Partitioning Problem

Steven Finch

May 3, 2013

Let us begin with a two-dimensional problem. Consider an equilateral triangular region $T$ with edges of unit length. What is the minimum length of a smooth curve that partitions $T$ into two subregions of equal area? Assuming the vertices of $T$ are $(-1/2, 0), (0, \sqrt{3}/2), (1/2, 0)$, a solution is given by one-sixth of the circumference of the circle

$$x^2 + \left(y - \frac{\sqrt{3}}{2}\right)^2 = r^2 = \frac{3\sqrt{3}}{4\pi}$$

and hence the desired length is 

$$\frac{1}{6}(2\pi r) = \frac{\pi}{3} \sqrt{\frac{3\sqrt{3}}{4\pi}} = 0.6733868435...$$

See Figure 1. The solution is a curve of constant curvature and meets the boundary $\partial T$ of $T$ orthogonally.

Let us now move up one dimension. Consider a regular tetrahedral region $T$ with edges of unit length. What is the minimum area of a smooth surface that partitions $T$ into two subregions of equal volume? A precise answer to this question evidently remains unknown, although a graph appears in [5] without elaboration. See Figure 2. The solution is a surface of constant mean curvature and meets the boundary $\partial T$ of $T$ orthogonally everywhere. We will not discuss this particular tetrahedron further; additional words are found in [6, 7, 8].

Consider instead the irregular tetrahedral region $T$ with vertices $(0, 0, 0), (1, 0, 0), (0, 0, 1), (0, 1, 1)$. We pose the same problem as before. This is a classical example [9], solved in 1872, and features a portion of what is known as the Schwarz P surface (P stands for “Primitive”). The surface has zero mean curvature and thus is a minimal surface in the same sense as the Schwarz D surface. In the following, the functions $F[\phi, m]$ and $K[m]$ are defined exactly as in [10].
Figure 1: Optimally partitioning an equilateral triangle in half

Figure 2: Optimally partitioning a regular tetrahedron in half (Smyth [5])
0.1. Tetrahedral Dissection. Unlike our treatment of the Schwarz D surface [10], an expression for the Schwarz P surface in \( x, y, z \) solely does not seem possible. We thus turn to a parametric approach using the Weierstrass-Enneper representation [11]:

\[
\begin{align*}
x(u, v) &= \kappa \text{Re} \int_0^{u+iv} \frac{1 - \omega^2}{\sqrt{1+14\omega^4 + \omega^8}} \, d\omega, \\
y(u, v) &= \kappa \text{Re} \int_0^{u+iv} \frac{i(1 + \omega^2)}{\sqrt{1+14\omega^4 + \omega^8}} \, d\omega, \\
z(u, v) &= \frac{1}{2} + \kappa \text{Re} \int_0^{u+iv} \frac{2\omega}{\sqrt{1+14\omega^4 + \omega^8}} \, d\omega
\end{align*}
\]

where the complex line integrals have endpoint \( u + iv \) satisfying

\[
u \geq 0, \quad v \leq 0, \quad (u + 1)^2 + v^2 \leq 2, \quad u^2 + (v - 1)^2 \leq 2
\]

– call this planar domain \( \Omega \) – and the normalization constant is

\[
\kappa = \frac{3}{2K[1/9]} = 0.9274219745 \ldots
\]

This is as far as Nitsche [6, 11] went in characterizing the surface; calculations based on [12] further yield that

\[
\begin{align*}
x &= \frac{\kappa}{4} \text{Re} \left( -i F\left(\theta(u, v), \frac{1}{4}\right) + F\left(\theta(u, v), \frac{3}{4}\right) \right), \\
y &= \frac{\kappa}{4} \text{Re} \left( i F\left(\theta(u, v), \frac{1}{4}\right) + F\left(\theta(u, v), \frac{3}{4}\right) \right), \\
z &= \frac{1}{2} + (2 - \sqrt{3})\kappa \text{Im} \left( \left. F\right|_{\arcsin\left( i\left(2 + \sqrt{3}\right)(u + iv)^2, \left(2 - \sqrt{3}\right)^4 \right)} \right)
\end{align*}
\]

where

\[
\theta(u, v) = \arcsin \left( \frac{2(1 + i)(u + iv)}{\sqrt{1 + 4i(u + iv)^2 - (u + iv)^4}} \right).
\]

See Figures 3, 4, 5. The four corners of \( \Omega \) are mapped to the surface as follows:

