Let $L$ denote the positive octant of the regular $d$-dimensional cubic lattice. Each vertex $(j_1, j_2, \ldots, j_d)$ of $L$ is adjacent to all vertices of the form $(j_1, j_2, \ldots, j_k + 1, \ldots, j_d)$, $1 \leq k \leq d$. A $d$-partition of a positive integer $n$ is an assignment of nonnegative integers $n_{j_1, j_2, \ldots, j_d}$ to the vertices of $L$, subject to both an ordering condition

$$n_{j_1, j_2, \ldots, j_d} \geq \max_{1 \leq k \leq d} n_{j_1, j_2, \ldots, j_k+1, \ldots, j_d}$$

and a summation condition $\sum n_{j_1, j_2, \ldots, j_d} = n$. The summands in the $d$-partition are thus nonincreasing in each of the $d$ lattice directions. We agree to suppress all zero labels. A 1-partition is the same as an ordinary partition; a 2-partition is often called a plane partition and a 3-partition is often called a solid partition. Three sample plane partitions of $n = 26$ are

$$\begin{bmatrix} 8 \\ 9 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 4 \\ 2 \\ 1 \\ 1 \\ 5 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad (7, 6, 4, 4, 3, 1, 1).$$

Let $p_d(n)$ denote the number of $d$-partitions of $n$. The generating functions [1]

$$1 + \sum_{n=1}^{\infty} p_1(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \cdots$$

$$= \prod_{m=1}^{\infty} (1 - x^m)^{-1},$$

$$1 + \sum_{n=1}^{\infty} p_2(n)x^n = 1 + x + 3x^2 + 6x^3 + 13x^4 + 24x^5 + 48x^6 + 86x^7 + 160x^8 + \cdots$$

$$= \prod_{m=1}^{\infty} (1 - x^m)^{-m}$$

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give rise to well-known asymptotics [2, 3, 4, 5]:

\[ p_1(n) \sim \frac{1}{4\sqrt{3}n} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) \]
\[ \sim (0.1443375672...)^{-n} \exp \left( (2.5650996603...)n^{1/2} \right), \]

\[ p_2(n) \sim \frac{\zeta(3)^{7/36}e^{\zeta(-1)}}{2^{11/36}\sqrt{3\pi n^{25/36}}} \exp \left( 3\zeta(3)^{1/3} \left( \frac{n}{2} \right)^{2/3} \right) \]
\[ \sim (0.2315168134...)^{-n} \exp \left( (2.0094456608...)n^{2/3} \right) \]

as \( n \to \infty \), where \( \zeta(3) = 1.2020569031... \) is Apéry’s constant [6] and \( \zeta'(-1) = -0.1654211437... = 2(-0.0827105718...) = \ln(0.8475366941...) \) is closely related to the Glaisher-Kinkelin constant [7]. Although an infinite product expression for the generating function [1]

\[ 1 + \sum_{n=1}^{\infty} p_3(n)x^n = 1 + x + 4x^2 + 10x^3 + 26x^4 + 59x^5 + 140x^6 + 307x^7 + 684x^8 + \cdots \]

remains unknown, it is conjectured that [8, 9]

\[ p_3(n) \sim C_{n^{61/96}} \exp \left( \frac{2^7/4\pi}{3^{5/4}5^{1/4}n^{3/4}} \right) \]
\[ \sim C n^{-61/96} \exp \left( (1.7898156270...)n^{3/4} \right) \]

for some constant \( C > 0 \). The evidence for this asymptotic formula includes exact enumerations (for \( n \leq 68 \)) and Monte Carlo simulation. See [10, 11, 12, 13] for more about planar partitions and [14, 15, 16, 17] for more about solid partitions.

