Ornstein-Uhlenbeck Process

Steven Finch

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We first define several words. A stochastic process \( \{ Y_t : t \geq 0 \} \) is

- **stationary** if, for all \( t_1 < t_2 < \ldots < t_n \) and \( h > 0 \), the random \( n \)-vectors \((Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})\) and \((Y_{t_1+h}, Y_{t_2+h}, \ldots, Y_{t_n+h})\) are identically distributed; that is, time shifts leave joint probabilities unchanged

- **Gaussian** if, for all \( t_1 < t_2 < \ldots < t_n \), the \( n \)-vector \((Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})\) is multivariate normally distributed

- **Markovian** if, for all \( t_1 < t_2 < \ldots < t_n \), \( P(Y_{t_n} \leq y | Y_{t_1}, Y_{t_2}, \ldots, Y_{t_{n-1}}) = P(Y_{t_n} \leq y | Y_{t_{n-1}}) \); that is, the future is determined only by the present and not the past.

Also, a process \( \{ Y_t : t \geq 0 \} \) is said to have **independent increments** if, for all \( t_0 < t_1 < \ldots < t_n \), the \( n \) random variables \( Y_{t_1} - Y_{t_0}, Y_{t_2} - Y_{t_1}, \ldots, Y_{t_n} - Y_{t_{n-1}} \) are independent. This condition implies that \( \{ Y_t : t \geq 0 \} \) is Markovian, but not conversely. The increments are further said to be **stationary** if, for any \( t > s \) and \( h > 0 \), the distribution of \( Y_{t+h} - Y_{s+h} \) is the same as the distribution of \( Y_{t} - Y_{s} \). This additional provision is needed for the following definition.

A stochastic process \( \{ W_t : t \geq 0 \} \) is a **Wiener-Lévy process** or **Brownian motion** if it has stationary independent increments, if \( W_t \) is normally distributed and \( E(W_t) = 0 \) for each \( t > 0 \), and if \( W_0 = 0 \). It follows immediately that \( \{ W_t : t > 0 \} \) is Gaussian and that \( \text{Cov}(W_s, W_t) = \theta^2 \min\{s,t\} \), where the variance parameter \( \theta^2 \) is a positive constant. For concreteness’ sake, we henceforth assume that \( \theta = 1 \). Almost all sample paths of Brownian motion are everywhere continuous but nowhere differentiable.

One technical stipulation is required for the following. A stochastic process \( \{ Y_t : t \geq 0 \} \) is **continuous in probability** if, for all \( u \in \mathbb{R}^+ \) and \( \varepsilon > 0 \), \( P(\{|Y_u - Y_v| \geq \varepsilon\} \rightarrow 0 \) as \( v \to u \). This holds if \( \text{Cov}(Y_s, Y_t) \) is continuous over \( \mathbb{R}^+ \times \mathbb{R}^+ \). Note that this is a statement about distributions, not sample paths.

Having dispensed with preliminaries, we turn to the central topic. A stochastic process \( \{ X_t : t \geq 0 \} \) is an **Ornstein-Uhlenbeck process** or a **Gauss-Markov**
**Ornstein-Uhlenbeck Process** if it is stationary, Gaussian, Markovian, and continuous in probability [1, 2]. A fundamental theorem, due to Doob [3, 4, 5], ensures that \( \{X_t : t \geq 0\} \) necessarily satisfies the following linear stochastic differential equation:

\[
dX_t = -\rho(X_t - \mu)dt + \sigma dW_t
\]

where \( \{W_t : t \geq 0\} \) is Brownian motion with unit variance parameter and \( \mu, \rho, \sigma \) are constants. We have moments

\[
E(X_t) = \mu, \quad \text{Cov}(X_s, X_t) = \frac{\sigma^2}{2\rho} e^{-\rho|s-t|}
\]

in the unconditional (strictly stationary) case and

\[
E(X_t \mid X_0 = c) = \mu + (c - \mu)e^{-\rho t}
\]

\[
\text{Cov}(X_s, X_t \mid X_0 = c) = \frac{\sigma^2}{2\rho} \left(e^{-\rho|s-t|} - e^{-\rho(s+t)}\right)
\]

in the conditional (asymptotically stationary) case, where \( X_0 \) is initially constant. The latter case encompasses Brownian motion when \( \mu = c = 0, \sigma = 1 \) and \( \rho \to 0^+ \). The former case encompasses idealized white noise \( \{dW_t/dt : t \geq 0\} \) when \( \mu = 0, \sigma = \rho \) and \( \rho \to \infty \).

