Gergonne-Schwarz Surface

STEVEN FINCH

April 12, 2013

We mentioned Plateau’s problem in [1] but did not give a nontrivial example. Let
\[ F[\phi, m] = \int_{0}^{\sin(\phi)} \frac{d\tau}{\sqrt{1 - \tau^2} \sqrt{1 - m \tau^2}} \]
denote the incomplete elliptic integral of the first kind and \( K[m] = F[\pi/2, m] \); the latter is admittedly incompatible with [2] but we purposefully choose formulas here to be consistent with the computer algebra package MATHEMATICA. The three basic Jacobi elliptic functions are defined via
\[
\begin{align*}
  u &= \int_{0}^{\text{sn}(u, m)} \frac{d\tau}{\sqrt{1 - \tau^2} \sqrt{1 - m \tau^2}} = \int_{\text{cn}(u, m)}^{1} \frac{d\tau}{\sqrt{1 - \tau^2} \sqrt{m \tau^2 + (1 - m)}} \\
  &= \int_{\text{dn}(u, m)}^{1} \frac{d\tau}{\sqrt{1 - \tau^2} \sqrt{\tau^2 - (1 - m)}}
\end{align*}
\]
and two (of nine) others we require are
\[ \text{sc}(u, m) = \frac{\text{sn}(u, m)}{\text{cn}(u, m)}, \quad \text{sd}(u, m) = \frac{\text{sn}(u, m)}{\text{dn}(u, m)}. \]

Our work supplements [3] very closely, even down to the level of notation. The setting is three-dimensional \( xyz \)-space.

0.1. Six Edges of a Cube. Consider a polygonal wire loop with six line segments:
\[
(0, 0, 0) \to (1, 0, 0) \to (1, 0, 1) \to (1, 1, 1) \to (0, 1, 1) \to (0, 1, 0) \to (0, 0, 0).
\]

What is the minimal area for any surface spanning this fixed boundary? Equivalently, what is the outcome of dipping the wire loop in a soap solution?

\(^{0}\)Copyright © 2013 by Steven R. Finch. All rights reserved.
Define
\[ \rho_0 = K[1/4] = 1.6857503548... \]
and let \( t = \mathcal{E}(\xi) \) denote the functional inverse of the elliptic integral
\[ \xi = \int_0^t \frac{d\tau}{\sqrt{1 + \tau^2 + \tau^4}}. \]

The desired minimal surface is given implicitly by the equation [3]
\[ \mathcal{E}(x)\mathcal{E}(y) = \mathcal{E}(z) \]
where \( 0 \leq x, y, z \leq \rho_0 \).

This is as far as Nitsche [3] went in describing his calculations. Solving for \( z \) and rescaling (so that the surface spans the \( 1 \times 1 \times 1 \) cube), we find that
\[ z = \frac{1}{2\rho_0} F \left[ \arccos \left( \frac{\text{cn} \left( 2\rho_0 x, \frac{1}{4} \right) + \text{cn} \left( 2\rho_0 y, \frac{1}{4} \right)}{1 + \text{cn} \left( 2\rho_0 x, \frac{1}{4} \right) \text{cn} \left( 2\rho_0 y, \frac{1}{4} \right)} \right), \frac{1}{4} \right], \quad 0 \leq x, y \leq 1 \]
and the surface area is
\[ 2 \int_0^1 \int_0^{1-x} \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dy \, dx = \frac{3 K[3/4]}{2 K[1/4]} = 1.9188923567..., \]
as predicted in [4]. See Figure 1.

0.2. Four Edges of a Regular Tetrahedron. Consider a polygonal wire loop with four line segments:
\[ (0,0,0) \rightarrow (1,0,1) \rightarrow (1,1,0) \rightarrow (0,1,1) \rightarrow (0,0,0). \]

Again, what is the minimal area for any surface spanning this fixed boundary?

With \( \rho_0 \) as before, let \( s = \mathcal{F}(\eta) \) denote the functional inverse of the elliptic integral
\[ \eta = \int_0^s \frac{d\sigma}{\sqrt{\frac{1}{4} + \frac{5}{2} \sigma^2 + \frac{3}{4} \sigma^4}}. \]

The desired minimal surface is given implicitly by the equation [3]
\[ \mathcal{F}(y)\mathcal{F}(z) + \mathcal{F}(z)\mathcal{F}(x) + \mathcal{F}(x)\mathcal{F}(y) + 1 = 0 \]
Figure 1: “Six edges” minimal surface
where \(0 \leq x, y \leq \rho_0\) and \(-\rho_0 \leq z \leq 0\). Dalpe [5] introduced one correction in the preceding: the cube has side \(\rho_0\), not \(2\rho_0\).