0.1. Hardy-Ramanujan-Rademacher. The Hardy-Ramanujan-Rademacher formula for \( p_1(n) \) is a spectacular exact result [18, 19, 20, 21, 22, 23, 24, 25, 26]:

\[ p_1(n) = \frac{\pi}{2^{5/4}3^{3/4}} \left( n - \frac{1}{24} \right)^{-3/4} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{3/2} \left( \sqrt{\frac{2\pi}{3k}} \sqrt{n - \frac{1}{24}} \right) \]

where

\[ I_{3/2}(x) = \sqrt{\frac{2x}{\pi}} \left( \cosh(x) - \frac{\sinh(x)}{x} \right) \]

is the modified Bessel function of order 3/2,

\[ A_k(n) = \sum_{\gcd(h,k)=1}^{\omega_{h,k} \exp \left( \frac{-2\pi inh}{k} \right),} \]
and \( \omega_{h,k} = \exp(\pi i s(h,k)) \) is the unique \( 24k^{\text{th}} \) root of unity with Dedekind sum

\[
s(h,k) = \sum_{m=1}^{k-1} \left( \frac{m}{k} - \left\lfloor \frac{m}{k} \right\rfloor - \frac{1}{2} \right) \left( \frac{hm}{k} - \left\lfloor \frac{hm}{k} \right\rfloor - \frac{1}{2} \right).
\]

For example,

\[
A_1(n) = 1, \quad A_2(n) = (-1)^n, \quad A_3(n) = 2 \cos \left( \frac{\pi(12n-1)}{18} \right),
\]

\[
A_4(n) = 2 \cos \left( \frac{\pi(4n-1)}{8} \right), \quad A_5(n) = 2 \cos \left( \frac{\pi(2n-1)}{5} \right) + 2 \cos \left( \frac{4\pi n}{5} \right).
\]

Defining

\[
c = \sqrt{\frac{2}{3}} \pi, \quad \lambda(n) = \sqrt{n - \frac{1}{24}},
\]

\[
\mu(n) = c\lambda(n), \quad A_k^*(n) = A_k(n)/\sqrt{k},
\]

we have the following variations:

\[
p_1(n) = \frac{1}{2^{1/2}\pi} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn} \left[ \frac{\sinh(c\lambda(n)/k)}{\lambda(n)} \right]
\]

\[
= 2 \frac{3^{1/2}}{24n-1} \sum_{k=1}^{\infty} A_k^*(n) \left[ \left( 1 - \frac{k}{\mu(n)} \right) \exp \left( \frac{\mu(n)}{k} \right) + \left( 1 + \frac{k}{\mu(n)} \right) \exp \left( -\frac{\mu(n)}{k} \right) \right].
\]

In contrast, the original Hardy-Ramanujan formula is only an asymptotic expansion:

\[
p_1(n) \sim \frac{1}{2^{3/2}\pi} \sum_{k=1}^{\infty} A_k(n) k^{1/2} \frac{d}{dn} \left[ \frac{\exp(c\lambda(n)/k)}{\lambda(n)} \right]
\]

\[
\sim 2 \frac{3^{1/2}}{24n-1} \sum_{k=1}^{\infty} A_k^*(n) \left( 1 - \frac{k}{\mu(n)} \right) \exp \left( \frac{\mu(n)}{k} \right),
\]

which was later proved to be divergent by Lehmer [27, 28, 29]. Therefore Rademacher’s contribution was the identification of a small additional term that forces the original Hardy-Ramanujan series to converge.

A third formula for \( p_1(n) \):

\[
p_1(n) \sim \frac{\pi}{2^{3/4}3^{3/4}} \lambda(n)^{-3/2} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{-3/2} \left( \frac{c\lambda(n)}{k} \right)
\]
appears in Almkvist [30, 31] and is a consequence of a more general theory (to be discussed shortly). The only difference between this formula and the Hardy-Ramanujan-Rademacher formula is that \( I_{-3/2} \) appears rather than \( I_{3/2} \). It is believed to be divergent, but this has not yet been proved. For practical purposes, using the modified Bessel function of order \(-3/2\):

\[
I_{-3/2}(x) = \sqrt{\frac{2x}{\pi}} \left( \frac{\sinh(x)}{x} - \frac{\cosh(x)}{x^2} \right)
\]
gives only slightly different numerical results (for large \( \sqrt{n/k} \)).