Before proceeding, we note the following simple algorithm for generating a sample path of the Ornstein-Uhlenbeck process (also known as colored noise) over the time interval \([0, T]\). Let \( N \) be a large integer and let \( z_0, z_1, \ldots, z_N \) be independent random numbers generated from a normal distribution with mean 0 and variance \( \sigma^2/(2\rho) \). Define \( x_0 = \mu + z_0 \) for the unconditional case and \( x_0 = c \) for the conditional case. Then define recursively

\[
x_n = \mu + \kappa_N(x_{n-1} - \mu) + \sqrt{1 - \kappa_N^2} z_n
\]

for \( 1 \leq n \leq N \), where \( \kappa_N = \exp(-\rho T/N) \). The sequence \( x_0, x_1, \ldots, x_N \) is called a first-order autoregressive sequence (a discrete analog of the OU process) with lag-one correlation coefficient \( \kappa_N \). Finally, interpolate linearly the values \( X(nT/N) = x_n \) for \( 0 \leq n \leq N \) to obtain the desired path [6, 7, 8]. More sophisticated simulation methods are found in [9, 10, 11].

For concreteness’ sake, we henceforth assume that \( \mu = 0, \rho = 1 \) and \( \sigma^2 = 2 \). (Some authors take \( \sigma^2 = 1 \) instead; the decision becomes apparent in any paper by seeing whether \( \text{Cov}(X_s, X_t) \) is \( e^{-|s-t|} \) or \( e^{-|s-t|}/2 \).) The conditional probability

\[
P(X_t \leq x \mid X_0 = c) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \int_{-\infty}^{x} \exp \left(-\frac{(\xi - ce^{-t})^2}{2(1-e^{-2t})}\right) d\xi
\]
tends to the standard normal distribution, of course, as \( t \to \infty \) (meaning that transients die out with time and don’t affect long-term behavior). Likewise, \( P(X_s \leq x \text{ and } X_t \leq y \mid X_0 = c) \) can be evaluated. One might believe that the solution of any problem involving the OU process would be similarly straightforward; the following sections serve, however, to eliminate such ideas [12, 13].

**0.1. First-Passage Times.** For \( a \in \mathbb{R} \), we wish to find the length of time required for an OU process to cross the level \( x = a \), given that it started at \( x = c \). Define the **first-passage time** or **hitting time** \( T_{a,c} \) by \( T_{a,c} = \inf \{ t \geq 0 : X_t = a \mid X_0 = c \} \). The random variable \( T_{a,c} \) is 0 if and only if \( a = c \). Let \( f_{a,c}(t) \) denote the density function of \( T_{a,c} \). In the special case when \( a = 0 \), it is known that [2, 12, 14, 15]

\[
f_{0,c}(t) = \sqrt{\frac{2}{\pi}} \frac{|c|e^{-t}}{1 - e^{-2t}}^{3/2} \exp \left( - \frac{2e^{-2t}}{2(1 - e^{-2t})} \right)
\]

but for \( a \neq 0 \), the formulas for \( f_{a,c}(t) \) are more complicated (as we shall soon see). For \( a > 0 \) and \( c > 0 \), Thomas [16] and Ricciardi & Sato [17, 18] demonstrated that

\[
E(T_{0,c}) = \frac{\sqrt{\pi}}{2} \int_{0}^{\infty} \left( 1 + \text{erf} \left( \frac{t}{\sqrt{2}} \right) \right) \exp \left( \frac{t^2}{2} \right) dt = \frac{1}{2} \sum_{k=1}^{\infty} \frac{\left( \sqrt{2a} \right)^k}{k!} \Gamma \left( \frac{k}{2} \right),
\]

\[
E(T_{0,c}) = \frac{\sqrt{\pi}}{2} \int_{-c}^{0} \left( 1 + \text{erf} \left( \frac{t}{\sqrt{2}} \right) \right) \exp \left( \frac{t^2}{2} \right) dt = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\left( \sqrt{2c} \right)^k}{k!} \Gamma \left( \frac{k}{2} \right)
\]

and, for example,

\[
E(T_{0,1}) = 2.0934066496..., \quad E(T_{0,1}) = 0.9019080126..., \quad E(T_{2,0}) = 10.4284093079..., \quad E(T_{2,0}) = 1.4252045655....
\]