This is as far as described in [3]. Solving for \(z\) and rescaling (so that the surface spans the \(1 \times 1 \times 1\) cube), we find that

\[
z = \frac{1}{\sqrt{3}\rho_0} F \left[ \arccos \left( \frac{\text{cn} \left( \sqrt{3}\rho_0 x, -\frac{1}{3} \right) \text{cn} \left( \sqrt{3}\rho_0 y, -\frac{1}{3} \right)}{1 + \text{sn} \left( \sqrt{3}\rho_0 x, -\frac{1}{3} \right) \text{sn} \left( \sqrt{3}\rho_0 y, -\frac{1}{3} \right)} \right), -\frac{1}{3} \right], \quad 0 \leq x, y \leq 1
\]

(note multiplication in the numerator and \(\text{sn}\) in the denominator, unlike before) and the surface area is

\[
2 \int_0^1 \int_0^{1-x} \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dy \, dx = \frac{K[3/4]}{K[1/4]} = 1.2792615711...,\]

as predicted in [4]. See Figure 2. This example and the first one feature portions of what is known as the *Schwarz D surface* (D stands for “Diamond”).

### 0.3. Two Diagonals and Free Boundaries.

Consider the soap film (resembling a twisted curtain) formed between two skew line segments:

\[
(2, 0, 0) \rightarrow (0, 2, 0) \quad \text{and} \quad (0, 0, 2) \rightarrow (2, 2, 2).
\]

Understanding that two remaining boundaries are unspecified, what is the minimal area for any surface spanning the diagonals? [6] This is a famous question due to Gergonne (1816) and answered by Schwarz (1872).

For fixed \(\kappa > 0\), let \(t = Q(\varphi, \kappa)\) and \(t = R(\psi, \kappa)\) denote functional inverses of the elliptic integrals

\[
\varphi = \int_0^t \frac{d\tau}{\sqrt{\kappa - \tau^2 - \tau^4}}, \quad \psi = \int_0^t \frac{d\tau}{\sqrt{\kappa + (1 + 2\kappa)\tau^2 + \kappa \tau^4}}.
\]

Define also

\[
\lambda(\kappa) = \frac{\sqrt{1 + 4\kappa} - 1}{2\sqrt{1 + 4\kappa}}, \quad \mu(\kappa) = \sqrt{\frac{\sqrt{1 + 4\kappa} - 1}{2}}.
\]

We have, in particular,

\[
\int_0^{\mu(\kappa)} \frac{d\tau}{\sqrt{\kappa - \tau^2 - \tau^4}} = \frac{K[\lambda(\kappa)]}{(1 + 4\kappa)^{1/4}},
\]
Figure 2: Tetrahedral “four edges” minimal surface
\[
\int_0^1 \frac{d\tau}{\sqrt{\kappa + (1+2\kappa)\tau^2 + \kappa \tau^4}} = \frac{K \left[ -\frac{1}{4\kappa} \right]}{2\sqrt{\kappa}}
\]

and these two expressions, when set equal, force \( \kappa = \kappa_0 = 0.2092861374 \). Denote the former integral by \( \varphi_0 \) and latter by \( \psi_0 \); consequently \( \varphi_0 = \psi_0 = 1.3970394887 \). The desired minimal surface is given implicitly by the equation [3]

\[
Q(x - \varphi_0)R(z - \psi_0) + Q(y - \varphi_0) = 0
\]

where \( 0 \leq x, y \leq 2\varphi_0 \) and \( 0 \leq z \leq 2\psi_0 \). We have introduced two corrections in the preceding: the upper integration limit of \( \psi_0 \) is \( 1 \) (not \( \mu(\kappa) \), which was a typographical error in [3]) and the denominator underlying \( K \left[ -\frac{1}{4\kappa} \right] \) is \( 2\sqrt{\kappa} \) (not merely \( 2 \), which was a computational error in [3]). More on the second correction will be mentioned shortly.