Analogous series exist for plane partitions. The terms involve neither exponentials nor Bessel functions, but rather a new function

\[
g(x, \gamma) = \sum_{\nu=0}^{\infty} \frac{x^{2\nu+\gamma-1}}{\nu!\Gamma(2\nu + \gamma)}
\]

that satisfies the third-order differential equation

\[
 xg'''(x, \gamma) - (\gamma - 3)g''(x, \gamma) - 2g(x, \gamma) = 0
\]

(the derivatives are taken with respect to \( x \)) as well as

\[
g'(x, \gamma) = g(x, \gamma - 1), \quad 2g(x, \gamma + 2) + (\gamma - 1)g(x, \gamma) = xg(x, \gamma - 1).
\]

A heuristic argument in [30, 31] gives that

\[
p_2(n) \sim \phi_1(n) + \phi_2(n) + \phi_3(n) + \cdots
\]
as \( n \to \infty \), where

\[
\phi_1(n) = \zeta(3)^{13/24} e^{(\zeta(3) - 1)} \sum_{k=0}^{\infty} a_{2k} \zeta(3)^k g \left( n \sqrt{\zeta(3)}, -\frac{1}{12} - 2k \right)
\]

and \( a_{2k} \) is the coefficient of \( x^{2k} \) in the Maclaurin series of

\[
h(x) = \exp \left( -\sum_{j=1}^{\infty} \frac{2(2j + 1)! \zeta(2j) \zeta(2j + 2)}{j(2\pi)^{4j+2}} x^{2j} \right),
\]

\[
\phi_2(n) = (-1)^n 2^{-5/3} \zeta(3)^{7/12} e^{2\zeta'(-1)} \sum_{k=0}^{\infty} b_{2k} \left( \frac{\zeta(3)}{8} \right)^k g \left( n \sqrt{\zeta(3)/8}, -\frac{1}{6} - 2k \right)
\]
and $b_{2k}$ is the coefficient of $y^{2k}$ in the Maclaurin series of

$$\frac{h(2y)^5}{h(y)h(4y)^2},$$

and so forth. The additional terms $\varphi_3(n)$, $\varphi_4(n)$ appear in [30] and $\varphi_5(n)$, $\varphi_6(n)$ appear in [31]. Taken together, these terms provide remarkably accurate estimates of $p_2(n)$. Govindarajan & Prabhakar [32] revisited Almkvist’s results, using a modified function

$$\tilde{g}(x, \gamma) = \frac{1}{2} \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu! \Gamma((3-\gamma+\nu)/2)}$$

that seems better behaved than $g(x, \gamma)$ and evidently does for $p_2(n)$ akin to what Rademacher’s modification of Hardy-Ramanujan did for $p_1(n)$.

0.2. Addendum. Recent Monte Carlo work indicates that [33]

$$\lim_{n \to \infty} n^{-3/4} \ln (p_3(n)) \approx 1.822 > 1.789... = \frac{27/4\pi}{3^{5/4}5^{1/4}},$$

contradicting [8, 9]. The asymptotics of solid partitions appear to differ sharply from those of line and plane partitions; in addition to sub-leading terms of order $n^{1/2}$, $n^{1/4}$ and $\ln(n)$, there seems to be an oscillatory function at the $n^{-1/4}$ level. Theory lags far behind numerical experimentation here.

Let us consider one of many possible variations on 1-partitions. Define $\hat{p}_1(n)$ to be the number of partitions of $n$ into integers, each of which may occur only an odd number of times. It can be shown that [34]

$$\hat{p}_1(n) \sim \frac{B}{2\pi n} \exp \left(2B\sqrt{n}\right)$$

where

$$B^2 = \frac{\pi^2}{12} + \int_0^1 \frac{\ln(1 + x - x^2)}{x} dx = \frac{\pi^2}{12} + 2 \ln(\varphi)^2$$

$$= \frac{\pi^2}{12} + 0.4631296411... = (1.1338415562...)^2$$

and $\varphi = (1 + \sqrt{5})/2$ is the Golden mean.
References


