

Information in Curvatures

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Abstract

In the world of mathematics, there are many "curvatures": principle curvature, mean curvature, Gaussian curvature, Riemannian curvature, sectional curvature, Ricci curvature, scalar curvature..etc. It is not too difficult to have an intuitive sense for principle curvatures, but even mathematicians cannot intuitively "see" Ricci curvature. This article is meant to explain various curvatures mentioned above with clarity. Emphasis is made on the motivation as to why one wants to define such curvatures and the information one can take out of curvatures.

1 Curvatures in R^3

1.0.1 Intuitive curvature

First of all, let's try to define a curvature according to our intuition. In order to have a curvature, we certainly need a curve. Consider a curve in R^3 , $p(t) : R \rightarrow R^3$. Since curvature should be the measurement of change (how much a curve bends) in the curve, we can look at the derivatives. The first derivative won't do; if we called the first derivative a curvature, then a straight line defined by $p(t) = t(a_x, a_y, a_z)$ where a_x s are constants would have a non-zero curvature. Then we might guess that the second derivative $\frac{d^2 p}{dt^2}$ might be a good candidate for defining a curvature. In order to see this more clearly, consider a normal unit vector $N(t_0)$ to the velocity vector $\frac{dp}{dt}$ at $q = p(t_0)$. Then how much a curve bends can be known by measuring how much $N(t)$ changes. But if we simply take $\left| \frac{N(t)}{dt} \right|$ as our curvature, it would be inconvenient, for the curvature would depend on the parametrisation t . For example, we can decide to go along the path $p(t)$ with doubled speed and we would measure $\frac{N(2t)}{dt} = 2 \frac{N(t)}{dt}$, even though we have the same curve. Thus, curvature should be measured (1) as the amount of change in $N(t)$ (2) during a certain distance s along a curve. In order to measure the distance along a curve, we'll use the standard metric in R^3 (I denote an inner product by (X, Y) .)

$$(\vec{x}_1, \vec{x}_2) = |(x_1, y_1, z_1) - (x_2, y_2, z_2)|$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \quad (1)$$

Using this metric, we can measure the distance along a curve s . In order to measure a curve, we want to add up infinitesimal distance at each point. The infinitesimal distance is $\frac{dp}{dt}$, therefore, we have

$$s = \int \left(\frac{dp}{dt}, \frac{dp}{dt} \right) dt. \quad (2)$$

We want to measure the change of N during a small s , that is, we want to parametrise N with s and differentiate $N(s)$ in terms of s . Now from eq.2 and remembering that the velocity vector is $V(s) = \frac{dp}{ds}$, we have $s = \int (V(s), V(s)) ds$. Differentiating both sides in terms of s gives $1 = (V(s), V(s))$. So the velocity vector of the curve $p(s)$ parametrised by the length of the curve always has unit length. If we differentiate both sides in terms of s one more time, we obtain $2 \left(\frac{V(s)}{ds}, V(s) \right) = 0$. So, $\frac{V(s)}{ds}$ is perpendicular to $V(s)$. Accordingly, we can define a unit normal vector $N(s)$ through $k(s)N(s) = \frac{V(s)}{ds}$ where $k(s) = \left| \frac{V(s)}{ds} \right|$. Since R^3 is oriented, we can unambiguously determine a unit vector $B(s)$ such that vectors $(T(s), N(s), B(s))$ have the same orientation as R^3 . Let $T(s) = F_1(s), N(s) = F_2(s), B(s) = F_3(s)$. Then they form an orthonormal basis for R^3 . Since we have $(F_i, F_j) = \delta_{ij}$, we obtain through differentiating both side in terms of s ,

$$\left(\frac{dF_i}{ds}, F_j(s) \right) + \left(\frac{dF_j}{ds}, F_i(s) \right) = 0$$

Moreover, since we can write $\frac{dF_i}{ds}$ as $\sum_k a_i^k F_k$, we obtain $a_i^j(s) + a_j^i = 0$. Since the matrix is skew-symmetric, we have the following Frenet-Serret formulas[Boothby 294].(Let $a_2^3 = \tau(s)$.)

$$\begin{aligned} \frac{dT}{ds} &= && k(s)N, \\ \frac{dN}{ds} &= &-k(s)T && +\tau(s)B \\ \frac{dB}{ds} &= && -\tau(s)N \end{aligned}$$

Here we can see explicitly that $k(s)$ is the actually "curvature," whereas τ is "twisting." So $\left(\frac{dN(s)}{ds}, T \right)$ should be our curvature. Note that our guess that the second derivative of $p(s)$ could be our curvature turns out to be almost correct. This is the description of "intuitive" curvature.

1.1 principle curvature and shape operator

Now that we know what a curvature of a line should be, let's move on to curvatures of a 2-dimensional surface M in R^3 . Restrict our attention to

some small open set U of a point q . We want to know the "curvature" of M at q . We can extend our definition of curvature of a curve. However, there is an infinite number of curves on M that go through the point q . We need to pick some special curves out of these in order to decrease the number of "curvatures" we deal with, and say something sensible. There are two special curves that we can think of:

Curve 1 *curves whose velocity vector at q , V_q , is in the same (or opposite, depending on the sign) direction as $\frac{dN(t)}{dt}$ where $N(t)$ is a unit normal vector to M at q*

Curve 2 *curves that have minimum and maximum curvature. More precisely, curves with a unit velocity vector V at q that give maximum and minimum values of $(\frac{dN}{dt}, V)$.*

We will see that these two curves are the same curves. Then we can "represent" the curvature on the surface with two numbers, maximum and minimum values of $(\frac{dN}{dt}, V)$.

To see that the two curves are actually the same, we need to define a linear function called shape operator.

Shape operator is a function that takes a vector on the tangent space $T_q(M)$ of M at q to $T_q(M)$. "Tangent space" and "tangent plane" are intuitively similar concepts. For details about tangent spaces, please see Boothby. I'll briefly describe a tangent space below. Suppose that given a 2-dimensional smooth surface M in R^3 , we can find a smooth function $\phi : M \subset R^3 \rightarrow N \subset R^2$ such that $\phi(q) = (u(q), v(q))$. The geometrical objects with this property are called "manifolds." In other words, manifolds are geometrical objects that locally look like R^n . Then, we can think of ϕ as "local coordinate of a point on tangent plane." The idea is that we view the surface M as a 2-dimensional object, rather than an object embedded in R^3 . Define the tangent vector space $T_q(M)$ at point q of M as a set of maps $F : C^\infty \rightarrow R$, where C^∞ is a smooth map defined on an open set that contains q . For example, we have $\frac{\partial}{\partial u}|_q$ in $T_{\phi(q)}(N)$. Then, given a vector, say, $\frac{\partial}{\partial u}|_{\phi(q)}$ in $T_{\phi(q)}(N)$, we obtain a smooth map through push forward $\phi_*^{-1} : T_{\phi(q)}(N) \rightarrow T_q(M)$ by $\phi_*^{-1}(\frac{\partial}{\partial u}) = \sum_i \frac{\partial f^i}{\partial u} \frac{\partial}{\partial x^i}$. where $\phi^{-1}(u, v) = (f^1, f^2, f^3)$. I claim the basis for $T_{\phi(q)}(N)$ is $\frac{\partial}{\partial u}|_{\phi(q)}$ and $\frac{\partial}{\partial v}|_{\phi(q)}$, the basis for $T_q(M)$ is $E_1 = \phi_*^{-1} \frac{\partial}{\partial u}|_{\phi(q)}$ and $E_2 = \phi_*^{-1} \frac{\partial}{\partial v}|_{\phi(q)}$. E_1 and E_2 are called coordinate frames. Then, there is a 2-dimensional tangent space for each point of 2-dimensional surface M . Similarly, there is a 3-dimensional tangent space for each point of R^3 . We can think of $T_q(M)$ as a part of $T_q(R^3)$ through isomorphism.

