1D Luttinger Liquid

Takuya Kitagawa

1 Spinless Luttinger Liquid

1.1 introduction

Luttinger Liquid is a generic 1D many-particle state. Historically, it has been a toy for theorists, but recently, experimentalists are getting close to realizing such a system. Plausible examples are carbon nanotubes, semiconductor wires, organic crystals and edge states, i.e. as in fractional quantum hall effect. Among many things that textbooks mention, (for example, in Bruus and Flensberg, Mahan, Giamarchi), there are several reasons that Luttinger Liquid (LL) is interesting to me; (1) The behavior of LL is drastically different from the higher dimensional counterparts, which can be described fairly well as Fermi Liquid (2) any generic 1D system can be characterized as LL and solved in terms of a few parameters (3) 1D system provides examples of qunatum phase transition (4) this topic nicely summarizes what I have learned this semseter. Even though LL is solved and investigated in numerous textbooks, I would like to summarize the result in the way I understand it, in the following.

1.2 General solution

Starting point for us is non-interactiong, 1D spinless fermions described by the following Hamiltonian.

$$H_0 = \sum_k \epsilon_k c_k^\dagger c_k \tag{1}$$

We introduce interactions to this Hamiltonian,

$$H_I = \sum_q{}' V(q)\rho(q)\rho(-q) \tag{2}$$

where $\rho(q) = \sum_k c_k^\dagger c_{k+q}$ and we substract the Hartree term ($q = 0$ in the sum) since we have an uniform background charge. We will deal with spinful case later. Now our purpose is to solve this Hamiltonian and find the eigenstate and eigenenergy of the hamiltonian. Before we present the solution, let me point out the signatures that Fermi theory does not work in 1D. This gives us the hint of how we should solve this Hamiltonian.

First of all, we do not have quasiparticles, namely, the life time of low energy 1 D particle is not long compared to the energy of excitation, i.e. $\zeta_k \approx \tau_k^{-1}$ (Bruus and Flensberg). Second, the dispersion relation has a gap and there is no gap between "plasmon excitation" and "electron-hole excitation." Intuitively, a single particle in 1D cannot be excited alone–if one particle tries to move, it has to push the particle next to it. Therefore, there cannot be a single particle excitation but there is only a collective excitation. So it makes sense to have these two modes of excitation, one being collective and another being single particle, merge on the dispersion relation. The second observation hints at how we can solve the 1D problem. A natural basis for this system is not a single particle operators ($c_k^\dagger$) but rather collective excitations such as electron-hole excitations ($c_{k+q}^\dagger c_k$).

The model we present here is called Tomonaga Model. In this model, we focus on low energy excitations. This means, we assume that excitations have a linear dispersion relation $\omega = qv_F$, as we can see from looking at the low energy electron-hole excitation $\frac{(k+q)^2-q^2}{2m}$. In other words, we assume that interaction energy V is such that $V << \epsilon_F$ where ϵ_F is Fermi energy. Then all the excitations and interactions involve only the states near Fermi surface. This implies that we have essentially two sectors of excitations, excitations with $k > 0$ and $k < 0$. We can conveniently call the former right moving particle and latter left moving particles. This model fails near $k \approx 0$ and this requires some care when we deal with Fourier transformations of, say, density of particles. Mathematically it is cleaner and easier to solve Luttinger Model, where we assume the existence of two different particles with linear dispersion relations everywhere. For details of Luttinger model, I refer to Giamarchi or Giuliani and Vignale.

Following the insight that collective excitations is the natural basis to use in 1D, in addition to the assumptions stated above, we would like to express our Hamiltonians in terms of the following operators.

$$\begin{aligned}\rho_R(q) &= \sum_{k>0} c_{k+q}^\dagger c_k \\ \rho_L(q) &= \sum_{k<0} c_{k+q}^\dagger c_k\end{aligned}$$

In order to promote these operators to creation and annihilation operators, we look at the commutation relations.

$$[\rho_r(q), \rho_{r'}(-q')] = \delta_{r,r'}\delta_{q,q'}\frac{rLq}{2\pi}$$

where $r, r' = (R = 1), (L = -1)$, depending on the context.