- \((u, v) = (0, 0) \mapsto (x, y, z) = (0, 0, \frac{1}{2})\) [front left]
- \((u, v) = \left(\frac{\sqrt{3} - 1}{2}, -\frac{\sqrt{3} - 1}{2}\right) \mapsto (x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\) [back right]
- \((u, v) = (\sqrt{2} - 1, 0) \mapsto (x, y, z) = (\xi, 0, 1 - \xi)\) [front right]
- \((u, v) = (0, -(\sqrt{2} - 1)) \mapsto (x, y, z) = (0, \xi, \xi)\) [back left]
where $\xi \approx 0.350$. Letting $x_u, y_u, z_u, x_v, y_v, z_v$ denote partial derivatives and

$$
e = (x_u, y_u, z_u) \cdot (x_u, y_u, z_u), \quad g = (x_v, y_v, z_v) \cdot (x_v, y_v, z_v),$$

$$f = (x_u, y_u, z_u) \cdot (x_v, y_v, z_v)$$

we have surface area

$$\iint_{\Omega} \sqrt{eg - f^2} \, dv \, du = \frac{1}{4} K[1/4] = \frac{1}{12} (2.3451028840...) = 0.1954...$$

as predicted in [12].

0.2. Four Edges of a Regular Octahedron. We return to a variation of Plateau’s problem in [10]. Consider a polygonal wire loop with four line segments:

$$(0, 0, 1/2) \rightarrow (1/2, -1/2, 1/2) \rightarrow (1/2, 0, 1) \rightarrow (1/2, 1/2, 1/2) \rightarrow (0, 0, 1/2).$$

What is the minimal area for any surface spanning this fixed boundary? Equivalently, what is the outcome of dipping the wire loop in a soap solution? [13, 14]

The same formulas for $x, y, z$ apply here, but a new domain $\tilde{\Omega}$ is needed:

$$u \geq |v|, \quad u^2 + (v + 1)^2 \leq 2, \quad u^2 + (v - 1)^2 \leq 2.$$

See Figure 6. The two corners of $\tilde{\Omega}$ not in $\Omega$ are mapped to the surface as follows:

$$(u, v) = \left( \frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}-1}{2} \right) \mapsto (x, y, z) = \left( \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right),$$

$$(u, v) = (1, 0) \mapsto (x, y, z) = \left( \frac{1}{2}, 0, 1 \right)$$

and the corresponding area is

$$\iint_{\tilde{\Omega}} \sqrt{eg - f^2} \, dv \, du = \frac{1}{2} K[1/4] = \frac{1}{6} (2.3451028840...) = 0.3908...$$

again as predicted in [12].

0.3. Integration Details. To prove our formulas for $x, y, z$, we must evaluate the hyperelliptic integral

$$I_p(\eta) = \int_0^\eta \frac{\omega^p}{\sqrt{1 - 14\omega^4 + \omega^8}} \, d\omega$$
Figure 3: Optimally partitioning an irregular tetrahedron in half
Figure 4: First closeup of tetrahedral partition
Figure 5: Second closeup of tetrahedral partition
Figure 6: Octahedral “four edges” surface
for \( p = 0, 1, 2 \). Note that the coefficient of \( \omega^4 \) in \( I_p(\eta) \) is \(-14\) whereas it is \(+14\) in the definitions of \( x, y, z \). This is chosen so that we may follow [12] closely and then, at the end, perform a transformation to align with [11].

Let \( t = \omega^2 \), then \( dt = 2\omega\,d\omega \) and

\[
I_p(\eta) = \frac{1}{2} \int_0^{\eta^2} \frac{t^{p/2}}{\sqrt{1 - 14t^2 + t^4}} \, dt.
\]

Let \( s = t + 1/t \), then assuming \( 0 < \text{Re}(t) < 1 \), we have

\[
t = \frac{1}{2} \left( s - \sqrt{s^2 - 4} \right),
\]

\[
t^4 - 14t^2 + 1 = (s^2 - 16) \, t^2,
\]

\[
dt = \frac{1}{2} \frac{\sqrt{s^2 - 4} - s}{\sqrt{s^2 - 4}} \, ds = - \frac{1}{\sqrt{s^2 - 4}} \, t \, ds
\]

hence

\[
\frac{dt}{t^{3/2}} = - \frac{1}{\sqrt{s^2 - 4}} \sqrt{\frac{2}{s - \sqrt{s^2 - 4}}} \, ds
\]

\[
= - \frac{1}{\sqrt{s - 2} + \sqrt{s + 2}} \, ds
\]

\[
= - \frac{1}{2} \left( \frac{1}{\sqrt{s - 2}} + \frac{1}{\sqrt{s + 2}} \right) \, ds
\]

hence

\[
\frac{1}{\sqrt{t^4 - 14t^2 + 1}} \, dt = \frac{1}{\sqrt{s - 4}\sqrt{s + 4}} t^{3/2} \, dt
\]