The asymmetry in going from 0 to \( x \), versus going from \( x \) to 0, is unsurprising: The process has mean 0, hence it tends to arrive at 0 more often than it departs from 0. For \( a > 0 \) and \( c > 0 \), we have [17, 18]

\[
\text{Var}(T_{a,0}) = \sqrt{2\pi} \int_{0}^{a} \int_{-\infty}^{a} \left( 1 + \text{erf} \left( \frac{r}{\sqrt{2}} \right) \right) \exp \left( \frac{r^2 + t^2 - s^2}{2} \right) dr ds dt - E(T_{a,0})^2
\]

\[
= E(T_{a,0})^2 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{\left( \sqrt{2a} \right)^k}{k!} \Gamma \left( \frac{k}{2} \right) \Psi \left( \frac{k}{2} \right),
\]
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\[ \text{Var}(T_{0,c}) = \sqrt{2\pi} \int_0^t \int_0^t \left(1 + \text{erf} \left( \frac{r}{\sqrt{2}} \right) \right) \exp \left( \frac{r^2 + t^2 - s^2}{2} \right) dr \, ds \, dt - \text{E}(T_{0,c})^2 \]

\[ = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \left( \frac{\sqrt{2}c}{k!} \right)^k \Gamma \left( \frac{k}{2} \right) \Psi \left( \frac{k}{2} \right) - \text{E}(T_{0,c})^2 \]

where \( \Psi(x) = \psi(x) - \psi(1) \) and \( \psi(x) \) is the digamma function [19]. In particular, \( \Psi(1) = 0 \) and

\[ \Psi(x) = \begin{cases} 
\sum_{j=1}^{x-1} \frac{1}{j} & \text{if } x \text{ is an integer } > 1 \\
-2 \log(2) + 2 \sum_{j=1}^{x-1/2} \frac{1}{2j-1} & \text{if } x \text{ is a half-integer } > 0.
\end{cases} \]

For example,

\[ \text{Var}(T_{1,0}) = 5.8420278024..., \quad \text{Var}(T_{0,1}) = 0.8510837032..., \]
\[ \text{Var}(T_{2,0}) = 105.2752035488..., \quad \text{Var}(T_{0,2}) = 1.0669454393.... \]

To compute \( f_{a,c}(t) \) exactly for arbitrary \( a \) and \( c \), we would need to invert the following (Laplace transform) identity due to Darling & Siegert [20, 21, 22, 23]:

\[ \text{E}(e^{-\lambda T_{a,c}}) = \int_0^{\infty} f_{a,c}(t)e^{-\lambda t} dt = \begin{cases} 
\frac{D_{-\lambda}(-c)}{D_{-\lambda}(-a)} \exp \left( \frac{c^2 - a^2}{4} \right) & \text{if } c < a \\
\frac{D_{-\lambda}(c)}{D_{-\lambda}(a)} \exp \left( \frac{c^2 - a^2}{4} \right) & \text{if } c > a
\end{cases} \]

where \( D_{\nu}(x) \) is the parabolic cylinder function or Weber function [24]:

\[ D_{\nu}(x) = \begin{cases} 
\sqrt{\frac{2}{\pi}} \exp \left( \frac{x^2}{4} \right) \int_0^{\infty} t^{\nu} \exp \left( -\frac{t^2}{2} \right) \cos \left( xt - \frac{\nu \pi}{2} \right) dt & \text{if } \nu > -1 \\
\frac{1}{\Gamma(-\nu)} \exp \left( -\frac{x^2}{4} \right) \int_0^{\infty} t^{-\nu-1} \exp \left( -\frac{t^2}{2} - xt \right) dt & \text{if } \nu < 0.
\end{cases} \]