It is not clear how we can express the Hamiltonian in terms of these opeartors. But here, we note that commutation relation determines the operator. In other words, two operators that have exactly the same commutation relations with any other operators in the Hilbert space should be regarded as the same operator. But in fact, combinations of the operators $\rho_R(q), \rho_L(q)$ are complete, and refer to Giormarchi Appendix B for rigorous proof. This statement is plausible because this just says, you can go from one state to any other state, by repeatedly acting $c^\dagger_{k+q}c_k$ (or killing a particle with momentum k and creating a particle with momentum $k+q$) on the state. Therefore, we can look at the commutators of H_0 with the operators $\rho_R(q), \rho_L(q)$, and try to guess what form H_0 should take. (Note that we have $\epsilon_k \approx v_F k$.)

$$[H_0, \rho_r(q)] = -rqv_F\rho_r(q)$$

With the commutation relations for ρ_r, we can easily guess and check that H_0 is the form

$$H_0 = \frac{2\pi v_F}{L}\sum_{q>0}[\rho_R(-q)\rho_R(q) + \rho_L(q)\rho_L(-q)]$$

plus some c-number.

In the similar fashion, we want to express H_I in terms of density operators. An obvious and straightforward way to do it is to express H_I in terms of creation and annihilation operators of right and left moving particles. That is, we write $c_k = \Theta(k)c_{k,R} + \Theta(-k)c_{k,L}$, where $\Theta(k)$ is a step function. Since we assume that V is small compared to Fermi energy, so $k+q$ and k are restricted to near the Fermi surface. Then there could be only two interaction terms; (1) forward scattering, with the exchange momentum $q \approx 0$ (2) backward scattering with the exchange momentum $q \approx 2k_For - 2k_F$. These interactions are, respectively,

$$\begin{aligned}
H_{I,1} &= \frac{1}{2L}\sum_{|q|<\Gamma} V_q(\rho_R(q) + \rho_L(q))(\rho_R(-q) + \rho_L(-q)) \\
H_{I,2} &= -\frac{1}{2L}\sum_{|q'|<\Gamma} V_{2k_F}(\rho_R(-q)\rho_L(q) + \rho_R(q)\rho_L(-q))
\end{aligned}$$

where Γ is the momentum cutoff. These terms make a very good intuitive sense.

Now looking at the commutation relations of ρ_r, we can introduce the follwoing natural creation and annihilation operators

$$b_{R,q} = \rho_R(q)\sqrt{\frac{2\pi}{qL}}, \qquad b^\dagger_{R,q} = \rho_R(-q)\sqrt{\frac{2\pi}{qL}}$$

$$b_{L,q} = \rho_L(-q)\sqrt{\frac{2\pi}{qL}}, \qquad b^\dagger_{L,q} = \rho_L(q)\sqrt{\frac{2\pi}{qL}}$$

With these operators, our Hamiltonian becomes

$$H = H_0 + H_I = \sum_{q>0} |q|(v_F + V_q)(b^\dagger_{R,q}b_{R,q} + b^\dagger_{L,q}b_{L,q}) + \sum_{q>0} |q|(V_q - V_{2k_F})(b^\dagger_{R,q}b^\dagger_{L,q} + b_{L,q}b_{R,q})$$

There are a several things to note. First of all, when expressed in terms of collection excitation operators, the Hamiltonian becomes quadratic in these operators. Therefore, using the usual Bogoliubov transformations, we can easily bring the Hamiltonian in the form $E_k(\alpha^\dagger_k\alpha_k + \beta^\dagger_k\beta_k)$. I don't explicitly express the Hamiltonian in quadratic form through Bogoliubov transformation here, but one can refer to Mahan for the details. Secondly, some of the interaction terms $H_{I,1}$ and $H_{I,2}$ are incorporated in free fremion terms and renormalized the energy. This form of Hamiltonian allows us to solve the problem exactly.

1.3 single-particle operator

Now that we know the Hamiltonian takes free-fermion form in a certain basis, we can in principle compute expectation value of any operator. In particular we can compute the correlation functions in this theory. However, the basis we took was the collective excitations, so single-particle operators now take a complicated form in this basis.