Now that we have a basis for $T_q(M)$, we can choose a unit vector N_q in $T_q(R^3)$ such that (E_1, E_2, N) is a basis for $T_q(M)$ with the same orientation as x, y, z . Then we have unambiguously determined N . We can pick N_q for each $q \in M$. Then this N is a map $M \rightarrow T_q(M)$ defined by $q \mapsto N_q$ that assigns each point of manifold to a vector in a tangent space. Such N without subscripts is called a vector field. Given a vector field N , we can see that the restriction of N to $p(t)$ for some curve p as a function of t through $p(t) = \phi^{-1}(u(t), v(t))$. So I denote the restriction of N onto $p(t)$ as $N(t)$. Note that since $1 = (N(t), N(t))$, we have $0 = 2 \left(N_t, \frac{dN(t)}{dt} \right)$, therefore, $\frac{dN}{dt}$ lies in $T_q(M)$. From this fact, we can define shape operator $S : T_q(M) \rightarrow T_q(M)$ as below.

Definition 1 Suppose $p(t_0) = q$. Given a vector X_q in $T_q(M)$, define $S : T_q(M) \rightarrow T_q(M)$ by $S(X_q) = - \left. \frac{dN(t)}{dt} \right|_{t_0}$. S is called a shape operator.

Theorem 1 The shape operator defined above linearly depends on X_q . In particular, $S(X_q)$ is independent of the choice of the curve $p(t)$.

Furthermore, we can prove the following[Boothby 358].

Theorem 2 S is self-adjoint transformation(that is, $(S(X), Y) = (X, S(Y))$ for any vectors X, Y). Therefore, S is symmetric and $[S]$ is diagonalisable through triangulation.

Sketch proof

I'll explicitly show that S is self-adjoint. Since S is linear, it would suffice to show that $(S(E_1), E_2) = (S(E_2), E_1)$. Note that $S(E_1)$ is the change of N along a curve that has a velocity vector E_1 at q . A coordinate curve given by $\phi^{-1}(u(t), v(t)) = \phi^{-1}(t, 0)$ is one of such curves. Then, we have $S(E_1) = -\frac{\partial N}{\partial u}$, $S(E_2) = -\frac{\partial N}{\partial v}$. Intuitively, we can see this by thinking that moving along E_1 is just like moving along $\frac{\partial}{\partial u}$ in local coordinates. So the derivative in terms of t is the same as the derivative in terms of u . Since we have $(E_2, N) = 0$, differentiating this in terms of u , we obtain, $(S(E_1), E_2) = -(N, \frac{\partial E_2}{\partial u})$. As above, if we write $\phi^{-1}(u, v) = (f^1, f^2, f^3)$, then $E_2 = \sum_i \frac{\partial f^i}{\partial v} \frac{\partial}{\partial x^i}$. Our assumption is that f is smooth, so we have $(S(E_1), E_2) = -(N, \sum_i \frac{\partial^2 f^i}{\partial v \partial u} \frac{\partial}{\partial x^i}) = (S(E_2), E_1)$. The self-adjointness of S comes from the interchangeability of partial differentiation when f is smooth.

□

Consequence of the diagonalizability of the matrix of S is that we have eigenvectors and eigenvalues. Suppose V_1, V_2 are eigenvectors corresponding to eigenvalues k_1, k_2 . Then the fact that curve 1 and curve 2 are equivalent follows from the corollary below.

Corollary 1 For a shape operator S , we have eigenvectors V_1, V_2 with unit length such that $S(V_1) = k_1V_1$ and $S(V_2) = k_2V_2$. Then k_1 and k_2 are minimum and maximum of $(S(X_q), X_q)$ for any unit vector X_q .

Sketch proof Without loss of generality, assume that $k_1 \leq k_2$. Since we know V_1 and V_2 are orthogonal (or we can always pick such V_1, V_2) and form a basis, we can express any X_q such that $X_q = aV_1 + bV_2$ with some constants a, b with $a^2 + b^2 = 1$. Then we have $(S(X_q), X_q) = a^2k_1 + b^2k_2 < k_2$.

□

This values k_1 and k_2 are called *principle curvatures* at point q and corresponding unit vectors V_1, V_2 are called *principledirections*. These are the directions along the 2-dimensional surface at which a curve bends the most. It's not very surprising that at any point of M , we can find such directions.

Moreover, the value $K = k_1k_2$ is called *Gaussian curvature* and $H = \frac{1}{2}(k_1 + k_2)$ is called *mean curvature*. From this definition, we can see K is negative when the manifold locally looks like a saddle, and K is positive when manifold locally looks like a sphere or a hole. In terms of principle curvature, when the curve bends toward the normal vector, the curvature is positive, and when the curve bends away from the normal, then the curvature is negative.

I do not go into details about mean curvature, but I mention one thing: when mean curvature is zero, the surface area is minimum among other surfaces that have the same boundary.

2 Gaussian Curvature

2.1 Intrinsic and extrinsic property

One of the discoveries that Gauss made was the invariance of Gaussian curvature under local isometries. A local isometry is a diffeomorphism F such that $(F_*X, F_*Y) = (X, Y)$. A property of manifolds is *intrinsic* iff it is invariant under isometries. Otherwise, a property is *extrinsic*. So Gauss found that Gaussian curvature is intrinsic.

To see what those properties are, consider the 2-manifolds (surface) in R^3 . In the section above, we considered a map from the surface into R^2 , or local coordinates. This R^2 is what the surface looks like to the residence on the surface. A change through distant-preserving map should not be noticeable by resident of the surface because locally everything should look the same if a distance between points is preserved. Then a property which is *intrinsic* to the 2-dimensional surface should be preserved under isometries.

On the other hand, *extrinsic* property is there only because the surface is embedded in R^3 .

