} \bigcup e^{2n}$, using Morse function ([Milnor]). Define a function $f : \mathbb{CP}^n \to \mathbb{R}$ by

$$f([z_1 \cdots z_{n+1}]) = \sum_i c_i |z_i|^2$$

where $0 < c_1 < c_2 \cdots < c_{n+1}$. Note that the function is well defined because $|z_i|$ is the same for all z_i in the equivalent class. I claim this is a Morse function. Now let's find the critical point of f. Suppose $z_1 \neq 0$. Then as we have seen above, the set of elements $U_1 = \{[z_1, \cdots, z_{n+1}] \in \mathbb{CP}^n : z_1 \neq 0\}$ maps into complex n-sphere diffeomorphically. Let each element of U_1 be $(\sqrt{1-K^2}, z_2, \cdots, z_{n+1})$ and assign the coordinates $(x_2, y_2 \cdots y_{n+1})$ by $z_i = x_i + iy_i$. Then f and Df becomes

$$f = c_1 + \sum_{i=2} (c_i - c_1)(x_i^2 + y_i^2)$$
$$Df = (0, (c_2 - c_1)x_2, (c_2 - c_1)y_2, \cdots, (c_{n+1} - c_1)y_{n+1})$$

Therefore, possible critical point is $[1, 0, \cdots, 0]$. Similarly, we can consider the set U_2 such that $z_2 \neq 0$, etc. Then we obtain n critical points, p_i Let $p_1 = [1, 0, \cdots, 0], \cdots p_n = [0, \cdots, 0, 1]$. Then from the expression of Df above and the definition of c_i, it's easy to see that index of f at p_1 is $2n$, $\cdots$ p_n is 0. Therefore, following Theorem 1, we find that $\mathbb{CP}^n$ is the same homotopy class as $e^0 \bigcup e^2 \cdots \bigcup e^{2n}$.

Now using the Corollary 1, we easily find that

$$H_i(\mathbb{CP}^n, Z) = \begin{cases} Z & \text{for } i = 0, 2, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

2.3 Proof of Morse inequality by Witten

It is interesting that Witten re-proved the Morse inequalities from physics concepts. Unfortunatelly, my knowledge of physics concerning supersymmetry is not sufficient to describe the full theory. Therefore one should refer to Witten's original paper for the correct and precise theory ([Witten]). The description of his idea below follows closely his paper [Witten]. I will only prove the weak form of the Morse inequality, i.e. $C_\lambda \geq R_\lambda$.

Now the rough idea of how physics can have anything to do with mathematics is the following. In quantum mechanics, we consider an operator (measurements) acting on a certain states. In mathematical equations, this can be thought of as differential (such as $K = \frac{d}{dx}$) acting on a function(such as $f = \cos x$). Then the result of the measurement gives you some quantity corresponding to the operator. The assumption in physics is that the eigenvalue of the operator corresponds to the observed quantity (so in our case, $Kf(x) = -f(x)$ so the observed quantity corresponding to the measurement (operator) is -1). Therefore, we measure the energy (operator represented by H) of a particle in a certain state, and obtain the energy, which must be the eigenvalue of the energy operator.

In the theory of supersymmetry, we consider two symmetry operators Q_1, Q_2 that are related to Energy operator (Hamiltonian) H. The relation is written as $Q_i^2 = H$. Now, the question that physicists are interested is whether the operator Q_i can have a zero eigenvalue, i.e. if there exists an eigenstate $|\phi >$ such that $Q_i|\phi >= 0$. Note that if $Q_i|\phi >= 0$, then $H|\phi >= 0$ too. If we consider the operator corresponding Q_i to be exterior derivative and the eigenstate to be differential forms, we can use differential topology to answer some of the questions posed by physics.

Following Witten's paper, let's consider

$$\begin{aligned} Q_1 &= d_t + d_t^*, \quad Q_2 = i(d_t - d_t^*), \quad H = d_t d_t^* + d_t^* d_t \\ d_t &= e^{-ft} d e^{ft}, \quad d_t = e^{ft} d^* e^{-ft} \end{aligned}$$

where f is a Morse function and d^* is d sandwitched by Hodge star operator $*$.

Now a result from Hodge theory tells us that the number of zero eigenvalues of laplacian $dd^* + d^*d$ is the same as the Betti number B_p, i.e. $dim(H_p(M))$, where M is the Hilbert space that our physical states lives. But the null space of laplacian is not changed by the conjugation by e^{ft}, so that the number of zero eigenvalues of H is equal to the Betti number B_p. The idea is to take $t \to \infty$ to constrain the number of zero eigenvalues.