In order to construct single particle operator out of ρ_r, we use the same trick as before, namely look at the commutation relations. To reproduce a single particle operator ψ_r, we note that we want ψ_r (1) to change the total number of r-moving particle by one, as well as (2) to change the total number of fermions by one. The former characteristic can be captured by finding an operator A such that $[A(x), \rho_{x'}] = i\delta(x - x')$ and constructing e^{iA}. This is so can be proved by noting that the expectation value of $\rho(x)$ in the state $e^{iA(x_0)}|0>=|x_0>$ is given by $<x_0|\rho(x)|x_0>=\delta(x - x_0)$ as we can check easily. Now, to find such an operator A, we note that the Fourier transform $\rho_r(x) = \frac{1}{L}\sum_q e^{iqx}\rho_r(q)$ satisfies the following commutation relations

$$[\rho_r(x), \rho_{r'}(x')] = \frac{r}{2\pi i}\delta_{rr'}\frac{\partial}{\partial x}\delta(x - x')$$

Then we find $A(x) = \int_{-\infty}^x \rho_r(x')dx'$.

The second property needs to be put in by hand, by introducing total fermion creation and annihilation operator U_r. The need for to introduce this operator independently can be seen by noting that $\rho_r(q)$ is just $\sum_k c^\dagger_{q+k}c_k$, and thus it conserves the total number of fermions.

Now we know the general form of the single particle operator, which is

$$\psi_r(x) \propto U_r e^{\int_{-\infty}^{x} \rho_r(x')dx'}$$

The exact form of this operator needs to be determined by analyzing various commutation relations, such as the anti-commutation relations of $\psi_r(x)$. A various care needs to be taken in this process, such as introducing momentum cutoff. However most of the physics is contained in the description above. To present the final form of single particle operator, it is convenient to introduce the "semi-classical" density operator and current operator through ϕ and θ given by

$$\begin{aligned} \partial\phi(x) &= -\pi(\rho_R(x) + \rho_L(x)) \\ \partial\theta(x) &= \pi(\rho_R(x) - \rho_L(x)) \end{aligned}$$

Note that looking at how $\rho_r(x)$ is defined and how Fourier transform is defined, $\rho_R(x) + \rho_L(x) \neq \rho(x)$.

Then our single particle operator becomes

$$\psi_r(x) = \frac{U_r}{\sqrt{2\pi\alpha}} e^{irk_F x} e^{-i(r\phi(x) - \theta(x))}$$

where α is the momentum cutoff.

1.4 Justification of single particle operator

It is always nice to double check if the single particle operator obtained above is the right one. One way to check this is to compute some objects in two basis, one in bosonic basis ($\psi_r(x)$, or $b_{k,r}$) and another in fermionic basis ($\psi(x)$ or c_k) and see if they agree. We can compute (1) single particle Green's function (for example, in Mahan) and (2) specific heat (Giomarchi). For example, Green's function is just $< \psi(x)\psi^\dagger(0) >$. The results for both quantity indeed agree with each other.

1.5 Correlation functions (superconducting phase and charge)

Using this expression, we obtain, for example, density operator in terms of ϕ and θ. Note that $\rho_R(x) + \rho_L(x) \neq \rho(x)$. We expand $\rho(x) = \psi^\dagger\psi = (\psi_R + \psi_L)^\dagger(\psi_R + \psi_L)$, then use the expression for the single particle operator.

$$\rho(x) = -\frac{1}{\pi}\partial\phi(x) + \frac{1}{2\pi\alpha}(e^{i2k_F x} e^{-i2\phi(x)} + h.c.)$$

Then density-density correlation function is given by

$$< \rho(x)\rho(0) > = \frac{1}{\pi^2} < \partial\phi(x)\partial\phi(0) > + \frac{1}{(2\pi\alpha)^2}(e^{-i2k_F x} < e^{i(2\phi(x) - 2\phi(0))} > + h.c.)$$

There are several ways to compute $< e^{i(2\phi(x)-2\phi(0))} >$. One way is to change to the basis such that Hamiltonian takes the free-fermion form, by Bogoliubov transformation. Then since we know the ground state in terms of the new creation and annihilation operators, it is easy to compute it. This is done in Mahan. Another way is more brute force, and it is to use Feynman Path integral. This method is straightforward and has the advantage that formally, the correlation function can be evaluated at non-zero temperature. This is done in Giomarchi. Either way, the computation is technical and I refer to the appropriate textbooks for details.