There's a good example to describe this. Consider a $x - y$ plane and a cylinder that can be created by folding the $x - y$ plane in x direction [O'Neill]. This "folding" is apparently (local) isometry. Consider the principle curvatures of these two manifolds. We can see immediately that principle curvatures of plane are both 0. However, the principle curvature of the cylinder in the direction that goes around the circle (x direction) is non-zero. So principle curvatures must be extrinsic. However the cylinder's another principle curvature in the axial direction is zero. Therefore, the Gaussian curvatures ($k_1 k_2$) for both manifolds are zero. Note that this example also illustrates that mean curvature is not preserved under (local) isometries.

2.2 Examples: Catenoid and helicoid

To see explicitly how we can calculate principle curvature, mean curvature as well as Gaussian Curvature, let's work out the curvatures of *Catenoid* and *helicoid* [Boothby 368]. These are also non-trivial example of isometric surfaces.

For convenience, let me define two matrixes. I denote an inner product through a metric by (X, Y) , coordinate frame by $E_1 = \phi_*^{-1} \left(\frac{\partial}{\partial u} \right) = X_u$, $E_2 = \phi_*^{-1} \left(\frac{\partial}{\partial v} \right) = X_v$ where ϕ, U is coordinate neighbourhood and shape operator by S . Here, the subscript of X indicates the partial differentiation and X is given by $\sum_{i=1}^3 f^i \frac{\partial}{\partial x^i}$ where $\phi^{-1}(u, v) = (f^1(u, v), f^2(u, v), f^3(u, v))$. If we write the unit normal vector to the surface M by N , then we have, for example, $\frac{\partial}{\partial u}(N, X_u) = 0$. This gives us $(S(X_u), X_u) = -(N, X_{uu})$, etc. Suppose that surfaces in R^3 are endowed with the induced metric from R^2 . Then define

$$g_{ij} = \begin{pmatrix} (X_u, X_u) & (X_u, X_v) \\ (X_v, X_u) & (X_v, X_v) \end{pmatrix} = \begin{pmatrix} (E_1, E_1) & (E_1, E_2) \\ (E_2, E_1) & (E_2, E_2) \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$- \begin{pmatrix} (N, X_{uu}) & (N, X_{uv}) \\ (N, X_{uv}) & (N, X_{vv}) \end{pmatrix} = \begin{pmatrix} (S(E_1), E_1) & (S(E_1), E_2) \\ (S(E_2), E_1) & (S(E_2), E_2) \end{pmatrix} = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

The first one is metric and called first fundamental form, while the second matrix is called second fundamental form. Then, it is easy to confirm that the following theorem holds [Boothby VIII, Theorem 2.1] from the definition $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$.

Theorem 3 $K = \frac{ln - m^2}{EG - F^2}$ and $H = \frac{1}{2} \frac{Gl - 2Fm + En}{EG - F^2}$

This is the prescription for calculating the Gaussian and mean Curvature.

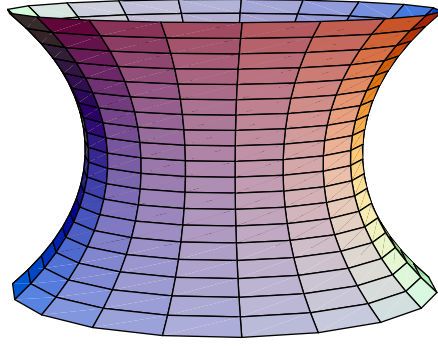


Figure 1: Catenoid[Mathematica]

2.2.1 Catenoid

The parametrisation (local coordinate) of Catenoid is given by

$$F : (z, \theta) \mapsto (\cos \theta \cosh z, \sin \theta \cosh z, z), \quad -\infty < z < \infty, 0 < \theta < 2\pi$$

Now

$$X_z = F_* \left(\frac{\partial}{\partial z} \right) = \cos \theta \sinh z \frac{\partial}{\partial x} + \sin \theta \sinh z \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$$X_\theta = F_* \left(\frac{\partial}{\partial \theta} \right) = -\sin \theta \cosh z \frac{\partial}{\partial x} + \cos \theta \cosh z \frac{\partial}{\partial y}$$

$$\lambda N = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} - \sinh z \frac{\partial}{\partial z}$$

$$\text{where } \lambda = \sqrt{1 + \sinh^2 z} = \cosh z$$

Furthermore,

$$X_{zz} = \cos \theta \cosh z \frac{\partial}{\partial x} + \sin \theta \cosh z \frac{\partial}{\partial y}$$

$$X_{z\theta} = -\sin \theta \sinh z \frac{\partial}{\partial x} + \cos \theta \sinh z \frac{\partial}{\partial y}$$

$$X_{\theta\theta} = -\cos \theta \cosh z \frac{\partial}{\partial x} - \sin \theta \cosh z \frac{\partial}{\partial y}$$

We want to know the matrix. (Induced metric is, of course, $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \delta_{ij}$)

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \cosh^2 z & 0 \\ 0 & \cosh^2 z \end{pmatrix}$$

$$\begin{pmatrix} l & m \\ m & n \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

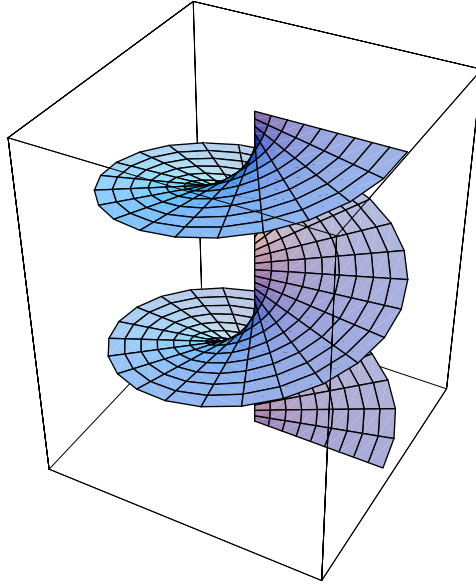


Figure 2: helicoid[Mathematica]

Therefore,

$$K = \frac{-1}{\cosh^4 z} \quad H = 0$$

This implies that Catenoid is a minimal surface. Indeed, the Catenoid is what you get if you have a bubble liquid in two circles face to face at first and separate them gradually. Object in this world, when they are free to move, tends to take the minimal surface! This is an interesting fact.

2.2.2 helicoid

Similarly, the parametrisation (local coordinate) of helicoid is given by

$$G : (u, v) \mapsto (u \cos v, u \sin v, v), v > 0, -\infty < v < \infty$$

The following calculation is routine.

$$\begin{aligned} X_u &= F_* \left(\frac{\partial}{\partial u} \right) = \cos v \frac{\partial}{\partial x} + \sin v \frac{\partial}{\partial y} \\ X_v &= F_* \left(\frac{\partial}{\partial v} \right) = -u \sin v \frac{\partial}{\partial x} + u \cos v \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \\ \gamma N &= -\sin v \frac{\partial}{\partial x} + \cos v \frac{\partial}{\partial y} - u \frac{\partial}{\partial z} \\ &\text{where } \gamma = \sqrt{1 + u^2} \end{aligned}$$

$$\begin{aligned}
X_{uu} &= 0 \\
X_{uv} &= -\sin v \frac{\partial}{\partial x} + \cos v \frac{\partial}{\partial y} \\
X_{vv} &= -u \cos v \frac{\partial}{\partial x} - u \sin v \frac{\partial}{\partial y}
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} E & F \\ F & G \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1+u^2 \end{pmatrix} \\
\begin{pmatrix} l & m \\ m & n \end{pmatrix} &= -\begin{pmatrix} 0 & \frac{1}{\sqrt{1+u^2}} \\ \frac{1}{\sqrt{1+u^2}} & 0 \end{pmatrix}
\end{aligned}$$

Therefore,

$$K = \frac{-1}{(1+u^2)^2} \quad H = 0$$

The mean curvature happens to be the same as Catenoid, but this is an accident.

