A map on a compact surface \( \Sigma \) without boundary is an embedding of a graph \( \Gamma \) into \( \Sigma \) such that all components of \( \Sigma - \Gamma \) are simply connected \([1]\). These components are thus homeomorphic to open disks and are called faces. The graph \( \Gamma \) is allowed to have both loops and multiple parallel edges (unlike those in \([2]\)). A map is rooted when an edge, a direction along that edge, and a side of the edge, are distinguished. The edge is called the root edge, and the face on the distinguished side is the root face. Two rooted maps are equivalent if there is a homeomorphism between the underlying surfaces that preserves all graph incidences and rootedness.

In the case when \( \Sigma \) is orientable, two rooted maps are equivalent if and only they are related by an orientation-preserving homeomorphism that (merely) preserves all graph incidences. Such thinking doesn’t apply, of course, when \( \Sigma \) is non-orientable. For orientable surfaces, the genus \( g \) is 0 for the sphere, 1 for the torus, 2 for the connected sum of two tori, and so forth. For non-orientable surfaces, the type \( h \) is 1/2 for the projective plane, 1 for the Klein bottle, 3/2 for the connected sum of three projective planes, and so forth.

The requirement that faces be simply connected implies that the graph \( \Gamma \) itself must be connected \([3]\). Proof: if \( \Gamma \) were to possess two components, then a curve drawn around one of the components could not be contracted to a point (because the other component would present an obstacle), which is a contradiction. The converse is true if the surface \( \Sigma \) is a sphere, but is false if \( \Sigma \) is a torus. Reason: consider the figure-eight graph \( \Gamma \) consisting of one vertex and two edges (orthogonal loops that together generate the torus). While \( \Sigma - \Gamma \) is simply connected, this is not true for any proper subgraph of \( \Gamma \).

Let \( T_g(n) \) denote the number of rooted maps with \( n \) edges on an orientable surface of genus \( g \). Let \( P_h(n) \) denote the number of rooted maps with \( n \) edges on a non-orientable surface of type \( h \). (\( T \) stands for “torus” and \( P \) stands for “projective plane”.) It is known that \( T_0(n) \) is the coefficient of \( x^n \) in the Maclaurin series expansion \([1, 4, 5]\)

\[
\frac{4(1 + 2r)}{3(1 + r)^2} = 1 + 2x + 9x^2 + 54x^3 + 378x^4 + 2916x^5 + 24057x^6 + 208494x^7 + 1876446x^8 + 17399772x^9 + 165297834x^{10} + \cdots,
\]

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$T_1(n)$ is the coefficient of $x^n$ in the expansion [6, 7]

$$\frac{(-1 + r)^2}{12r^2(2 + r)} = x^2 + 20x^3 + 307x^4 + 4280x^5 + 56914x^6$$
$$+ 736568x^7 + 9370183x^8 + 117822512x^9 + 1469283166x^{10} + \cdots,$$

$P_{1/2}(n)$ is the coefficient of $x^n$ in the expansion [1, 6]

$$\frac{-q}{(-1 + r)(1 + r)} = x + 10x^2 + 98x^3 + 982x^4 + 10062x^5 + 105024x^6$$
$$+ 1112757x^7 + 11934910x^8 + 129307100x^9 + 1412855500x^{10} + \cdots$$

and $P_1(n)$ is the coefficient of $x^n$ in the expansion [8, 9, 10]

$$\frac{(1 + r)q}{2r^2(2 + r)} = 4x^2 + 84x^3 + 1340x^4 + 19280x^5 + 263284x^6$$
$$+ 3486224x^7 + 45247084x^8 + 579150012x^9 + 7338291224x^{10} + \cdots$$

where $r = \sqrt{1 - 12x}$ and $q = 2 + 4r - 2\sqrt{3}\sqrt{r(2 + r)}$ throughout. Moreover [11],

$$T_g(n) \sim t_g n^{5(g-1)/2} 12^n, \quad P_h(n) \sim p_h n^{5(h-1)/2} 12^n$$

as $n \to \infty$, where $t_g$ is the orientable map asymptotics constant:

$$t_0 = \frac{2}{\sqrt{\pi}}, \quad t_1 = \frac{1}{24}, \quad t_2 = \frac{7}{4320\sqrt{\pi}}, \quad t_3 = \frac{245}{15925248}, \quad t_4 = \frac{37079}{96074035200\sqrt{\pi}}$$

and $p_h$ is the non-orientable map asymptotics constant:

$$p_{1/2} = \frac{\sqrt{3}}{2\pi} \Gamma(1/4) = -\frac{2\sqrt{6}}{\Gamma(-1/4)}, \quad p_1 = \frac{1}{2}, \quad p_{3/2} = \frac{\sqrt{6}}{3\Gamma(1/4)} = \frac{5}{8\sqrt{6}\Gamma(9/4)}$$

Since the status of $t_g$ is quite different from the status of $p_h$, we shall treat them separately.