In any way, the result is

$$< \rho(x)\rho(0) >= \frac{K}{2\pi^2}\frac{y_\alpha^2 - x^2}{(y_\alpha^2 + x^2)^2} + \frac{2}{(2\pi\alpha)^2)}\cos(2k_F x)\left(\frac{\alpha}{r}\right)^{2K}$$

where we defined $K = \left(\frac{1+V_0/2-V_{2k_F}/2}{1+V_0/2-V_{2k_F}/2}\right)^{1/2}$. Intuitively, $K < 1$ if the interaction is repulsive and $K > 1$ if it is attractive. Important point about this expression of density-density correlation function is that the first term is just as in Fermi liquid and the second term characterizes the Luttinger Liquid. If we take the Fourier transform of this correlation function and look at the singularity of susceptibility, we notice that density wave occurs when $K < 1$, repulsive interaction. This wave is called charge density wave (CDW). In this regime, ϕ tries to take a constant value, but prevented from doing so by qunatum fluctuation. These statements can be confirmed by looking at the Hamiltonian expressed in terms of ϕ and θ.

$$H = \frac{u}{2\pi}\int dx(K(\partial\theta)^2 + \frac{1}{K}(\partial\phi)^2) \tag{3}$$

For $K < 1$, the energy is minimized by making $\partial\phi$ smaller relative to $\partial\theta$. In the extreme case, when $\phi = constant$, we obtain the density wave $\rho(x) \propto \cos(2k_F x - 2\phi)$, that is, perfect density correlation, as we would have expected from the divergence. This constant ϕ is prevented because there is a conjugate relation between ϕ and $\partial\theta$.

Looking at the Hamiltonian above, we certainly expect that the opposite case, $K > 1$, would have a similar consequence. Indeed, we can work out the correlation function of the pairing operator $O_P = \psi^\dagger(r)\psi^\dagger(r + a)$ in the limit $a \to 0$. We find, at the end of computation, that

$$< O_p(x)O_p 0) >\propto \left(\frac{\alpha}{r}\right)^{1/2K}$$

Therefore, indeed, we have the singularity when $K > 1$. This corresponds to constant θ and complete pair of singlet. Therefore, this θ is like superconducting phase.

2 Spin 1/2 Luttinger Liquid

We can exntend our analysis to spinful case. In this section, we will see how we can do so with spin 1/2 system. The kinetic term of Hamiltonian will be changed in a natural way

$$H^0 = H^0_\uparrow + H^0_\downarrow$$

where $H^0_\sigma = \sum_k \epsilon_k c^\dagger_{k,\sigma} c_{k,\sigma}$. For the notation of spin indices, I will be using $\sigma = (\uparrow\ or + 1), (\downarrow\ or - 1)$ depending on the context. To write interaction Hamiltonian, it's helpful to think of it diagramatically. For example, refer to Giamarchi, p18, Fig 1.9. First of all, consider the exchange of momentum $q \approx 0$. Then it is a product of two $\rho_{r,\sigma}$, such as $\rho_{R,\uparrow}\rho_{L,\uparrow}$. Considering all the combinations and expecting each interaction to have different interaction strength, we write them as (in position space)

$$\begin{aligned} H_4 &= \int dx \sum_r \sum_\sigma \left(\frac{g_{4\parallel}}{2} \rho_{r,\sigma}(x)\rho_{r,\sigma}(x) + \frac{g_{4\perp}}{2} \rho_{r,\sigma}(x)\rho_{r,-\sigma}(x) \right) \\ H_2 &= \int dx \sum_\sigma \left(g_{2\parallel} \rho_{R,\sigma}(x)\rho_{L,\sigma}(x) + g_{2\perp} \rho_{R,\sigma}(x)\rho_{L,-\sigma}(x) \right) \end{aligned}$$

where we denoted the momentum exchange between the same spin (or parallel spin) to be $g_\parallel$.