On the other hand, Gaussian curvature ought to be the same. The reason is the following

Claim 1 *I claim Catenoid and helicoid are isometric surfaces by a local isometry I^{-1} given by $(v, u) = (\theta, \sinh z)$.*

I is a local isometry iff $(I_*E_i, I_*E_j) = (E_i, E_j)$. Here, we have

$$\begin{aligned}
I_* \left(\frac{\partial}{\partial \theta} \right) &= \frac{\partial}{\partial v} \\
I_* \left(\frac{\partial}{\partial z} \right) &= \cosh z \frac{\partial}{\partial u}
\end{aligned}$$

Now it is easy to check the equation above, $(I_*E_i, I_*E_j) = (E_i, E_j)$ holds. Also, by substituting $(v, u) = (\theta, \sinh z)$, we see immediately that Gaussian curvature, K , for Catenoid and helicoid are the same. For the intuitive understanding of this isometry, please see Figure. 3 below.

2.3 Riemannian Curvature Tensor and The Theorema Egregium of Gauss

Now let's show that Gaussian curvature is, in fact, an intrinsic property. However, it is not easy to see that Gaussian curvature defined by k_1k_2 is actually an intrinsic property of a surface M . There are two issues we face. One is that the definition of shape operator I gave above explicitly depends

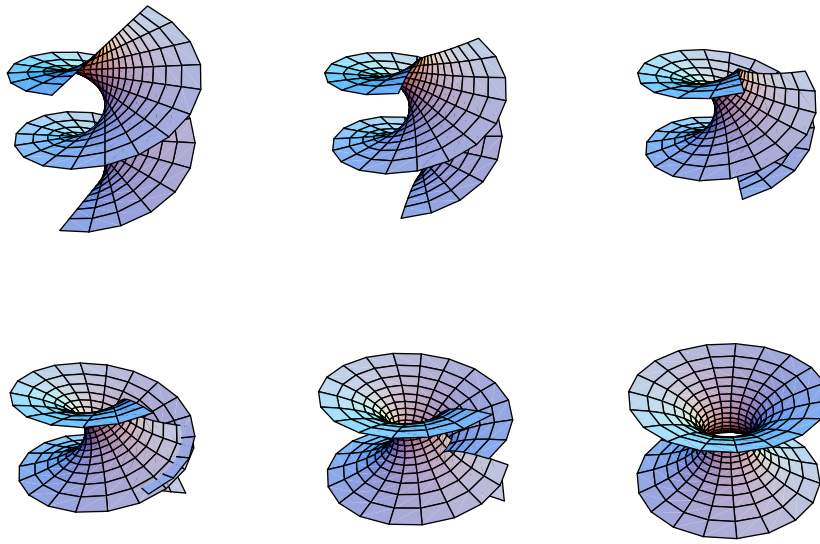


Figure 3: Transformation of helicoid into Catenoid[Mathematica]

on the embedding of a surface M in R^3 . That is, choosing N is possible because M is embedded in R^3 .

Second is that in general, the differentiation of a vector X in $T_q(M)$ does not yield a vector in $T_q(M)$. Therefore, derivative of X means nothing to the resident of the surface. This is a problem because curvature had better be defined in terms of differentiation.

To resolve the latter problem, we can project the derivative on the tangent space of M . More generally, given a vector X_q at q and vector fields Y , we can define a derivative along X_q to be $\frac{DY}{dt}$ by $\pi \frac{dY}{dt}$ where π is a projection on the tangent space of M . In the definition, it is implicitly assumed that we pick $p(t)$ such that $p(t_0) = q$ and $\left. \frac{dp}{dt} \right|_{t_0} = X_q$ and we differentiate Y along $p(t)$. Then we can show that this derivative does not depend on the choice of $p(t)$. This allows us to write the derivative as $D_{X_q}Y$. Furthermore, the following theorem holds [Boothby 306]

Theorem 4 X, X', Y, Z are vector fields on a manifold M and f, g are C^∞ mappings. We can see $X_q \rightarrow D_{X_q}Y$ to be a mapping $T_q(M) \rightarrow T_q(M)$. If we define the value of $D_X Y$ at q to be $D_{X_q}Y$, then an operator D_X on Y defined by $D_X Y$ is a mapping from a vector field to a vector fields. Then, the derivative D (called covariant derivative) has the following properties.

(1) D is linear in X_q and f .

$$D_{fX_q+gX'_q}Y = fD_{X_q}Y + gD_{X'_q}Y$$

(2) D acts on fY in the following way.

$$D_{X_q}fY = (X_qf)Y + fD_{X_q}Y$$

(3) $[X, Y] = D_XY - D_Y(X)$

(4) $X_q(Y, Z) = (D_{X_q}Y, Z) + (Y, D_{X_q}Z)$

Yet our definition of derivative still depends on the fact that a surface is embedded in R^3 . The issue is the same one as that we had to use N to define the shape operator. Here's a cute way to get around it: we *define* the derivative of vector fields (called *Riemannian connection*) in such a way that (1) allows us to define derivative without depending on the surroundings (2) retains the properties of covariant derivative $D_{X_q}Y$ defined above.

So here we define a *Riemannian connection*.

Definition 2 X, X', Y, Z are vector fields on a manifold M and f, g are C^∞ mappings. Define a *Riemannian connection* ∇ to be a mapping from a vector field to a vector field that satisfies

(1) ∇ is linear in X and f .

$$\nabla_{fX+gX'}Y = f\nabla_XY + g\nabla_{X'}Y$$

(2) ∇ acts on fY as

$$\nabla_{X_q}fY = (X_qf)Y + f\nabla_{X_q}Y$$

(3) $[X, Y] = \nabla_XY - \nabla_Y(X)$

(4) $X(Y, Z) = (\nabla_XY, Z) + (Y, \nabla_XZ)$

When we define something axiomatic way, the problem is the existence of it. The existence and uniqueness of Riemannian connections on manifolds is guaranteed by Fundamental theorem of Riemannian geometry.