For many years, the values of $t_g$ for $g > 2$ were unknown, owing to difficulties in their formulation. Impressive progress has been made recently. Define a sequence

$$u_0 = 1, \quad u_n = \frac{25(n - 1)^2 - 1}{48} u_{n-1} - \frac{1}{2} \sum_{k=1}^{a-1} u_k u_{n-k} \quad \text{for } n \geq 1$$

then provably

$$t_g = -\frac{1}{2g-2\Gamma((5g-1)/2)} u_g$$
for all integers $g \geq 0$. The formal power series $u(z) = \sum_{n=0}^{\infty} u_n z^{-(5n-1)/2}$ satisfies the Painlevé I differential equation

\[ u''(z) = 6u(z)^2 - 6z \]

which makes possible the following asymptotics:

\[ t_g \sim \frac{40 \sin(\pi/5) K}{\sqrt{2\pi}} \left( \frac{1440g}{e} \right)^{-g/2} \]

as $g \to \infty$ and

\[ K = \sqrt{\frac{3\Gamma(1/5)\Gamma(4/5)}{4\pi^2}} = 0.1048689877\ldots. \]

We explain further: Bender, Gao & Richmond [12] discovered the preceding approximation for $t_g$ but with only a rough numerical estimate 0.1034 for $K$. The connection with Painlevé I, streamlined $u_n$ recursion and exact $K$ expression are due to Garoufalis, Lê & Mariño [13]. A (somewhat different) full asymptotic series is also possible. We give the first term only:

\[ u_n \sim -\frac{1}{2\pi} \frac{3^{1/4}}{\sqrt{\pi}} \left( \frac{8\sqrt{3}}{5} \right)^{-2n+\frac{1}{2}} \Gamma \left( 2n - \frac{1}{2} \right) \]

as $n \to \infty$, quoting [14]. This is reminiscent of other quadratic recurrence studies [15, 16].

Likewise, the path to understanding $p_h$ for $h > 2$ is fraught with peril. Define a sequence

\[ v_0 = -\sqrt{3}, \quad v_n = \frac{1}{2\sqrt{3}} \left(-3u_{n/2} + \frac{5n - 6}{2} v_{n-1} + \sum_{k=1}^{n-1} v_k v_{n-k} \right) \quad \text{for } n \geq 1 \]

(the dependence of $v_n$ on $u_{n/2}$ from before is striking: if $n$ is odd, let $u_{n/2} = 0$). Conjecturally, we have [14]

\[ p_h = \frac{1}{2^{h-2}} \frac{1}{\Gamma(\frac{(5h - 3)/2)}{2^{2h-1}}} \]

for all integers/half-integers $h \geq 1/2$. Evidence for this equality comes from quantum physics. As consequences,

\[ p_2 = \frac{5}{36\sqrt{\pi}}, \quad p_{5/2} = \frac{1033}{1024\sqrt{6\Gamma(19/4)}}, \quad p_3 = \frac{3149}{442368}, \quad p_{7/2} = \frac{1599895}{294912\sqrt{6\Gamma(29/4)}}. \]
The formal power series \( v(z) = \sum_{n=0}^{\infty} v_n z^{-(5n-1)/4} \) satisfies the differential equation
\[
2v'(z) = v(z)^2 - 3u(z)
\]
and a full asymptotic series is again possible. We give the first term only:
\[
v_n \sim \frac{C}{2\pi} \left( \frac{4\sqrt{3}}{5} \right)^{-n} \Gamma(n)
\]
as \( n \to \infty \), where the Stokes constant \( C \) is conjectured to be \( \sqrt{6} \). See [17, 18] for a bivariate analog of the preceding theory.

References