Now we consider the exchange of momentum $q \approx 2k_F$. These are given by

$$H_2 = \int dx \sum_\sigma \left(g_{1\parallel} \psi^\dagger_{L,\sigma} \psi^\dagger_{R,\sigma} \psi_{L,\sigma} \psi_{R,\sigma} + g_{2\perp} \psi^\dagger_{L,\sigma} \psi^\dagger_{R,-\sigma} \psi_{L,-\sigma} \psi_{R,\sigma} \right)$$

Now we can proceed just as in spinless case to construct single particle operaors. Then, just as before, we would get the same expression for the single particle operator of the form,

$$\psi_{r,\sigma}(x) \propto U_{r,\sigma} e^{\int_{-\infty}^x \rho_{r,\sigma}(x')dx'}$$

To understand the physics of spin better ,we replace the two density operators $\rho_\uparrow$ and $\rho_\downarrow$ with total density operator ρ and spin operatorσ by

$$\begin{aligned} \rho(x) &= \frac{1}{\sqrt{2}}(\rho_\uparrow(x) + \rho_\downarrow(x)) \\ \sigma(x) &= \frac{1}{\sqrt{2}}(\rho_\uparrow(x) - \rho_\downarrow(x)) \end{aligned}$$

This change of basis naturally introduces the following semi-classical expressions for θ and ϕ.

$$\begin{aligned} \phi_\rho(x) = \frac{1}{\sqrt{2}}(\phi_\uparrow(x) + \phi_\downarrow(x)) \quad \phi_\sigma(x) = \frac{1}{\sqrt{2}}(\phi_\uparrow(x) - \phi_\downarrow(x)) \\ \theta_\rho(x) = \frac{1}{\sqrt{2}}(\theta_\uparrow(x) + \theta_\downarrow(x)) \quad \theta_\sigma(x) = \frac{1}{\sqrt{2}}(\theta_\uparrow(x) - \theta_\downarrow(x)) \end{aligned}$$

In this notation, we find that

$$\psi_r(x) = \frac{U_{r,\sigma}}{\sqrt{2\pi\alpha}} e^{irk_F x} e^{-\frac{i}{\sqrt{2}}(r\phi_\rho(x) - \theta_\rho(x) + \sigma(r\phi_\rho(x) - \theta_\rho(x)))}$$

Moreover, the Hamiltonian simplifies enormously

$$\begin{aligned} H^0 &= \frac{1}{2\pi}\int dx v_F((\partial\phi_\rho(x))^2 + (\partial\theta_\rho(x))^2 + (\partial\phi_\sigma(x))^2 + (\partial\theta_\sigma(x))^2) \\ H_4 &= \frac{1}{4\pi^2}\int dx(g_{4\|} + g_{4\perp})((\partial\phi_\rho(x))^2 + (\partial\theta_\rho(x))^2) + (g_{4\|} - g_{4\perp})((\partial\phi_\sigma(x))^2 - (\partial\theta_\sigma(x))^2) \\ H_2 &= \frac{1}{2\pi^2}\int dx(g_{2\|} + g_{2\perp})((\partial\phi_\rho(x))^2 + (\partial\theta_\rho(x))^2) + (g_{2\|} - g_{2\perp})((\partial\phi_\sigma(x))^2 - (\partial\theta_\sigma(x))^2) \\ H_1 &= \frac{1}{4\pi^2}\int dx(-g_{1\|})((\partial\phi_\rho(x))^2 + (\partial\theta_\rho(x))^2) + (-g_{4\|})((\partial\phi_\sigma(x))^2 - (\partial\theta_\sigma(x))^2) \\ &\quad + \frac{2g_\perp}{(2\pi\alpha)^2}\cos(2\sqrt{2}\phi_\sigma(x)) \end{aligned}$$

In this form, we see that spin σ and density ρ degree of freedom completely separates. This implies that spin and charge degree of freedom have separated dynamics (uncoupled) in the system. The situation is nicely described in Fig.3.4 of Giamarchi. In the textbook, Giamarchi also illustrates why this separation of charge and spin is unfavorable in higher dimensions. Note that all the Hamiltonians combine to give the Hamiltonian of the form (eq.1) except for the last term of H_1. The coefficients are renormalized by interaction terms. Therefore, if we don't incorporate the last term in H_1, we can compute correlation functions just as for spin-less case.

The last term cos will introduce a very interesting physics. In Giamarchi, they analyze this term through renormalization group flow and found a phase diagram. For details, see p56 to p69 of Giamarchi.

2.1 Other interesting 1D system

Even though the analysis of Luttinger Liquid given above is completely general for spinless fermionic systems, we can consider other 1D systems, such as Fermion and Boson mixture (Adilet) or particles with higher spins. Adilet suggested to me to analyze the latter system, and it is a potential future research project.

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