Modular Forms on $\text{SL}_2(\mathbb{Z})$

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Let $k \in \mathbb{Z}$ and let $\text{SL}_2(\mathbb{Z})$ denote the special linear group

$$\text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$ 

A modular form of weight $k$ is an analytic function $f$ defined on the complex upper half plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ that transforms under the action of $\text{SL}_2(\mathbb{Z})$ according to the relation [1]

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

and whose Fourier series $f(z) = \sum_{n=-\infty}^{\infty} \gamma_n e^{2\pi inz}$ satisfies $\gamma_n = 0$ for all $n < 0$. In particular, we have

$$f(z + 1) = f(z), \quad f(-1/z) = (-z)^k f(z).$$

If, additionally, we have $\gamma_0 = 0$, then $f$ is a cusp form of weight $k$. Every nonconstant modular form has weight $k \geq 4$, where $k$ is even, and every nonzero cusp form has weight $k \geq 12$. The set $M_k$ of modular forms and the set $S_k$ of cusp forms are finite-dimensional vector spaces over $\mathbb{C}$ with [2]

$$\dim(M_k) = \left\{ \begin{array}{ll} \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 2 \mod 12, \\
\left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{if } k \equiv 0, 4, 6, 8, 10 \mod 12 \end{array} \right.$$ 

and $\dim(S_k) = \dim(M_k) - 1$ if $k \geq 12$. We will focus primarily on a specific basis element of $S_{12}$, leaving other aspects of this huge research area for later.

The discriminant function $\Delta : \mathbb{H} \to \mathbb{C}$, defined via

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{m=1}^{\infty} \tau(m)q^m$$

where $q = e^{2\pi iz}$ and $\tau : \mathbb{Z}^+ \to \mathbb{Z}$ is the Ramanujan tau function [3, 4, 5, 6, 7], can be proved to be a cusp form of weight 12. Nobody knows whether $\tau(m) \neq 0$.

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for all \( m \geq 1 \), but Mordell [8] proved that \( \tau \) is a multiplicative function and Deligne [9, 10, 11] proved that \( |\tau(p)| \leq 2p^{11/2} \) for any prime \( p \). This implies that [12]

\[
\tau(m) = O(m^{11/2+\varepsilon})
\]
as \( m \to \infty \), for any \( \varepsilon > 0 \); further [13, 14, 15, 16, 17],

\[
\liminf_{m\to\infty} m^{-11/2}\tau(m) = -\infty, \quad \limsup_{m\to\infty} m^{-11/2}\tau(m) = \infty.
\]

Let the Hecke L-series be

\[
L_\Delta(z) = \sum_{m=1}^\infty \tau(m)m^{-z} = \prod_p \frac{1}{1 - \tau(p)p^{-z} + p^{11-2z}}, \quad \text{Re}(z) > \frac{13}{5},
\]

and its modification be

\[
L_\Delta^*(z) = (2\pi)^{-z}\Gamma(z)L_\Delta(z).
\]

Then \( L_\Delta(z) \) can be extended to an entire function and the functional equation \( L_\Delta^*(z) = L_\Delta^*(12-z) \) is satisfied everywhere. One can compute \( L_\Delta(6) = 0.7921228386... \), for example, but it turns out that more can be said.

Define two constants [18, 19, 20]

\[
\xi = 30L_\Delta^*(6) = 0.0463463808... = 960(0.0000482774...) = 5(0.0092692761...),
\]

\[
\eta = 28L_\Delta^*(5) = 28L_\Delta^*(7) = 0.0457516089... = \frac{32}{9}(0.0214460667...) = \frac{2}{5}(0.1143790224...).
\]

It can be shown that the values of \( L_\Delta^*(n) \) at even \( 2 \leq n \leq 10 \) are rational multiples of \( \xi \):

\[
L_\Delta^*(4) = L_\Delta^*(8) = \frac{1}{24}\xi, \quad L_\Delta^*(2) = L_\Delta^*(10) = \frac{2}{5}\xi,
\]

and that the values of \( L_\Delta^*(n) \) at odd \( 1 \leq n \leq 11 \) are rational multiples of \( \eta \):

\[
L_\Delta^*(3) = L_\Delta^*(9) = \frac{1}{18}\eta, \quad L_\Delta^*(1) = L_\Delta^*(11) = \frac{90}{609}\eta.
\]