Now, the eigenvector with zero eigenvalue for H is constrained by the value of t. In fact, we can compute from the definition that [1].

$$H = dd^* + d^*d + t^2(df)^2 + \sum_{i,j} \frac{\partial f}{\partial x_i \partial x_j}[dx_j, \frac{\partial}{\partial x_i}]$$

where x_i is a local coordinate and $\frac{\partial}{\partial x_i}$ represents a interior product operator on p-form, dx_j represent exterior product (wedge product). Therefore, in order to have zero eigenvalue as $t \to \infty$, we need to have $(df)\omega = 0$. This is the connection with Morse theory. Therefore, we look points close to a critical points, say q whose index of f is λ. Using the Morse Lemma introduced above, we express the function f in local coordinates as $f =$

[1] I got a help from a grad student in physics department, John, to compute this

$f(0) + \frac{1}{2}\sum_i \epsilon_i x_i^2$, where $\epsilon_1 \cdots \epsilon_\lambda$ are -2 and $\epsilon_{\lambda+1} \cdots \epsilon_n$ are 2. In this local coordinates, we obtain

$$\sum_i \left(-\frac{\partial^2}{\partial x_i^2} + t^2 \epsilon_i^2 x_i^2 + t\epsilon_i [\frac{\partial}{\partial x_i}, dx_i] \right)$$

This is where physicists get exicted. Apparently, the first expression, $-\frac{\partial^2}{\partial x_i^2} + t^2 \epsilon_i^2 x_i^2$ is the expression of Hamiltonian for quantum harmonic oscillator, and anybody with decent physics background knows the eigenvalues of it. For example, see [Griffith]. The eigenvalue of this operator is $t|\epsilon_i|\ (1 + 2N_i)$ with any $N_i = 0, 1, 2, 3, \cdots$. The eigenvalue of the latter operator $[dx_i, \frac{\partial}{\partial x_i}]$ is the operation on "forms". It is easy to check if the forms that it acts on contains dx_i, then we obtain 1, and if it does not, then we get -1. Note that the harmonic oscillator operator is the operation on the "function" in front of "forms", so we can choose the eigenfunctions of both operators simultaneously. To summarize, we can obtain the following eigenvalues for operator H.

$$t \sum_i (|\epsilon_i|(1 + 2N_i) + \epsilon_i n_i)$$

where N_i can take any value $0, 1, 2 \cdots$ and n_i can take $1, -1$. Note that if the action is on p-form, then the number of n_i that will take 1 is precisely p. In order to make this expression equal to zero, we need to have $N_i = 0$ for all i and $n_1 = \cdots = n_\lambda = -1$, $n_{\lambda+1} = \cdots = n_n = 1$. In particular, there is at most one zero eigenvalue of H acting on p form for each critical point with index p of f, so that $C_\lambda \geq R_\lambda$.

3 Further application: Selection Rule in crystals

Since I study physics, I ought to present some application of Morse Theory to physics. So here we go. The phenomenon we try to analyze is the selection rule that takes place when the symmetry group of crystal changes. Crystal has symmetry and we can describe the symmtry of the crystal as the invariance of density function $\rho(x)$ under the action of some group G (rotations and reflection, a subgroup of O_n). Now it is a physical fact that as we change the temperature of crystals, some crystal go through phase transition and change its symmetry group. The surprising fact is that even though the density function changes continuously with temperature, the symmetry group of the crystal changes abruptly at some temperature, say, T_0. We consider the case when this symmetry group after the transition is smaller than the original group G. It is not hard to convince yourself that this smaller symmetry group is likely to be the subgroup H of the original group G. However, we know from experiments that only particular subgroups of G are realized as symmetry group after the transition. Therefore, there seems to be a "selection" of subgroups at the time of transition. Morse Theory applied to this problem somewhat solves the mystery. I'll describe the method, following [Nash, Sen].

The key parameter we deal with is the density $\rho(x)$ of crystal. We consider a symmetry group G acting on a vector x, and say crystal is invariant under the action of the group G if $\rho(gx) = \rho(x)$ whenever $g \in G$. This allows us to describe the symmetry group of the crystal. In addition, we need some methods to determine which density distribution $\rho(x)$ is allowed (or preferred) in physical situation. For this, thermodynamics provides us with a function called Gibbs free energy, which nature tries to minimize when temperature τ and pressure p are constants. Explicitly, this function $Gibb$ is given by

$$Gibb = U + pV - \tau\sigma$$

where U is internal energy, p is pressure, V is volume, τ is temperature and σ is entropy, the number of available states for the system. It's not hard to see that differential of $Gibb$ is zero when $d\tau = 0, dp = 0$ and with $dU = \tau d\sigma - pdV$ (assuming that there's no particle exchange) . The density of crystal will be determined such that it minimizes the Gibbs free energy. So Gibbs free energy is a function of $\rho(x), \tau, p$. This Gibbs free energy will play the role of Morse function.The minimum of $Gibb$ is of course the critical points of $Gibb$ with index 0.