Theorem 5 *Let M be a Riemannian manifold. Then there exists a uniquely determined Riemannian connection on M*

sketch proof A Riemannian manifold is a manifold on which a metric g_{ij} is defined. If a uniqueness and existence of the connection on a local coordinate neighbourhood is proved, then, we can patch the connection and obtain

a unique connection on all M . On the overlap of the coordinate neighbourhood, connection must agree from the uniqueness requirement. Then we only need to prove the uniqueness and existence of the connection in a single coordinate neighbourhood. This is essentially proved by the following claim.

Claim 2 *If we define Γ_{ij}^k by $\nabla_{E_i} E_j = \Gamma_{ij}^k E_k$, then*

$$\Gamma_{ij}^k = \frac{1}{2} \sum_s g^{ks} \left(\frac{\partial g_{si}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i} \right)$$

where g^{ks} is the inverse of the metric g_{ks} .

For the detailed proof of this statement, please see [Boothby 313].

□

Now back to Gaussian curvature. We now have a way to differentiate vector fields without depending on the surroundings. From the construction of Riemannian connection above (Claim 2), we see that a connection is determined by Γ_{ij}^k , where Γ_{ij}^k is determined by a metric g_{ij} . Therefore, the connection is invariant under local isometries. Therefore, if we can define a curvature using the connection, and show the equivalence with Gaussian curvature, we could prove that Gaussian curvature is intrinsic property. So we define Riemannian curvature.

Definition 3 *Riemannian Curvature tensor is defined to be $R(E_i E_j E_k, E_l) = (R(E_i E_j E_k), E_l)$ where $R(E_i E_j E_k) = \nabla_{E_i} \nabla_{E_j} E_k - \nabla_{E_j} \nabla_{E_i} E_k - \nabla_{[E_i, E_j]} E_k$.*

This Riemannian curvature tensor has a very nice properties

Theorem 6 *Let f be a C^∞ mapping and X, Y, Z, W vector fields. Then Riemannian curvature tensor satisfies*

(1) *Linearity*

$$R(fXYZ) = R(XfYZ) = R(XYfZ) = fR(fXYZ)$$

(2) *antisymmetry*

$$\begin{aligned} R(XYZW) &= -R(YXZW) \\ R(XYZW) &= -R(YXWZ) \end{aligned}$$

(3) *symmetry*

$$R(XYZW) = R(ZWXY)$$

(4) *cyclic*

$$R(XYZW) + R(YZWX) + R(ZXYW) = 0$$

Then we have the following

Theorem 7 *The Gaussian curvature (determinant of S) is equal to $\frac{R(E_1, E_2, E_2, E_1)}{(E_1, E_1)(E_2, E_2) - (E_1 E_2)^2}$, where E_1, E_2 are coordinate frames.*

Sketch proof Since we had $K = \frac{ln - m^2}{EG - F^2}$, we only need to show $ln - m^2 = R(E_1, E_2, E_2, E_1)$. Remember that using the previous notation, we have $ln - m^2 = (N_u, X_u)(N_v, X_v) - (N_u, X_v)(N_u, X_u)$. Also note that $[E_i, E_j]f = \frac{\partial^2}{\partial u \partial v} f \phi^{-1} - \frac{\partial^2}{\partial v \partial u} f \phi^{-1} = 0$. Here is the important point. Since a Riemannian connection on a given manifold is unique, ∇ in this case is equivalent to D . Therefore, we can calculate ∇ by projecting it on $T_q M$. Then $\nabla_{X_u} X_v = X_{vu} - (N, X_{vu})N$. Therefore,

$$\begin{aligned} R(E_1 E_2 E_2, E_1) &= (\nabla_{X_u} \nabla_{X_v} X_v - \nabla_{X_v} \nabla_{X_u} X_v, X_u) \\ &= X_{vvv} - (N, X_{vv}) (N_v, X_u) - X_{vvu} + (N, X_{vv}) (N_u, X_u) \\ &= ln - m^2 \end{aligned}$$

□

Corollary 2 *If F_1 and F_2 are orthonormal coordinate frames, then $K = R(F_1, F_2, F_2, F_1)$.*

3 Sectional curvature, Ricci curvature and scalar curvature

3.1 definitions

Now that we know K is expressible in terms of R . On the other hand, if we can define the expression $R(F_1, F_2, F_2, F_1)$ to be a curvature, we would expect to get an analogue of Gaussian curvature for $\dim(M) \geq 2$.

Definition 4 *Let M be a n dimensional manifold and $T_q(M)$ be a tangent space at q on M . Let E_1, E_2 be any two different vectors in an orthonormal basis of $T_q(M)$. Then define a sectional curvature by $K(E_1, E_2) = R(E_1, E_2, E_2, E_1)$. Then $K(X, Y)$ is symmetric in its argument.*

Then the following theorem supports my statement above [see Boothby 381 or Lee 146]

Theorem 8 *Pick two independent vectors X, Y in $T_q(M)$. Call the surface N created by geodesics whose velocity vectors at q is $\text{span}(X, Y)$. Then Gaussian curvature of N is exactly $K(X, Y)$.*

Consider geodesic as "straight curve uniquely determined by a vector Z_q ." Theorem 8 is subtle and non-trivial in spite of the ease to understand it. The subtlety comes from the difference between derivative and covariant derivative: the differentiation in R^3 being different from differentiation on the surface. The book by [Lee] has a good proof of this.

In the case of 2-dimensional manifold(surface), Gauss curvature is the only intrinsic curvature that we know. Since principle curvatures can be obtained from K and H , and we know H is not intrinsic property, then naively we can think that K must be the only intrinsic curvature on M .

This intuition is not totally off the point. The following theorem gives us an idea of how much information Gaussian curvature and sectional curvature contain[Boothby 373].

Theorem 9 *Suppose we know all sectional curvatures at point q . Then the Riemannian curvature tensor at q is uniquely determined.*

sketch proof Suppose there are two Riemannian curvature tensor that gives you the sectional curvatures at q . Write them as R and \tilde{R} . Then define $R - \tilde{R} = A$. Then A has all the properties in Theorem 6. From $A(X + Z, Y, X + Z, Y) = 0$, we have $A(XY, ZY) = 0$ for any vector fields X, Y, Z . Similarly, forming $A(X, Y + W, Z, Y + W) = A(XYZW) - A(YZXW) = A(ZXYW) = 0$.

□

Therefore, we can conclude that in the case of 2-manifolds, knowing Gaussian curvature at each point suffices to determine Riemannian curvature. We'll see this explicitly later (Theorem 12).

If sectional curvature at all points of M is constant t , then the manifold is said to have constant curvature t . Examples are R^n with constant curvature 0 and S^n with constant curvatures $\frac{1}{R^2}$ where R is the radius.

The advantage of considering sectional curvature rather than Riemannian curvature is that since it's 2-covariant tensor, it contains less information and tractable compared to 4-covariant tensor. If we look at the proof of Theorem 9, we can see that there is another curvature that might come in handy.

Definition 5 *Let $F_i : 1 \leq i \leq n$ be an orthonormal basis for $T_q(M)$. Define a Ricci curvature by $Ric(X, Y) = \sum_{i=1}^n R(F_i, X, Y, F_i)$.*

The following theorem gives you an idea how Ricci curvature behaves.

