Periodicity in Sequences Mod 3

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September 5, 2002

0.1. Puzzle. For each positive integer \( k \), consider the ternary sequence \( \{x_n\} \) given initially by

\[
x_n = \begin{cases} 
0 & \text{if } 1 \leq n < k \\
1 & \text{if } n = k
\end{cases}
\]

and thereafter determined by the quadratic recurrence

\[
x_n = x_{n-1} + x_{2n-k}^2 \mod 3.
\]

Define \( N \) to be the smallest positive integer for which \( x_{n+N} = x_n \) for all sufficiently large \( n \). For example, if \( k = 4 \), then \( N = 9 \) is obvious by looking at the following \( 6 \times 10 \) table of the first 60 terms of \( \{x_n\} \):

\[
\begin{array}{ccccccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 1 \\
2 & 0 & 0 & 1 & 2 & 2 & 2 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 & 2 & 0 & 1 & 2 & 0 & 1 \\
0 & 1 & 2 & 2 & 2 & 0 & 1 & 2 & 0 & 1 \\
1 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
2 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\
\end{array}
\]

Examine \( N \) as a function of \( k \). What patterns can be seen? Can a formula for \( N(k) \) be written down and proved?

Repeat for the recurrence

\[
x_n = x_{n-1}^2 + x_{n-k} \mod 3.
\]

Repeat for the recurrence

\[
x_n = 2 \left( x_{n-1}^2 + x_{n-1} + x_{n-k}^2 + x_{n-k} \right) \mod 3.
\]
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0.2. Solution by Bob Harder (Wed, 19 Jan 2000 and Thu, 5 Apr 2000).
Using a Fortran program, I found the following results. Corresponding to \( x_n = x_{n-1} + x_{n-k}^2 \mod 3 \), the values of \( N(k) \) for \( 1 \leq k \leq 300 \) are

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Corresponding to \( x_n = x_{n-1}^2 + x_{n-k} \mod 3 \), the values of \( N(k) \) for \( 1 \leq k \leq 23 \) are

\[
\begin{array}{cccccc}
k & 1 & 8 & 22 & 45 & 138 \\
6 & 415 & 916 & 3998 & 13142 & 38763 \\
11 & 60718 & 44686 & 121298 & 2068731 & 11214378 \\
16 & 25158877 & 3909879 & 299954193 & 977046702 & 3028468981 \\
21 & 1107563239 & 2983913960 & 91973871622
\end{array}
\]

and corresponding to \( x_n = 2\left(x_{n-1}^2 + x_{n-1} + x_{n-k}^2 + x_{n-k}\right) \mod 3 \), the values of \( N(k) \) for \( 1 \leq k \leq 36 \) are

\[
\begin{array}{cccccc}
k & 1 & 3 & 7 & 15 & 21 \\
6 & 63 & 127 & 63 & 73 & 889 \\
11 & 1533 & 3255 & 7905 & 11811 & 32767 \\
16 & 255 & 273 & 253921 & 413385 & 761763 \\
21 & 5461 & 4194303 & 2088705 & 2097151 & 10961685 \\
26 & 298935 & 125829105 & 17895697 & 402653181 & 10845877 \\
31 & 2097151 & 1023 & 1057 & 255652815 & 3681400539 \\
36 & 22839252821
\end{array}
\]

I am still looking for patterns in this data....