Now we describe the phase transition in the following way; we write the density function of crystal as $\rho(x) = \rho_0(x) + \delta\rho(x)$ where $\rho_0(x)$ is invariant under the action of the group G and $\delta\rho(x)$ is invariant only under the action of the group H. Since H is a subgroup of G, $\rho_0(x)$ is of course invariant under the action of H. The phase transition is the transition from $\delta\rho(x) = 0 \to \delta\rho(x) \neq 0$.

In order to describe the group action on density function, we use the representation of the group. We prepare the complete set of k-functions $\Phi_i(x)$ for the crystal. Then the group acts on this function as $g\Phi_i(x) = \Phi_i(gx) = \sum_j D_{ij}(g)\Phi_j(x)$. Since the functions $\Phi_i(x)$ are complete, we can write $\delta\rho(x)$ in terms of these functions, therefore, we can express this $\delta\rho(x)$ as a vector, i.e. $\delta\rho(x) = \sum_j C_j \Phi_j(x)$ with $C_j \in R$. Let me write this vector $(C_1, \cdots C_k) = C$. This vector space V is the underlying vector space of our representation. Then the fact that H fixes $\delta\rho(x)$ can be written as $hC = C, \forall h \in H$.

A great physicist Landau formulated his theory of phase transition and noted that from physical argument (symmetry), the Gibbs free energy depends on $|C|^2$ and $|C|^4$, rather than other powers of $|C|$. At the transition temperature, $|C|$ is a small quantity. The dependance is power of $|C|$ so that the density function changes smoothly. Supposing $Gibb(C)$ is degree 4 function in terms of C, the physically realizable C is determined by solving the degree 3, k-equations, $F'(C) = 0$. By solving one equation for one variable (there are three choices of expression) and plugging into another, we see that there are at most 3^k solutions to this equation. Therefore, if the number of critical points with index i is denoted by n_i, we have the upper bound, $\sum_{i=0} n_i \leq 3^k$.

In order to use the Morse inequalities, we compactify the space R^k that C lives in, and obtain S^k. From this, the upper bound increases possibly by one (infinity), so that $\sum_{i=0} n_i \leq 3^k+1$. Note that just as cohomology group, the homology group $H_i(S^k)$ of S^k is 0 unless $i = 0, k$ and when $i = 0, k$, the dimension of the group over the underlying field is 1. Therefore, we obtain the Morse inequality that bounds n_i.

Now we have the upper bound of $\sum_i n_i$ and some relation about lower bound of n_is. If we could obtain the relation between k (dimension of the vector space) and given symmery group G, as well as the relation between n_i and G, H, we can figure out what H is allowed by determining what n_i is allowed. These relations are a bit tricky.

First of all, Landau argues that the density function of crystal can be expressed by the functions $\Phi_j(x)$ that corresponds to the irreducible representation of G. Therefore, he argues we can restrict V to its irreducible space V_0. This reduces k by substantial amount. Let me assure you that this argument is not totally out of nowhere; in many situation of quantum mechanics, irreducible spaces do not "mix" each other.

The relation of G, H and n_i is obtained in the following way. Pick a critical point with index p, C_p. For this vector C_p, find a subgroup of G that fixes this vector, H_p. Now the Gibbs free energy is physical entity and invariant under rotations or reflection of crystal, so that $Gibb(GC) = Gibb(C)$. In particular, the action of G on C_p carries C_p to another critical point C'_p with index p. Now the number of distinct critical points with index p that we can obtain in this way is obviously $\frac{|G|}{|H_p|}$ by considering the quotient group. Now from here, we can conclude that $n_p = k\frac{|G|}{|H_p|}$ with $k \in Z$. Given a particular group G, we can find the possible the subgroup of H, therefore possible n_p.

With these inequalities for n_i and possible values of n_i, we can figure out the possible symmetry group after the transition $H = H_0$ as a subgroup of G. An example with a group with order 48 is worked out in detail in [Nash, Sen].

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