Bipartite, $k$-Colorable and $k$-Colored Graphs

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A labeled graph $G$ is bipartite if its vertex set $V$ can be partitioned into two disjoint subsets $A$ and $B$, $V = A \cup B$, such that every edge of $G$ is of the form $(a, b)$, where $a \in A$ and $b \in B$.

Let $k$ be a positive integer and $K = \{1, 2, \ldots, k\}$. A labeled graph $G$ is $k$-colorable if there exists a function $V \rightarrow K$ with the property that adjacent vertices must be colored differently. Clearly $G$ is bipartite if and only if $G$ is 2-colorable.

Define $c_{n,k}$ to be the number of $k$-colorable graphs with $n$ vertices. We have $c_{n,1} = 1$ for $n \geq 1$ since a 1-colorable graph $G$ cannot possess any edges. We also have $c_{1,k} = 1$ for $k \geq 1$, $c_{2,k} = 2$ for $k \geq 2$, $c_{3,2} = 7$ by Figure 1, $c_{3,3} = 8$, $c_{4,2} = 41$ by Figure 2, and $c_{4,3} = 63$. More generally, $c_{n,n-1} = 2^{n(n-1)/2} - 1$ since the total number of labeled graphs with $n$ vertices is $2^{n(n-1)/2}$ and, of these, only the complete graph cannot be $(n-1)$-colored.

Does there exist a formula for $c_{n,k}$? The answer is yes if $k = 2$, but evidently no for $k \geq 3$. We’ll examine this issue momentarily, but first define a related notion.

A $k$-colored graph is a labeled $k$-colorable graph together with its coloring function. Let $\gamma_{n,k}$ be the number of $k$-colored graphs with $n$ vertices. The point is that a $k$-colorable graph counts several times as a $k$-colored graph. Clearly $\gamma_{n,1} = 1$, $\gamma_{1,k} = k$, $\gamma_{2,2} = 6$ by Figure 3, $\gamma_{2,3} = 15$ by Figure 4, and $\gamma_{3,2} = 26$ by Figure 5.

When $k = 2$, the following formulas can be proved [1, 2, 3]:

$$\gamma_{n,2} = \sum_{j=0}^{n} \binom{n}{j} 2^{j(n-j)}$$

$$c_{n,2} = n! \cdot \left(\text{the } n^{th} \text{ degree Maclaurin series coefficient of } \sqrt{\Gamma(x)}\right)$$

where

$$\Gamma(x) = \sum_{i=0}^{\infty} \gamma_{i,2} \frac{x^i}{i!}$$

For arbitrary $k$, we have the following recursion [4, 5]:

$$\gamma_{n,k} = \sum_{j=0}^{n} \binom{n}{j} 2^{j(n-j)} \gamma_{j,k-1}$$
with initial conditions $\gamma_{0,k} = 1$ and $\gamma_{n,0} = 0$ for $n \geq 1$. Alternatively, we have a closed-form expression involving multinomial coefficients:

$$
\gamma_{n,k} = \sum_N \binom{n}{n_1, n_2, \ldots, n_k} 2^{\frac{k}{2}(n^2 - n_1^2 - n_2^2 - \cdots - n_k^2)}
$$

where the summation is over all nonnegative integer $k$-vectors $N = (n_1, n_2, \ldots, n_k)$ satisfying $n_1 + n_2 + \cdots + n_k = n$. There is, however, no known analogous formula for $c_{n,k}$ when $k \geq 3$.

Computations show that [4, 6]

$$
\{\gamma_{n,2}\}_{n=1}^\infty = \{2, 6, 26, 162, 1442, 18306, 330626, 8488962, \ldots\}
$$

$$
\{c_{n,2}\}_{n=1}^\infty = \{1, 2, 7, 41, 376, 5177, 103237, 2922446, \ldots\}
$$

and suggest that $\gamma_{n,2}/c_{n,2} \to 2$ as $n \to \infty$. We also have

$$
\{\gamma_{n,3}\}_{n=1}^\infty = \{3, 15, 123, 1635, 35043, 1206915, 6662083, 5884188675, \ldots\}
$$

$$
\{c_{n,3}\}_{n=1}^\infty = \{1, 2, 8, 63, 958, 27554, \ldots\}
$$

but there is insufficient data on $c_{n,3}$ to clearly suggest the asymptotic behavior of $\gamma_{n,3}/c_{n,3}$. Prömel & Steger [7], however, proved that

$$
\lim_{n \to \infty} \frac{\gamma_{n,k}}{c_{n,k}} = k!
$$

for each $k \geq 2$. In words, a random $k$-colorable graph is almost surely uniquely $k$-colorable (up to a permutation of colors). This is an important result since it allows us to utilize at least one term of the $\gamma_{n,k}$ asymptotics to estimate the growth of $c_{n,k}$. 

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**Figure 1:** There are 7 labeled bipartite graphs with 3 vertices.
Figure 2: There are 41 labeled bipartite graphs with 4 vertices.

Figure 3: There are 6 labeled 2-colored graphs with 2 vertices.
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Figure 4: There are 15 labeled 3-colored graphs with 2 vertices (these 9 plus the preceding 6).