The two branches of this formula agree for \(-1 < \nu < 0\). A differential equation

\[ \frac{d^2y}{dx^2} - \left( \frac{x^2}{4} - \nu - \frac{1}{2} \right) y(x) = 0 \]
is satisfied by \( D_\nu(x) \) and, if \( \nu \) is not an integer, independently by \( D_{-\nu}(x) \). A series representation in terms of confluent hypergeometric functions [0.4] is also useful. Unfortunately a closed-form expression for the inverse Laplace transform seems not to be possible; only a numerical approach is feasible at present. Keilson & Ross [25] tabulated the distribution of \( T_{a,c} \) for a number of values \( a \) and \( c \). For example, the median time for an OU process \( X_t \) to reach \( a = 1 \), given that \( X_0 = c = 0 \), is 1.1892.... This corresponds to the 50\(^{th} \) percentile of the distribution of \( T_{1,0} \). The median of \( T_{2,0} \), by contrast, is 7.2521....

We turn to a more complicated problem involving two (absorbing) boundaries rather than just one. Given \( a < c < b \), what is the length of time required for the process to escape the interval \((a,b)\), given that it started at \( x = c \)? Define \( T_{a,b,c} = \inf \{ t \geq 0 : X_t = a \text{ or } X_t = b \mid X_0 = c \} \) and let \( f_{a,b,c}(t) \) denote the density function of \( T_{a,b,c} \). Efforts have focused on the scenario in which \( -a = b > 0 \). The Laplace transform of \( f_{-b,b,c}(t) \) satisfies [22]

\[
E(e^{-\lambda T_{-b,b,c}}) = \frac{D_{-\lambda}(c) + D_{-\lambda}(-c)}{D_{-\lambda}(b) + D_{-\lambda}(-b)} \exp \left( \frac{c^2 - b^2}{4} \right)
\]

assuming \(-b < c < b\). From another table in [25], the median of \( T_{-1,1,0} \) is found to be 0.4449.... The reason that this is less than 1.1892... is clear: Each direction of travel leads to a potential crossing. The median of \( T_{-2,2,0} \) is 3.2439....

Keilson & Ross’ approach to evaluating such probabilities was based on finding zeroes and residues in the complex plane of the parabolic cylinder functions. Alternative approaches for numerically computing \( f_{a,c}(t) \) and \( f_{-b,b,c}(t) \) include [26, 27, 28, 29]. We report on some related asymptotics in [0.4].

There is an obvious connection between first-passage times and extreme values of a process (in the conditional case). We simply summarize:

\[
\begin{align*}
 P \left( \max_{0 \leq t \leq T} X_t \leq a \mid X_0 = c \right) & \quad \text{if } c < a \\
 P \left( \min_{0 \leq t \leq T} X_t \geq a \mid X_0 = c \right) & \quad \text{if } c > a
\end{align*}
\]

and, if \( a < c < b \),

\[
P \left( a \leq \min_{0 \leq t \leq T} X_t \leq \max_{0 \leq t \leq T} X_t \leq b \mid X_0 = c \right) = P(T_{a,b,c} > T) = 1 - F_{a,b,c}(T)
\]

where \( F_{a,c}(t) \), \( F_{a,b,c}(t) \) are the cumulative distribution functions of \( T_{a,c} \), \( T_{a,b,c} \). In the special case when \(-a = b > 0\), the latter formula becomes a statement about
max_{0 \leq t \leq T} |X_t|, given X_0 = c. Also, the range of the process satisfies [22]

\[ P \left( \max_{0 \leq t \leq T} X_t - \min_{0 \leq t \leq T} X_t \leq r \mid X_0 = c \right) = \int_0^r \int_{c-q}^{c+q} \frac{\partial^2}{\partial a \partial b} F_{a,b,c}(T) \bigg|_{b=a+q} \, da \, dq \]

but no one apparently has calculated this probability.