\[
= - \frac{1}{2} \sqrt{s - 4} \sqrt{s + 4} \left( \frac{1}{\sqrt{s - 2}} + \frac{1}{\sqrt{s + 2}} \right) \, ds
\]

hence

\[
I_p(\eta) = -\frac{1}{4} \int_{s=\infty}^{\eta^2 + 1/\eta^2} \frac{1}{\sqrt{s} - 4\sqrt{s} + 4} \left( \frac{1}{\sqrt{s} - 2} + \frac{1}{\sqrt{s} + 2} \right) t^{p/2} \, ds
\]

\[
= \frac{1}{2^{2+p/2}} \int_{\eta^2 + 1/\eta^2}^{\infty} \left( \frac{s - \sqrt{s - 2}\sqrt{s + 2}}{\sqrt{s - 4}\sqrt{s - 2}\sqrt{s + 4}} \right)^{p/2} \left( \frac{s - \sqrt{s - 2}\sqrt{s + 2}}{\sqrt{s - 4}\sqrt{s + 2}\sqrt{s + 4}} \right)^{p/2} \, ds.
\]
Define $\zeta = \eta^2 + 1/\eta^2$. For the case $p = 0$, we need \[15\]
\[
\int_\zeta^\infty \frac{1}{\sqrt{s - 4\sqrt{s - 2\sqrt{s + 4}}} \, ds} = \frac{1}{\sqrt{2}} F \left[ \arcsin \left( \frac{2\sqrt{2}}{\sqrt{\zeta + 4}} \right), \frac{3}{4} \right],
\]
\[
\int_\zeta^\infty \frac{1}{\sqrt{s - 4\sqrt{s + 2\sqrt{s + 4}}} \, ds} = \frac{1}{\sqrt{2}} F \left[ \arcsin \left( \frac{2\sqrt{2}}{\sqrt{\zeta + 4}} \right), \frac{1}{4} \right],
\]
which together imply that $I_0(\eta)$ is equal to
\[
\frac{1}{4\sqrt{2}} \left( F \left[ \arcsin \left( \frac{2\sqrt{2}\eta}{\sqrt{\eta^4 + 4\eta^2 + 1}} \right), \frac{1}{4} \right] + F \left[ \arcsin \left( \frac{2\sqrt{2}\eta}{\sqrt{\eta^4 + 4\eta^2 + 1}} \right), \frac{3}{4} \right] \right).
\]
Similar work implies that $I_2(\eta)$ is equal to
\[
\frac{1}{4\sqrt{2}} \left( -F \left[ \arcsin \left( \frac{2\sqrt{2}\eta}{\sqrt{\eta^4 + 4\eta^2 + 1}} \right), \frac{1}{4} \right] + F \left[ \arcsin \left( \frac{2\sqrt{2}\eta}{\sqrt{\eta^4 + 4\eta^2 + 1}} \right), \frac{3}{4} \right] \right).
\]
For the case $p = 1$, it is best to factor an earlier representation of $2I_1(\eta)$:
\[
\int_0^{\eta^2} \frac{dt}{\sqrt{t - (2 + \sqrt{3})\sqrt{t - (2 - \sqrt{3})\sqrt{t + (2 - \sqrt{3})\sqrt{t + (2 + \sqrt{3})}}}}}
\]
and employ \[15\] to simplify this integral to
\[
\left( 2 - \sqrt{3} \right) F \left[ \arcsin \left( (2 + \sqrt{3})\eta^2 \right), (2 - \sqrt{3})^4 \right].
\]
Our expression for $2I_1(\eta)$ corrects an error that appears in \[12\].
From
\[
\int_0^\eta \frac{\omega^p}{\sqrt{1 + 14\omega^4 + \omega^8}} \, d\omega = \left( \frac{1 - i}{\sqrt{2}} \right)^{p+1} I_p \left( \frac{1 + i}{\sqrt{2}} \eta \right)
\]
(that is, a rotation of the domain by $45^\circ$) and
\[
\arcsin \left( \frac{2\sqrt{2}\omega}{\sqrt{\omega^4 + 4\omega^2 + 1}} \right) \bigg|_{\omega = \frac{1+i}{\sqrt{2}}(u+i)} = \theta(u,v),
\]
we deduce that

\[
\frac{x}{\kappa} = \text{Re}\left\{ \left( \frac{1-i}{\sqrt{2}} \right) I_0 \left( \frac{1+i}{\sqrt{2}} (u+iv) \right) - \left( \frac{1-i}{\sqrt{2}} \right)^3 I_2 \left( \frac{1+i}{\sqrt{2}} (u+iv) \right) \right\}
\]