Claim 3 *Ricci curvature has the following properties*

(1) *Ric is independent of the orthonormal basis chosen*

(2) symmetric $Ric(X Y) = Ric(Y X)$

(3) If we write $R(E_i E_j E_k, E_l) = R_{ijkl}^l$ where E_i s are coordinate frame, then $Ric_{jk} = Ric(E_j E_k) = R_{ijk}^i$. In other words, Ric_{jk} is the trace of the linear function defined by $Z \mapsto R(Z X Y)$ with coordinate frame.

Note that when there is the same indices in subscript and superscript, it implies summation (Einstein convention.) (1) is just a confirmation that Ric is well-defined. The consequence of (2) is that $Ric(V V)$ for all V in $T_q(M)$ determines Ric completely. This follows from $Ric(X Y) = \frac{1}{2}(Ric(X + Y X + Y) - Ric(X X) - Ric(Y Y))$. Then we can interpret Ricci curvature as following. Pick up a vector V in $T_q(M)$ and add all the sectional curvatures $K(V F_i)$ where $\{F_i\}$ is an orthonormal basis. Then you obtain Ricci curvature.

With the idea of (3), if we further take the trace

Definition 6 A scalar curvature R is the trace of a linear function $X \mapsto R(X X)$. Therefore, it is defined by $R = Ric_{jk} g^{jk}$.

Then, the scalar curvature is the sum of Ricci curvature of the form $Ric(F_i, F_i)$ where F_i s are orthonormal coordinate frame. R is a function on M that assigns each point a value R .

These curvatures are, in a sense, "natural" method to take information out of the most general curvature called Riemannian curvature. However, it is true that Ricci curvature or scalar curvature does not come as intuitively as Gaussian curvature or its analogue sectional curvature does. I would like to give more motivation to know these two curvatures.

Definition 7 Let M be a manifold. M is called Einstein Manifold if Ric_{ij} is a scalar multiple of g_{ij} at each point. So iff $Ric_{ij} = \lambda(q)g_{ij}$ where $\lambda(q)$ is a function on M . Then the metric g is called Einstein metric.

This name "Einstein" comes from Physics. The Einstein equation is

$$Ric - \frac{1}{2}Rg = T$$

where T is a stress-energy tensor. When T is equal to zero, then in components, we have $Ric_{ij} = \frac{1}{2}Rg_{ij}$. Since $g_{ij}g^{ji} = \sum_i \sum_l g_{ij}g^{jl} = \sum_i \sum_l \delta_i^l = dim(M) = 4$, we have $R = 2R$. Which implies $R = 0$ and $Ric = 0$. Therefore, the solution of Einstein equation in vacuum would be Einstein metric.

The following theorem will provide examples of Einstein manifolds.

Theorem 10 A manifold M with constant curvature t is Einstein manifold with Einstein constant $t(n-1)$ where $n = dim(M)$. That is, $Ric = t(n-1)g$.

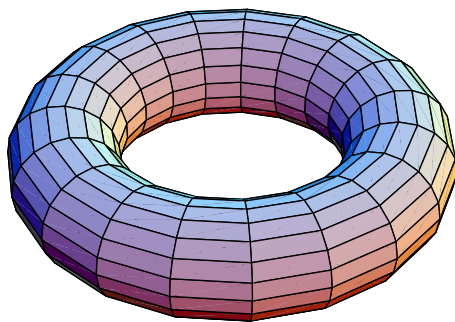


Figure 4: Torus[Mathematica]

proof First note that if F_i and F_j are orthonormal vectors and $X = aF_i + bF_j$, $Y = cF_i + dF_j$, then $R(X Y Y, X) = R(aF_i dF_j Y, X) - R(cF_i bF_j Y, X) = (ad - bc)R(F_i F_j Y, X) = (ad - bc)^2 K(F_i F_j)$. So we have $K(F_i F_j) = \frac{R(X Y Y, X)}{(X, X)(Y, Y) - (X, Y)^2}$. Let A, B, C denote arbitrary vectors and $\{F_i\}$ be an orthonormal basis. Now, from the condition of constant curvature, we have $\frac{R(F_i B+C B+C, F_i)}{(B+C, B+C) - (F_i, B+C)^2} = t$. Expanding the LHS, we obtain

$$\begin{aligned} 2R(F_i B C, F_i) &= \{(B + C, B + C) - (F_i, B + C)^2\}t - R(F_i B B, F_i) - R(F_i C C, F_i) \\ &= t\{(B + C, B + C) - (F_i, B + C)^2 - (B, B) + (F_i, B)^2 - (C, C) + (F_i, C)^2\} \\ &= t\{2(B, C) - 2(F_i, B)(F_i, C)\} \end{aligned}$$

Summing up the equation through $1 \leq i \leq n$ (note that $\sum_i (F_i, B)(F_i, C) = (B, C)$), we obtain the relation $Ric(X Y) = t(n - 1)(X, Y)$ as we claimed.

□

Therefore, we can conclude that S^n and R^n are examples of Einstein manifold. Then the question is, is there Einstein manifold that does not have constant curvature? The answer turns out to be yes. Claim 4 below will show our search for such an example should start from 4-manifolds. An example is $S^n \times S^n$ with metric $\sum_i (dx_i)^2 + \sum_i (dy_i)^2$.

3.2 Example: Torus

I provide an example of calculating all the curvatures mentioned above here.

Here's a very important theorem in doing the actual calculation of Riemannian curvature [Boothby 380].

Theorem 11 *If $R(E_k E_l E_i) = \sum_j R_{ikl}^j E_j$, then*

$$R_{ikl}^j = \frac{\partial \Gamma_{il}^j}{\partial x^k} - \frac{\partial \Gamma_{ik}^j}{\partial x^l} - \sum_h \left(\Gamma_{ik}^h \Gamma_{hl}^j - \Gamma_{il}^h \Gamma_{hk}^j \right)$$

and we have $R_{ijkl} = R(E_k E_l E_i, E_j) = \sum_h R_{ikl}^h g_{hj}$

This theorem and this equation $\Gamma_{ij}^k = \frac{1}{2} \sum_s g^{ks} \left(\frac{\partial g_{si}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i} \right)$ allows us to calculate Riemannian curvature.