0.2. Historical Maximums. If the condition \( X_0 = c \) is discarded, what then can be said about \( \max_{0 \leq t \leq T} X_t \) or \( \max_{0 \leq t \leq T} |X_t|? \) We focus solely on the former expression and write \( M_T = \max_{0 \leq t \leq T} X_t \). It can be shown that [30, 31, 32]

\[ P(M_T \leq 0) = \frac{1}{\pi} \arcsin (e^{-T}) \]

which is a beautiful (but isolated) result. More generally [32],

\[ \int_0^\infty P(M_t \leq y) e^{-\lambda t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \frac{1}{\lambda} \left( 1 - \frac{D_{-\lambda}(-x)}{D_{-\lambda}(-y)} \exp \left( \frac{x^2 - y^2}{4} \right) \right) \exp \left( -\frac{x^2}{2} \right) dx \]

for arbitrary \( y \), or

\[ \int_0^\infty g_t(y) e^{-\lambda t} dt = \frac{1}{\sqrt{2\pi}} \frac{D_{-\lambda-1}(-y)^2}{D_{-\lambda}(-y^2)} \exp \left( -\frac{y^2}{2} \right) \]

where \( g_t(y) \) is the density function of \( M_t \). For example, the median value of \( M_1 \) is 1.0393... and the median value of \( M_{10} \) is 2.2202.... It can be inferred from [0.3] that the median of \( M_T \) is asymptotically \( \sqrt{2 \ln(T)} \) as \( T \to \infty \).

An alternative approach for numerically computing \( P(M_t \leq y) \) via the Mellin transform is due to DeLong [33, 34, 35]. We hope to report on this later. An interesting application to computer science, involving the maximum size reached by a dynamic data structure over a long span of time, is described in [36].

0.3. Pickands’ Constants. Assume that \( \{Y_t : t \geq 0\} \) is a stationary Gaussian process with zero mean, unit variance and covariance function of the form

\[ r(|s-t|) = \text{Cov}(Y_s, Y_t) = 1 - C |s-t|^{\alpha} + o(|s-t|^{\alpha}) \]

as \( |s-t| \to 0 \), where \( 0 < \alpha \leq 2 \) and \( C > 0 \) are constants. Assume further that \( r(\tau) \ln(\tau) \to 0 \) as \( \tau \to \infty \). Pickands [37, 38, 39, 40, 41] demonstrated that \( M_T = \max_{0 \leq t \leq T} Y_t \) has the Gumbel limiting distribution [42]

\[ \lim_{T \to \infty} P \left( \sqrt{2 \ln(T)} (M_T - k_T) \leq x \right) = \exp(-e^{-x}) \]
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where

\[ k_T = \sqrt{2\ln(T)} + \frac{1}{\sqrt{2\ln(T)}} \left\{ \frac{2 - \alpha}{2\alpha} \ln(\ln(T)) + \ln \left( (2\pi)^{-\frac{1}{2}} 2^{\frac{2\alpha}{2\alpha}} e^{\frac{1}{2} H_\alpha} \right) \right\} \]

and \( H_\alpha \) is a positive constant independent of \( C \). It is known that \( H_1 = 1 \) (corresponding to the OU process) and \( H_2 = 1/\sqrt{\pi} \). No other exact values for \( H_\alpha \) are known.

An alternative characterization of \( H_\alpha \) is

\[ H_\alpha = \lim_{T \to \infty} \int_0^\infty P(M_T > y) e^y dy \]

where \( \{ \tilde{Y}_t : t \geq 0 \} \) is a nonstationary Gaussian process with

\[ \mathbb{E}(\tilde{Y}_t) = -|t|^\alpha, \quad \text{Cov}(\tilde{Y}_s, \tilde{Y}_t) = |s|^\alpha + |t|^\alpha - |s-t|^\alpha \]

but this does not seem to help. Shao [43] and Debicki, Michna & Rolski [44] gave bounds on \( H_\alpha \); for example,

\[ 0.009 \leq H_{1/2} \leq 715.94, \quad 0.208 \leq H_{3/2} \leq 3.04. \]

A conjecture that \( H_\alpha = 1/\Gamma(1/\alpha) \) remains unproved. There is also a connection with the Gaussian correlation conjecture and with estimating small ball probabilities [45], topics which we hope to address later.