These can alternatively be written in terms of \( L_\Delta(1) \) and \( L_\Delta(2) \); see Table 1. Similar collapsing occurs at integer arguments \( < k \) for the unique cusp forms of weight \( k = 16 \) and \( k = 18 \) [7]. An integral expression for \( L_\Delta^*(n) \) is [21]

\[
L_\Delta^*(n) = \frac{1}{\pi^{n-1}\pi^{11}} \int_0^1 \left( \int_v^1 \frac{du}{\sqrt{u(u-1)(u-v)}} \right)^{n-1} \left( \int_1^\infty \frac{du}{\sqrt{u(u-1)(u-v)}} \right)^{11-n} v(1-v) dv
\]
where \( n = 1, 2, \ldots, 11 \) and \( i \) is the imaginary unit. The product \( \xi \eta = 0.0021204214 \ldots \) also appears in the following [18, 19, 22, 23, 24]:

\[
\lim_{x \to \infty} \frac{1}{x^{12}} \sum_{m \leq x} \tau(m)^2 = \frac{2^{3 \pi^{11}}}{3^{45 \pi^{11}}} = 0.0320070045 \ldots
\]

which is an interesting asymptotic mean square result. In contrast, we know that [25, 26]

\[
\sum_{m \leq x} \tau(m) = O \left( x^{35/6+\varepsilon} \right)
\]

as \( x \to \infty \), for any \( \varepsilon > 0 \), and that [27, 28]

\[
\liminf_{x \to \infty} x^{-23/4} \sum_{m \leq x} \tau(m) = -\infty, \quad \limsup_{x \to \infty} x^{-23/4} \sum_{m \leq x} \tau(m) = \infty,
\]

but a more precise estimate of the mean apparently remains open. Moreover [0.2],

\[
\sum_{m \leq x} |\tau(m)| = o \left( x^{13/2} \right)
\]

as \( x \to \infty \). See also [29, 30, 31].

Table 1. Values of \( L_f(1), L_f(2) \); \( f \) is the unique cusp form of weight \( k = 12, 16, 18 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>12</th>
<th>16</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_f(1) )</td>
<td>0.0374412812 \ldots</td>
<td>0.5870144080 \ldots</td>
<td>-3.5316483054 \ldots</td>
</tr>
<tr>
<td>( L_f(2) )</td>
<td>0.1463745420 \ldots</td>
<td>1.6654560382 \ldots</td>
<td>-8.6783515629 \ldots</td>
</tr>
</tbody>
</table>

0.1. Congruence Subgroups. Given \( N \) to be a positive integer, define the following subgroup of the full modular group \( SL_2(\mathbb{Z}) \):

\[
\Gamma_0(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}
\]

and define a weight \( k \) modular form of level \( N \) exactly as before, with \( SL_2(\mathbb{Z}) \) replaced by \( \Gamma_0(N) \). Clearly the preceding discussion applies to the case \( N = 1 \) and \( k \) free; we focus henceforth on the case \( k = 2 \) and \( N \) free. The first nonzero weight 2 cusp form has level 11:

\[
f(z) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2
\]
whose Fourier coefficients coincide [32] with those of the L-series for the elliptic curve isogeny class 11A. The next two cusp forms have level 14 and 15, corresponding to 14A and 15A. On the one hand, not all cusp forms are linked to elliptic curves: the first counterexamples have level 22 and 23. On the other hand, the Taniyama-Shimura conjecture (proved by Wiles, Taylor, Diamond, Conrad & Breuil [33]) asserts that every elliptic curve $E$ is linked to a cusp form with level $N$ equal to the conductor of $E$.

Let $S_2(N)$ denote the vector space of weight 2 cusp forms of level $N$. The dimension $\delta_0(N)$ of $S_2(N)$ over $\mathbb{C}$ possesses a more complicated formula than earlier [34, 35, 36, 37, 38, 39]:

$$\delta_0(N) = 1 + \frac{\psi(N)}{12} - \frac{\nu_2(N)}{4} - \frac{\nu_3(N)}{3} - \frac{\chi(N)}{2},$$

where

$$\psi(N) = N \prod_{p|N} \left(1 + \frac{1}{p}\right), \quad \chi(N) = \sum_{d|N} \varphi\left(\gcd\left(d, \frac{N}{d}\right)\right),$$

$$\nu_2(N) = \begin{cases} 0 & \text{if } 4|N, \\ \prod_{p|N} \left(1 + \frac{(-4)}{p}\right) & \text{otherwise}; \end{cases} \quad \nu_3(N) = \begin{cases} 0 & \text{if } 9|N, \\ \prod_{p|N} \left(1 + \frac{(-3)}{p}\right) & \text{otherwise}; \end{cases}$$