Figure 5: There are 26 labeled 2-colored graphs with 3 vertices.
We turn now to a result due to Wright [8, 9, 10, 11, 12]: if \( n \equiv a \mod k \), where \( 0 \leq a < k \), then
\[
\gamma_{n,k} \sim C(k, a) \cdot 2^{k(1-\frac{1}{k})n^2} \cdot k^n \cdot \left( \frac{k}{\ln(2) \cdot n} \right)^{\frac{k-1}{2}}
\]
as \( n \to \infty \), where \( C(k, a) \) is a constant that depends on \( n \) only via its residue modulo \( k \). In fact,
\[
C(k, a) = \frac{k^2}{\pi} \cdot (\ln(2))^{\frac{k-1}{2}} \cdot (2\pi)^{-\frac{k-1}{2}} \cdot L_k(a)
\]
and the infinite series \( L_k(a) \) will be defined for \( k = 2, 3 \) and \( 4 \) shortly.

0.1. 2-Colored Graph Asymptotics. To characterize the growth of \( \gamma_{n,k} \), by the above, it is sufficient to determine \( C(k, a) \) for each \( 0 \leq a < k \). We have here
\[
L_2(a) = \sum_{r=-\infty}^{\infty} 2^{-\frac{1}{2}r^2 - \frac{1}{2}(a-r)^2 + \frac{1}{4}a^2} = \sum_{r=-\infty}^{\infty} 2^{-\frac{1}{4}(a-2r)^2} = \begin{cases} 1.2189368272 \ldots & \text{if } a = 0 \\ 2.1289312505 \ldots & \text{if } a = 1 \end{cases}
\]
These two constants also appear with regard to the asymptotic enumeration of partially ordered sets [13] and of linear subspaces of \( \mathbb{F}_2^n \) [14], where \( \mathbb{F}_2 \) is the binary field with arithmetic modulo 2. Therefore
\[
C(2, a) = \begin{cases} 1.0000013097 \ldots = 1 + \varepsilon & \text{if } a = 0 \\ 0.9999986902 \ldots = 1 - \varepsilon & \text{if } a = 1 \end{cases}
\]
where \( \varepsilon = 1.3097396978 \ldots \times 10^{-6} \). In fact, all of the constants \( C(k, a) \) we examine are close to 1; thus we shall focus on difference with 1 henceforth.

0.2. 3-Colored Graph Asymptotics. We have here
\[
L_3(a) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} 2^{-\frac{1}{2}r^2 - \frac{1}{2}s^2 - \frac{1}{2}(a-r-s)^2 + \frac{1}{4}a^2} = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} 2^{-\frac{1}{4}(a^2 - 3ar + 3r^2 - 3as + 3rs + 3s^2)}
\]
and therefore
\[
C(3, a) = \begin{cases} 1 + 2\varepsilon & \text{if } a = 0 \\ 1 - \varepsilon & \text{if } a = 1 \text{ or } 2 \end{cases}
\]
where \( \varepsilon = 1.7060611047 \ldots \times 10^{-8} \).
0.3. 4-Colored Graph Asymptotics. All planar graphs are 4-colorable by the famous Four Color Theorem. We have here \([4, 6]\)

\[ \left\{ \gamma_{n,4} \right\}_{n=1}^{\infty} = \{4, 28, 340, 7108, 254404, 15531268, 1613235460, 284556079108, \ldots \} \]

\[ \left\{ c_{n,4} \right\}_{n=1}^{\infty} = \{1, 2, 8, 64, 1023, 32596, \ldots \} \]

\[ L_4(a) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} 2^{\frac{1}{2}r^2 - \frac{1}{2}s^2 - \frac{1}{2}t^2 - \frac{1}{8}(a-r-s-t)^2 + \frac{5}{8}a^2} \]

and therefore

\[ C(4, a) = \begin{cases} 
1 + \delta & \text{if } a = 0 \\
1 - \varepsilon & \text{if } a = 1 \text{ or } 3 \\
1 - \delta + 2\varepsilon & \text{if } a = 2 
\end{cases} \]

where \( \delta = 4.2421496651 \ldots \times 10^{-9} \) and \( \varepsilon = 2.5731271141\ldots \times 10^{-12} \). A simple relationship between \( \delta \) and \( \varepsilon \) is not apparent.

Higher-order asymptotics for \( \gamma_{n,k} \) are possible, due to Wright [8]; we hope to examine the corresponding constants later. Observe that terms beyond the first need not necessarily apply for \( c_{n,k} \).

A random \( k \)-colorable graph is almost surely connected [10, 12, 15] and is almost surely \( k \)-chromatic (meaning that \( k - 1 \) colors won’t suffice to color all \( n \) vertices). The asymptotics discussed above therefore apply to these important subclasses as well.

Enumerating unlabeled \( k \)-colorable graphs (that is, non-isomorphic types of labeled \( k \)-colorable graphs) is also a difficult computational problem [16]. A general result due to Prömel [17] provides that \( c_{n,k}/n! \) is the associated asymptotic formula.

References


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