This, again, is as far as described in [3]. Let

\[
\theta_0 = (1 + 4\kappa_0)^{1/4} \varphi_0, \quad \lambda_0 = \lambda(\kappa_0), \quad \varepsilon(x,y) = \begin{cases} \frac{1}{2} & \text{if } (x-1)(y-1) > 0, \\ -1 & \text{otherwise}. \end{cases}
\]

Solving for \( z \) and rescaling (so that the surface spans the \( 2 \times 2 \times 2 \) cube), we find that

\[
z = 1 + \frac{\varepsilon(x,y)}{2\sqrt{\kappa}\psi_0} F \left[ \arccos\left( \frac{\text{sd}(\theta_0(x-1),\lambda_0)^2 - \text{sd}(\theta_0(y-1),\lambda_0)^2}{\text{sd}(\theta_0(x-1),\lambda_0)^2 + \text{sd}(\theta_0(y-1),\lambda_0)^2} \right), -\frac{1}{4\kappa_0} \right]
\]

assuming \((y > x \text{ and } x < 2 - y) \) or \((y < x \text{ and } x > 2 - y)\); elsewhere on \( 0 \leq x, y \leq 2 \), no definition for \( z \) is given. The surface area is

\[
4 \int_0^1 \int_0^{1-x} \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dy \, dx = 4.9348196582... = 4(1.2337049145...)
\]

and a closed-form expression remains open. See Figure 3. We have not attempted to establish consistency with [7].

0.4. Details of Elliptic Functions. We can compute \( E(\xi) \) and \( F(\eta) \) using results in [8]:

\[
\xi = \int_0^t \frac{d\tau}{\sqrt{1 + \tau^2 + \tau^4}} = \frac{1}{2} F \left[ \arccos\left( \frac{1-t^2}{1+t^2} \right), \frac{1}{4} \right],
\]

\[
\eta = \int_0^s \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}} = \frac{1}{\sqrt{3}} F \left[ \arccos\left( \frac{1-s^2}{1+s^2} \right), -\frac{1}{3} \right]
\]
Figure 3: “Two diagonals” minimal surface
since each quartic has four imaginary zeroes; hence

\[ t = \sqrt{\frac{1 - \cn(2\xi, 1/4)}{1 + \cn(2\xi, 1/4)}}, \]

\[ s = \sqrt{\frac{1 - \cn(\sqrt{3}\eta, -1/3)}{1 + \cn(\sqrt{3}\eta, -1/3)}} \]

and thus

\[ z = \frac{1}{2} F \left[ \arccos \left( \frac{1 - \mathcal{E}(x)^2 \mathcal{E}(y)^2}{1 + \mathcal{E}(x)^2 \mathcal{E}(y)^2} \right), \frac{1}{4} \right] \]

gives the “six edges” result. From

\[ \mathcal{F}(z) = -\frac{1 + \mathcal{F}(x)\mathcal{F}(y)}{\mathcal{F}(x) + \mathcal{F}(y)} \]

we obtain

\[ z = \frac{1}{\sqrt{3}} F \left[ \arccos \left( \frac{1 - \left(1 + \frac{\mathcal{F}(x)\mathcal{F}(y)}{\mathcal{F}(x) + \mathcal{F}(y)}\right)^2}{1 + \left(1 + \frac{\mathcal{F}(x)\mathcal{F}(y)}{\mathcal{F}(x) + \mathcal{F}(y)}\right)^2} \right), -\frac{1}{3} \right] \]

and, because \( \text{sn}(u, m)^2 + \text{cn}(u, m)^2 = 1 \), the “four edges” result follows.

Computing \( Q(\varphi, \kappa) \) is somewhat different [9]:

\[ \varphi = \int_{0}^{t} \frac{d\tau}{\sqrt{\kappa - \tau^2 - \tau^4}} \]

\[ = \frac{1}{(1 + 4\kappa)^{1/4}} \left\{ K[\lambda(\kappa)] - F \left[ \arcsin \left( \frac{\sqrt{1 + 4\kappa - 2t^2 - 1}}{\sqrt{1 + 4\kappa - 1}} \right), \lambda(\kappa) \right] \right\} \]

since the quartic has two real zeroes and two imaginary zeroes. Observe that, when \( t = \mu(\kappa) \), the second term vanishes. Inverting, we obtain

\[ t = \frac{\kappa}{(1 + 4\kappa)^{1/4}} \text{sd} \left( (1 + 4\kappa)^{1/4} \varphi, \lambda(\kappa) \right) \]

and therefore

\[ -\frac{Q(y - \varphi_0, \kappa)}{Q(x - \varphi_0, \kappa)} = \frac{\text{sd} \left( (1 + 4\kappa)^{1/4} (y - \varphi_0), \lambda(\kappa) \right)}{\text{sd} \left( (1 + 4\kappa)^{1/4} (x - \varphi_0), \lambda(\kappa) \right)}. \]
Only the inverse of $R(\psi, \kappa)$ is required:

$$
\psi = \int_0^t \frac{d\tau}{\sqrt{\kappa + (1 + 2\kappa)\tau^2 + \kappa \tau^4}} = \frac{\text{sign}(t)}{2\sqrt{\kappa}} F\left[\arccos\left(\frac{1 - t^2}{1 + t^2}\right), -\frac{1}{4\kappa}\right]
$$

which generalizes the earlier cases $\kappa = -1$ and $\kappa = 3/4$. Note the specialization $t = 1$, as well as the need here to track whether $t = -Q(y - \varphi_0, \kappa)/Q(x - \varphi_0, \kappa)$ is positive or negative.

