Golay-Littlewood Problem

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Two independent streams of investigation, one from digital communications engineering and the other from complex analysis on the unit circle, come together in this essay [1, 2, 3, 4, 5].

0.1. Merit Factor of Binary Sequences. Given a sequence $a_0, a_1, a_2, \ldots, a_n$ where each $a_j = \pm 1$, define the $k^{th}$ acyclic autocorrelation to be

$$c_k = \sum_{j=0}^{n-k} a_j a_{j+k} \quad \text{for } 0 \leq k \leq n; \quad c_k = c_{-k} \quad \text{for } -n \leq k < 0$$

and the merit factor to be the ratio

$$F = \frac{c_0^2}{2 \sum_{k=1}^{n} c_k^2} = \frac{(n + 1)^2}{2 \sum_{k=1}^{n} c_k^2}.$$

Identifying binary sequences $\{a_j\}$ whose autocorrelations $\{c_k\}$ are jointly as small as possible, for fixed $n$, is important for engineering design purposes. The “best” sequences are those with the largest merit factor $F$. As an example, the sequence $1, -1, 1, -1, 1, 1, -1, 1, 1, 1, 1, 1$ has the largest $F$ value $169/12 = 14.0833\ldots$ among all such with $n = 12$. As another example, the sequence $1, -1, 1, 1, -1, 1, 1, 1, 1, -1, -1, -1$ has the largest $F$ value $121/10 = 12.1$ among all such with $n = 10$. No other merit factor exceeding 10 is known for any $n$; a proof that $169/12$ and $121/10$ are the maximum possible values for $F$ is still open.

0.2. $L_4$ Norm of Polynomials on Unit Circle. Given a polynomial of complex variable $z$:

$$f(z) = \sum_{j=0}^{n} a_j z^j$$

the $L_p$ norm of $f$ over the unit circle for $p \geq 1$ is

$$\|f\|_p = \left[ \frac{1}{2\pi} \int_{0}^{2\pi} |f(e^{i\theta})|^p \, d\theta \right]^{1/p}.$$
Since the complex conjugate $\bar{z}$ is equal to $1/z$ and all polynomial coefficients $a_j$ are real, we have $f(z) = f(\bar{z}) = f(1/z)$. 

$$|f(z)|^2 = f(z) f\left(\frac{1}{z}\right) = c_0 + \sum_{k \neq 0} c_k \bar{z}^k$$

and, after integrating, $||f||^2_2 = c_0 = n + 1$ because each $a_j = \pm 1$. Also, we have

$$|f(z)|^4 = f(z)^2 f\left(\frac{1}{z}\right)^2 = \sum_k c_k^2 + \sum_{k+\ell \neq 0} c_k c_{\ell} \bar{z}^{k+\ell}$$

and, after integrating, $||f||^4_4 = \sum c_k^2 = (n+1)^2(1 + 1/F)$. Thus Littlewood's question [6, 7] about how closely the ratio $||f||_4/||f||_2$ can approach 1 as $n \to \infty$ translates into Golay's question [8, 9, 10, 11, 12, 13] about the limit supremum of $F$.

0.3. Bounds on Asymptotic Behavior. On the one hand, let $\xi = 1.157677...$ denote the smallest zero of $27x^3 - 498x^2 + 1164x - 722$. Jedwab, Katz & Schmidt [14] proved that there is a Littlewood polynomial sequence $\{f_n\}$ such that $\deg(f_n) \to \infty$ and

$$\frac{||f_n||_4}{||f_n||_2} \to 4\sqrt{\xi} = 1.037282...$$

as $n \to \infty$. As a consequence,

$$\limsup_{n \to \infty} F_n \geq \eta = \frac{1}{\xi - 1} = 6.342061....$$

The preceding best result, namely $\xi = 7/6 = 1.16...$ ($\eta = 6$), had remained in place for more than twenty years [15, 16]. Recent numerical computations indicate that $\xi = 1.1553...$ ($\eta = 6.4382...$) is feasible. We might have to wait a long time for rigorous verification of this result because, in the words of [17], “inclusion of the steep descent algorithm ... would seem to make a proof much more difficult”. Theory lags considerably behind experiment here: there is good evidence that $\eta > 8$ or even $\eta > 8.5$. Merit factors exceeding 9 are not uncommon for sequence lengths $\approx 100$, but it is difficult to project whether such extremities will continue to grow slowly or level off [18, 19].

On the other hand, no one has proved that the limit supremum of $F$ is necessarily finite. (An argument in [11, 20] that it is approximately 12.32 is only heuristic.) This would be good to see someday.

Imagine the set of all sequences of length $n + 1$, endowed with the uniform distribution. Draw one such sequence and compute $F$. The mean value of $1/F$ is exactly
\[ E \left( \frac{1}{F} \right) = \frac{n}{n+1} \to 1 \]
as \( n \to \infty \). An exact expression for \( \text{Var}(1/F) \) is not available, but it is \( O(1/n) \) according to [4]. Thus most sequences should have merit factor close to 1 [23]. What else can be said about the distribution of \( 1/F \) or, indeed, of \( F \) itself?

We hope to report on [24, 25] later. The survey [4] mentions Mahler’s measure and Lehmer’s conjecture surrounding a certain polynomial of degree 10 (with largest zero 1.1762808182...) [26]. Related problems involving \( \pm 1 \) sequences appear in [27, 28, 29].

0.4. Addendum. Choi [30] supplemented the result \( E(\|f\|_4^4) = (n + 1)(2n + 1) \) with a new one:

\[
\text{Var}(\|f\|_4^4) = \frac{8}{3} (n + 1) \left( 2n^2 - 2n + 3 \right) - 8 \left( \frac{n^2 + 2n + 2}{2} \right)
\]
giving a formula for \( \text{Var}(1/F) \) as a corollary. Golay’s constant is, to higher precision,

\[
12.3247958363... = \frac{2y^2}{2y - \ln(2y + 1)}
\]
where \( y \) is the unique positive solution of the equation \((y + 1) \ln(2y + 1) = 2(1 + \ln(2))y\) [20].

References


[6] J. E. Littlewood, On polynomials $\sum^n z^m, \sum^n e^{i\alpha_m} z_m, z = e^{\theta i}$, J. London Math. Soc. 41 (1966) 367–376; MR0196043 (33 #4237).