We will work out Torus' curvatures with induced metric in R^3 . First of all, we'll figure out g and second fundamental form. For simplicity, we'll consider a torus with radius 1. Then the parametrisation is given by $F : (\phi, \theta) \mapsto \{(1 + \cos \phi) \cos \theta, (1 + \cos \phi) \sin \theta, \sin \phi\}$ where $0 \leq \phi < \pi, \pi < \phi \leq 2\pi$ and $0 \leq \theta \leq 2\pi$. I excluded $\phi = \pi$, but one can always avoid this unpleasant setting by having a torus with radius a, b where $a \neq b$. Then,

$$\begin{aligned} X_\phi &= F_* \left(\frac{\partial}{\partial \phi} \right) = -\sin \phi \cos \theta \frac{\partial}{\partial x} - \sin \phi \sin \theta \frac{\partial}{\partial y} + \cos \phi \frac{\partial}{\partial z} \\ X_\theta &= F_* \left(\frac{\partial}{\partial \theta} \right) = -(1 + \cos \phi) \sin \theta \frac{\partial}{\partial x} + (1 + \cos \phi) \cos \theta \frac{\partial}{\partial y} \\ N &= \cos \phi \cos \theta \frac{\partial}{\partial x} + \cos \phi \sin \theta \frac{\partial}{\partial y} + \sin \phi \frac{\partial}{\partial z} \end{aligned}$$

and

$$\begin{aligned} X_{\phi\phi} &= -\cos \phi \cos \theta \frac{\partial}{\partial x} - \cos \phi \sin \theta \frac{\partial}{\partial y} - \sin \phi \frac{\partial}{\partial z} \\ X_{\phi\theta} &= \sin \phi \sin \theta \frac{\partial}{\partial x} - \sin \phi \cos \theta \frac{\partial}{\partial y} \\ X_{\theta\theta} &= -(1 + \cos \phi) \cos \theta \frac{\partial}{\partial x} - (1 + \cos \phi) \sin \theta \frac{\partial}{\partial y} \end{aligned}$$

From here, we obtain

$$\begin{aligned} g_{ij} &= \begin{pmatrix} (1 + \cos \phi)^2 & 0 \\ 0 & 1 \end{pmatrix} & g^{ij} &= \begin{pmatrix} \frac{1}{(1 + \cos \phi)^2} & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} l & m \\ m & n \end{pmatrix} &= \begin{pmatrix} \cos \phi (1 + \cos \phi) & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore,

$$K = \frac{\cos \phi}{1 + \cos \phi}$$

Notice that K is positive in the inner part of torus, K is zero at the top and bottom part of torus, and K is negative in the outer part of torus.

Now here comes the calculation of Γ . Note that $\Gamma_{ij}^k = \Gamma_{ji}^k$ and all the g_{ij} does not have θ . Let me set $\theta = 1$ and $\phi = 2$. This is not too bad.

$$\begin{aligned}\Gamma_{11}^1 &= 0 & \Gamma_{12}^1 = \Gamma_{21}^1 &= -\frac{\sin \phi}{1 + \cos \phi} & \Gamma_{22}^1 &= 0 \\ \Gamma_{11}^2 &= \sin \phi(1 + \cos \phi) & \Gamma_{12}^2 = \Gamma_{21}^2 &= 0 & \Gamma_{22}^2 &= 0\end{aligned}$$

Next, we want to calculate R_{ijkl}^j . Since R is highly symmetric, most of them are zero. If three of them are the same, then R_{ijkl} is zero. Also, if ij or kl are the same, then R_{ijkl} is zero. Then, the only value that might not be zero is R_{1122} or R_{1212} . Since $R_{1122} = R_{122}^j g_{2j}$ and $R_{1212} = R_{112}^j g_{2j}$, we need only those. In fact, we have $g_{21} = 0$, we want only R_{122}^2 and R_{112}^2 . Therefore,

$$R_{122}^2 = 0 \quad R_{112}^2 = -\frac{\partial \Gamma_{11}^2}{\partial \phi} + \Gamma_{12}^1 \Gamma_{11}^2 = -\cos \phi(1 + \cos \phi)$$

Then, now we know all the Riemannian curvature.

$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121} = R_{112}^2 g_{22} = -\cos \phi(1 + \cos \phi), \text{ otherwise, zero.}$$

Let's calculate Ricci curvature. Ricci curvature is trace, so given by $Ric_{il} = R_{ijkl} g^{jk}$.

$$Ric_{11} = R_{1221} g^{22} = \cos \phi(1 + \cos \phi) \quad Ric_{21} = Ric_{12} = 0 \quad Ric_{22} = R_{2112} g^{11} = \frac{\cos \phi}{1 + \cos \phi}$$

Finally, scalar curvature is

$$R = \sum_{ij} R_{ij} g^{ij} = 2 \frac{\cos \phi}{1 + \cos \phi}$$

3.3 2-manifold and 3-manifold

Through the example above, one thing we notice is that most terms of R_{ijkl} vanishes. This is not an accident. It comes from the fact that the manifold is 2-dimensional and R_{ijkl} is highly symmetric. In fact, in the case of 2-manifolds, information in scalar curvature is large enough that scalar curvature alone characterises all the other curvatures. (Lee 144).

Theorem 12 *Let M be a 2-dimensional manifold and X, Y, Z, W be vectors in $T_q(M)$. Let K be Gaussian curvature. Then on M*

- (1) $R(X, Y, Z, W) = \frac{1}{2}R((X, W)(Y, Z) - (X, Z)(Y, W))$
- (2) $Ric(X, Y) = \frac{1}{2}R(X, Y)$
- (3) $K = \frac{1}{2}R$

proof Since Riemannian curvature is linear, we only need to prove the case when X, Y, Z, W are elements in the basis. Let E_1, E_2 be orthonormal basis for $T_q(M)$. Then in this basis, we have $R_{11} = R_{22} = K$, and $R_{12} = R_{21} = 0$. Therefore, $R = R_{11} + R_{22} = 2K$. Finally, the only terms of Riemannian curvature that does not vanish are $R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = K$.

□

This indeed agrees with the calculation we carried out above. Note that Theorem 12 says we can characterise curvatures in terms of Gaussian curvature too. So, in a sense, Gaussian curvature is the only intrinsic curvature.

Similar result can be obtained for 3-manifolds. We saw that the values of sectional curvature $K(X, Y)$ completely determines Riemannian curvature. On 3-manifolds, the values of Ricci curvature completely determines the values of sectional curvature. Therefore, the information contained in Ricci curvature is the same as the information contained in Riemannian curvature in 3-manifolds [Petersen 31].

Claim 4 *Let E_1, E_2, E_3 be an orthonormal basis of a tangent space $T_q(M)$ of 3-dimensional manifold M . Let $K(X, Y)$ be sectional curvature and Ric be Ricci curvature. Then*

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} K(E_1, E_2) \\ K(E_1, E_3) \\ K(E_2, E_3) \end{pmatrix} = \begin{pmatrix} Ric(E_1, E_1) \\ Ric(E_2, E_2) \\ Ric(E_3, E_3) \end{pmatrix}$$

The matrix on the left has determinant = -2, therefore, invertible. So Ricci curvature determines sectional curvatures and vice versa. In particular, M is Einstein Manifold iff it has constant curvature.

4 Gauss-Bonnet Theorem

4.1 proof

So far, we saw how we can define curvatures of a curve $p(t)$ and generalise the idea to n-dimensional manifolds. The curvature we obtained in this way (Riemannian curvature) is a 4-covariant tensor. Ricci curvature, sectional curvature, and scalar curvature are ways to extract tractable information from Riemannian curvature. The relationships between those curvatures are described in the last section.