0.5. Approximations of Minimal Surfaces. A surprisingly good fit to the “four edges” surface is provided by the hyperbolic paraboloid

$$
z = x + y - 2xy
$$

and the corresponding surface area is

$$
2 \int_0^1 \int_0^{1-x} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dy \, dx = 1.2807... > 1.2792...
$$

See [10] for more on approximating the Schwarz D surface, which (upon suitable transformation) should enable a reasonable fit to the “six edges” surface.

Fairly coarse fits to the “two diagonals” surface are provided by

$$
z = 1 + \frac{y - 1}{x - 1}, \quad z = 1 + \frac{4}{\pi} \arctan\left(\frac{y - 1}{x - 1}\right)
$$

if $(y > x$ and $x < 2 - y)$ or $(y < x$ and $x > 2 - y)$, and the corresponding surface areas are $5.1231...$ and $5.0307...$, respectively. We mentioned earlier that Nitsche [3] mistakenly solved the equation

$$
\frac{K[\lambda(\kappa)]}{(1 + 4\kappa)^{1/4}} = \frac{K\left[\frac{-1}{4\pi}\right]}{2};
$$

the denominator underlying $K\left[\frac{-1}{4\pi}\right]$ is missing a factor $\sqrt{\kappa}$. It is nevertheless instructive to follow through to the end. We find $\kappa = \tilde{\kappa}_0 = 6.6061877190...$ and consequently $\tilde{\varphi}_0 = \tilde{\psi}_0 = 0.7781217795...$. The surface obtained is a minimal surface (with mean curvature everywhere equal to zero) and correctly spans the diagonals. The two free contours, however, are not best possible: the surface area for $\tilde{\kappa}_0$ is $4.9480...$, which is larger than the surface area $4.9348...$ for $\kappa_0$.

The constant $1.9188...$ appears in [11, 12], $1.2792...$ in [13, 14] and a rough estimate for $\frac{1}{\pi}(4.9348...)$ in [15]. See [16, 17] for introductory materials, as well as Schwarz’s complete works [18]. Other polygonal wire loops, with more solutions of Plateau’s problem, are surveyed in [19].
0.6. Acknowledgements. Ulrike Bücking was so kind as to point out two errors in [3]; I also appreciate correspondence with Djurdje Cvijovič and Stefan Hildebrandt.

0.7. Addendum. Another portion of the Schwarz D surface arises as a soap film spanning two parallel equilateral triangles with vertices
\{(1, -1, -1), (-1, 1, -1), (-1, -1, 1)\} \quad \text{and} \quad \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}.

One triangle is a copy of the other, rotated 60° about its center. Each of the six edges has length \(2\sqrt{2}\) and the perpendicular distance between triangular centers is \(2/\sqrt{3}\); the ratio of these is \(\sqrt{6}\). Define \(\zeta_0 = K[8/9]\). The desired minimal annulus is given implicitly by [18, 20]

\[
\text{sc}(\zeta_0 y, \frac{8}{9}) \text{sc}(\zeta_0 z, \frac{8}{9}) + \text{sc}(\zeta_0 z, \frac{8}{9}) \text{sc}(\zeta_0 x, \frac{8}{9}) + \text{sc}(\zeta_0 x, \frac{8}{9}) \text{sc}(\zeta_0 y, \frac{8}{9}) + 3 = 0
\]

where \(-1 \leq x, y, z \leq 1\) and its surface area is \(6K[3/4]/K[1/4]\). See Figure 4. (This result contradicts a statement in [21] that, for Schwarz D to appear, the ratio of edge length to distance should be \(2\sqrt{3}\).)

A more difficult task is to represent the minimal annulus corresponding to parallel triangles that are aligned [22, 23, 24, 25, 26], that is, with no rotation. This is a member of the family of Schwarz H surfaces (H stands for “Hexagonal”). Assistance on such representations, for a range of perpendicular distances between triangular centers, and on numerical calculation of surface areas, would be deeply appreciated.

References
Figure 4: “Two twisted triangles” minimal surface