\[
= \text{Re}\left\{ \left( \frac{1-i}{\sqrt{2}} \right) F \left[ \theta(u,v), \frac{1}{4} \right] + F \left[ \theta(u,v), \frac{3}{4} \right] \right. \right. \frac{4\sqrt{2}}{4\sqrt{2}} + \left( \frac{1+i}{\sqrt{2}} \right)^3 F \left. \left[ \theta(u,v), \frac{3}{4} \right] \frac{4\sqrt{2}}{4\sqrt{2}} \right\}
\]

\[
= \text{Re}\left\{ -2i F \left[ \theta(u,v), \frac{1}{4} \right] + 2 F \left[ \theta(u,v), \frac{3}{4} \right] \right\} = \text{Re}\left\{ -i F \left[ \theta(u,v), \frac{1}{4} \right] + F \left[ \theta(u,v), \frac{3}{4} \right] \right\},
\]

\[
\frac{y}{\kappa} = \text{Re}\left\{ i \left( \frac{1-i}{\sqrt{2}} \right) I_0 \left( \frac{1+i}{\sqrt{2}} (u+iv) \right) + i \left( \frac{1-i}{\sqrt{2}} \right)^3 I_2 \left( \frac{1+i}{\sqrt{2}} (u+iv) \right) \right\}
\]

\[
= \text{Re}\left\{ \left( \frac{1+i}{\sqrt{2}} \right) F \left[ \theta(u,v), \frac{1}{4} \right] + F \left[ \theta(u,v), \frac{3}{4} \right] \right. \right. \frac{4\sqrt{2}}{4\sqrt{2}} + \left( \frac{1-i}{\sqrt{2}} \right)^3 F \left. \left[ \theta(u,v), \frac{3}{4} \right] \frac{4\sqrt{2}}{4\sqrt{2}} \right\}
\]

\[
= \text{Re}\left\{ 2i F \left[ \theta(u,v), \frac{1}{4} \right] + 2 F \left[ \theta(u,v), \frac{3}{4} \right] \right\} = \text{Re}\left\{ i F \left[ \theta(u,v), \frac{1}{4} \right] + F \left[ \theta(u,v), \frac{3}{4} \right] \right\},
\]

\[
\frac{z - \frac{1}{\kappa}}{\kappa} = \text{Re}\left\{ \left( \frac{1-i}{\sqrt{2}} \right)^2 2I_1 \left( \frac{1+i}{\sqrt{2}} (u+iv) \right) \right\}
\]

\[
= \text{Re}\left\{ -i \left( 2 - \sqrt{3} \right) F \left[ \arcsin \left( \left( 2 + \sqrt{3} \right) \left( \frac{1+i}{\sqrt{2}} (u+iv) \right)^2 \right), \left( 2 - \sqrt{3} \right)^4 \right] \right\}
\]

\[
= \text{Im}\left\{ \left( 2 - \sqrt{3} \right) F \left[ \arcsin \left( i \left( 2 + \sqrt{3} \right) (u+iv)^2 \right), \left( 2 - \sqrt{3} \right)^4 \right] \right\}
\]

as was to be shown.

0.4. Approximations. With regard to tetrahedral dissection, a reasonable approximation is provided by the plane containing \( V_1 = (0, 0, \frac{1}{2}) \), \( V_2 = (\xi, 0, 1 - \xi) \), \( V_3 = (0, \xi, \xi) \), which also contains

\[ V_4 = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) = \left( 1 - \frac{1}{\xi} \right) V_1 + \frac{1}{2\xi} V_2 + \frac{1}{2\xi} V_3. \]

The plane cuts the tetrahedron into two polyhedra of equal area, and the area of the quadrilateral slice is written in terms of the cross-product of its diagonals:
\[
\frac{1}{2} |(V_3 - V_1) \times (V_4 - V_2)| = \frac{1}{2 \sqrt{2}} \sqrt{(1 - 2\xi)^2 + 2\xi^2} = 0.2046... > 0.1954....
\]

With regard to the octahedral “four edges” surface, an excellent approximation is given in [16]:

\[
z = \frac{1}{\pi} \arccos (\cos(\pi x) - \cos(\pi y)), \quad x \geq |y|, \quad -\frac{1}{2} \leq y \leq \frac{1}{2}
\]

and the corresponding surface area is

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{\frac{\pi}{2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dy \, dx = 0.3920... > 0.3908....
\]

The constant 2.3451... appears in [17, 18, 19, 20, 21, 22].

0.5. Addendum. The surface in Figure 2 has, in fact, zero mean curvature, but its Weierstrass-Enneper representation is unknown. An easy upper bound for the surface area is $1/4 = 0.25$, given by the planar square with vertices coinciding with edge midpoints. A purely numerical approach for computing the minimal surface [23] yields that the area is 0.2172341554...; I am grateful to Kenneth Brakke for this precise estimate. As far as is known, this particular constant is new.

References