0.4. Upper Tail Asymptotics. We revisit the single-boundary first-passage time distribution and ask about the limiting value

\[ \lambda(a) = \lim_{t \to \infty} \frac{1}{t} \ln \{ P(T_{a,0} > t) \} \]

as a function of \( a > 0 \). In words, what can be said about the upper tail of the distribution of the first hitting time \( T_{a,0} \) for an OU process \( X_t \) across the level \( x = a \), given that \( X_0 = 0 \)? Mandl [46, 47] and Beekman [48] demonstrated that \(-1 < \lambda(a) < 0 \) and that \( \lambda(a) \) is the zero of \( D_{-\lambda}(-a) \) closest to 0. Sample values include [17, 49, 50]

\[ \lambda(0.7649508673...) = -\frac{1}{2}, \quad \lambda(1) = -0.3882382947... = 2(-0.1941191473...), \quad \lambda(2) = -0.0972745958... = 2(-0.0486372979...). \]
For the symmetric double-boundary first-passage time distribution, we examine
\[ \lambda(-b, b) = \lim_{t \to \infty} \frac{1}{t} \ln \{ \mathbb{P}(T_{-b,b,0} > t) \} \]
as a function of \( b > 0 \). Breiman [51] proved that \( -\infty < \lambda(-b, b) < 0 \) and that \( \lambda(-b, b) \) is the zero of \( \Phi(\lambda/2, 1/2, b^2/2) \) closest to 0, where
\[ \Phi(u, v, w) = 1 + \sum_{k=1}^{\infty} \frac{u(u+1)(u+2)\cdots(u+k-1)w^k}{v(v+1)(v+2)\cdots(v+k-1)k!} \]
is the confluent hypergeometric function of the first kind. For simplicity, define \( \mu(b) = \lambda(-b, b) \). Sample values include [50, 51, 52]
\[
\begin{align*}
\lim_{b \to 0^+} \mu(b) &= -\infty, \\
\lim_{b \to \infty} \mu(b) &= \frac{-1}{\sqrt{2\pi}}, \\
\mu(1) &= -2, \\
\mu(1.3069297277\ldots) &= -1, \\
\mu(1.6438001904\ldots) &= -\frac{1}{2}, \\
\mu\left(\sqrt{3 - \sqrt{6}}\right) &= \mu(0.7419637843\ldots) = -4, \\
\mu(2) &= -0.2429928807\ldots, \\
\mu(3) &= -0.0239463006\ldots, \\
\mu\left(\sqrt{2}\right) &= -0.7984598320\ldots, \\
\mu\left(2\sqrt{2}\right) &= -0.0374612092\ldots.
\end{align*}
\]
The latter two values come from [52], where a different time scaling was chosen. Also, the constant \((3 - 6^{1/2})^{1/2}\) appears in [53, 54, 55] with regard to stopping rules in statistical sequential analysis.

For completeness’ sake, here is the expression for \( D_{-\lambda}(x) \) in terms of confluent hypergeometric functions:
\[
D_{-\lambda}(x) = \frac{\sqrt{\pi}2^{-3/2}e^{-x^2/4}}{\Gamma((1 + \lambda)/2)} \Phi\left(\frac{\lambda}{2}, \frac{1}{2}, \frac{x^2}{2}\right) - 2\sqrt{\pi}2^{-(1+\lambda)/2}xe^{-x^2/4}\Phi\left(\frac{1}{2}, \frac{3}{2}, \frac{x^2}{2}\right)
\]
which gives rise to the values \( \lambda(1), \lambda(2) \) and \( \lambda^{-1}(-1/2) \) listed earlier. The constant \( \mu^{-1}(-1) \) is important in the study of sample path behavior of Brownian motion [50, 56, 57] and first appeared in [54], as far as is known. Some higher dimensional results are given in [50, 58]. Csáki [59, 60] recently outlined the distributional asymptotics of the maximal path behavior \( \mu_{T} \), but we cannot discuss this topic further.

0.5. Addendum. New numerical transform inversion algorithms [61, 62, 63] make enhancement of the tables in [25, 32] possible. Also, the distribution of the \( L_2 \)-norm of \( X_t \) on \([0, T]\) can be inferred from closed-form expressions in [64, 65]. We wonder about corresponding results for \( L_1 \) and \( L_\infty \)-norms. The conjectured formula for \( H_\alpha \) in terms of the gamma function is probably false [66, 67, 68, 69]; simulation-based point estimates \( H_{3/2} \approx 0.77 \) and confidence bounds \( 0.768 \leq H_{3/2} \leq 0.786 \) do not carry over well to \( H_{1/2} \) since the underlying algorithm becomes unreliable for \( 0 < \alpha < 1 \).
REFERENCES