In this section, I'd like to describe Gauss-Bonnet Theorem, which connects Gaussian curvature of 2-manifolds M and Euler characteristic of M . This is the first theorem we see that relates local property (Gaussian curvature) and global, topological property (Euler characteristic). Here's the theorem (Lee 167).

Theorem 13 (The Gauss-Bonnet Theorem) *Let M be a triangulated, compact, oriented, Riemannian 2-manifolds, then*

$$\int_M K dA = 2\pi\chi(M)$$

Sketch Proof The essential part of the proof comes from the following theorem.

Theorem 14 (Rotation Angle Theorem) *If p is a positively oriented curved polygon in the plane, the rotation angle of p is exactly 2π .*

Sketch proof First suppose the closed curve p is C^∞ and $p(a) = p(b)$. Let's extend the domain to all of R such that for any t , we have $p(t) = p(t+b-a)$. Given a smooth closed curve p , we can put the curve in R^2 . Then we can grab the lowest point $p(c) = p(c+b-a) = p(d)$ (e.g. the point which has the smallest y value) and translate the curve so that the point $p(a)$ comes to origin. Now we have the whole curve in $0 \leq y$ region. This translation and choice of origin does not change the rotation angle that I will describe now. At each point q of p , we can draw a velocity vector. Measure the angle between the velocity vector and x axis and call it $\theta(q)$. We can take this function $\theta(q)$ to be smooth. Define rotation angle as $\int_c^d \dot{\theta} dt$ or $\theta(d) - \theta(c)$. Since $p(d) = p(c)$, the rotation angle must be an integer multiple of 2π . Now we only need to show this is actually 2π .

Define a function of angle $F(a, b) \mapsto \phi$ that assigns two points $p(a), p(b)$ on the curve p to an angle ϕ . The prescription is the following. Consider the vector $V = \overline{p(b) - p(a)}$ and take the angle between the vector V and x axis. Also define $F(t, t) = \dot{p}(t)$. Then we can show the function F is smooth. Then the rotation angle is now $\theta(d) - \theta(c) = F(d, d) - F(c, c)$. Since it's always that $0 \leq F(c, t) \leq \pi$ and $\pi \leq F(t, d) \leq 2\pi$ where $c \leq t \leq d$, it follows that $0 \leq F(d, d) - F(c, c) \leq 2\pi$. So, the rotation angle must be 2π .

When p is not smooth, we can chop it up into finite number of smooth curves. Suppose $p(e)$ is the edge. Then define $p(e)$ from right, that is, $\theta(e) = \lim_{t \rightarrow e^+} \theta(t)$. Then there's going to be an angle $\epsilon_i = \theta(e) - \lim_{t \rightarrow e^-} \theta(t)$. The idea is that we can smooth out this edge in such a way that does not change ϵ_i . Then we obtain a smooth closed curve with the same rotational angle as the original polygon.

□

The next step is to show the relationships between θ and curvature. We can do so by using the R^2 analogue of shape operator $S(D\dot{p}(t), N) = (X_q, \frac{N}{dt}) = -(X_q, S(X_q))$, where $X_q = \left. \frac{dp}{dt} \right|_{atq}$. The key point of using shape

operator in this proof is that this value depends only on X_q , as we saw before. For details please see [Lee 164]. Then, we obtain the Gauss-Bonnet Formula

Theorem 15 (The Gauss-Bonnet Formula) *Suppose p is a curved polygon on an oriented Riemannian 2-manifold, and p is positively oriented as the boundary of an open set Ω with compact closure, Then*

$$\int_{\Omega} K dA + \int_p -(X_q, S(X_q)) ds + \sum_i \epsilon_i = 2\pi \quad (3)$$

We apply this formula to each triangles obtained from triangulation. Let N_t be the number of triangles, N_e be the number of edges and N_v be the number of vertexes. Each ϵ_i corresponds to exterior angle of triangle. Let me denote the interior angle of triangle by T_j^i where $0 \leq i \leq N_t$ and $j = 1, 2, 3$. Since the integration around triangles takes place twice in the opposite direction, the middle term in eq.3 vanishes. Then the formula apply to all triangles become

$$\int_{\Omega} K dA - \sum_{i=1}^{N_t} \sum_{j=1}^3 (\pi - T_j^i) = 2\pi N_t$$

Note that the sum of T_j^i should be 2π times the number of vertexes. Also, since each triangle has 3 edges, it should be that $3N_t = N_e$. But in this calculation we count the same edge twice, so in the calculation of Euler characteristic, it must be that $3N_t = 2N_e$. These consideration and the fact that $\chi = N_v - N_e + N_t$ lead to the result $\int_{\Omega} K dA = 2\pi\chi(M)$.

□

4.2 Example:Torus

Using the formula we obtained in the torus example, we can calculate $\int_{\Omega} K dA$ (O'Neill 293). If the radius of torus a, b are equal, there would a singularity in Gaussian curvature, so let's avoid by setting $a \neq b$. Then the Gaussian curvature for torus is, with slight modification,

$$K = \frac{\cos \phi}{b(a + b \cos \phi)}$$

The area element can be obtained through $\sqrt{\|X_{\theta} \times X_{\phi}\|} d\phi \wedge d\theta = (a + b \cos \phi) d\phi \wedge d\theta$. Therefore,

$$\begin{aligned} \int_{\Omega} K dA &= \left(\int_0^{2\pi} \left(\int_0^{2\pi} \frac{\cos \phi}{b(a + b \cos \phi)} \cdot (a + b \cos \phi) d\phi \right) d\theta \right) \\ &= 0 \end{aligned}$$

Euler characteristic of torus is, using the formula $\chi = 2 - 2g$ where g is genus, 0.

5 Classification of 2 and 3 manifolds

The result of The Gauss-Bonnet Theorem tells us that for closed, compact 2-manifolds, Gaussian curvature (or any other curvatures mentioned above) contains information about not only how much curves bend on the manifolds, but also how many genus the manifold should have. Then this implies even if we have the same topological object, say Torus, and endow it with different metric, say induced metric in R^3 and $S^1 \times S^1$, we would obtain the same total curvature ($\int_{\Omega} K dA$). Therefore, total curvature is invariant under the choice of metric.

Then, if we could pick a metric such that K is constant all over M , then the sign of K is uniquely determined by the topological properties of 2-manifolds. This would motivate us to classify 2-manifolds in terms of the sign of the constant curvatures. It turns out that any 2-manifold indeed admits a Riemannian metric of constant curvature (it is called Uniformization Theorem).

We could say that classification of 2-manifolds in terms of Gaussian curvature is possible because according to Theorem 12, Gaussian curvature contains all the information that Riemannian curvature contains. With the similar idea, I would think that for the classification of 3-manifolds, Ricci curvature is the key player (Claim 4). In fact, the study of an equation involving Ricci curvature yields a promising result to classify 3-manifolds (for details, please see [Chow and Knopf]).

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