$\varphi(N) = N \prod_{p|N} (1 - 1/p)$ is the Euler totient function [40], and $(-4/p), (-3/p)$ are Kronecker-Jacobi-Legendre symbols [41]. We have asymptotic extreme results [36, 42]

$$\liminf_{N \to \infty} \frac{\delta_0(N)}{N} = \frac{1}{12}, \quad \limsup_{N \to \infty} \frac{\delta_0(N)}{N \ln(\ln(N))} = \frac{e^\gamma}{2\pi^2}$$

and average behavior

$$\sum_{N \leq y} \delta_0(N) = \frac{5}{8\pi^2} y^2 + o(y^2)$$

as $y \to \infty$. Similar dimension estimates can be found for the vector space $M_2(N)$ of weight 2, level $N$ modular forms [43].

Define also the subgroup

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \mod N \text{ and } c \equiv 0 \mod N \right\}$$

and the corresponding weight 2 cuspidal vector space dimension $\delta_1(N)$. An analogous formula for $\delta_1(N)$ is known [36, 37, 43], with extreme results

$$\liminf_{N \to \infty} \frac{\delta_1(N)}{N^2} = \frac{1}{4\pi^2} < \frac{1}{24} = \limsup_{N \to \infty} \frac{\delta_1(N)}{N^2}$$
Modular Forms on $\text{SL}_2(\mathbb{Z})$ and average behavior
\[
\sum_{N \leq y} \delta_1(N) = \frac{1}{72\zeta(3)} y^3 + o(y^3)
\]
as $y \to \infty$. Generalization to arbitrary integer weight $k$ is also possible.

Let $D = 1$ or $D$ be a fundamental discriminant [44]. A **level** $N$, **weight** $k$
**modular form** $f : \mathbb{H} \to \mathbb{C}$ with **Nebentypus character** $(D/\cdot)$ transforms according to
\[
f \left( \frac{az+b}{cz+d} \right) = \left( \frac{D}{d} \right) (cz+d)^k f(z) \quad \text{for all} \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N).
\]
The trivial case $D = 1$ reduces to the earlier definition. For example, we have
\[
(-15/d)|_{d=1,2,\ldots,15} = \{1,1,0,1,0,0,-1,1,0,0,-1,0,-1,-1,0\}.
\]
It turns out that the vector space of cusp forms corresponding to $(N,k,D) = (15,3,-15)$ is two-dimensional, and that a certain basis element is given by [38, 45, 46, 47]
\[
f(z) = q \prod_{n=1}^{\infty} (1 - q^{3n})^3 (1 - q^{5n})^3 + q^2 \prod_{n=1}^{\infty} (1 - q^{n})^3 (1 - q^{15n})^3.
\]
This will be useful later [0.3]. Also, the vector space of cusp forms corresponding to $(N,k,D) = (6,4,1)$ is one-dimensional with basis element
\[
g(z) = q \prod_{n=1}^{\infty} (1 - q^{n})^2 (1 - q^{2n})^2 (1 - q^{3n})^2 (1 - q^{6n})^2,
\]
which we likewise will see again.

**0.2. Ramanujan Tau Function.** Let us continue where we stopped earlier. It is conjectured that [48, 49, 50, 51, 52]
\[
\sum_{m \leq x} |\tau(m)| \sim A x^{13/2} (\ln(x))^{-1+8/(3\pi)}
\]
as $x \to \infty$, for some constant $0 < A < \infty$, whereas it is known that [50, 53]
\[
\sum_{m \leq x} \tau(m)^4 \sim B x^{23} \ln(x)
\]
for some constant $0 < B < \infty$. Numerical estimates of $A$ and $B$ would be good to see someday. We cannot hope for similar accuracy in estimating $\sum_{m \leq x} \tau(m)$ until the
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The correct order of magnitude – conjectured to be $O \left( x^{23/4+\varepsilon} \right)$ – is established. Evidence that $23/4$ is the best exponent includes the formula \[54, 55, 56, 57, 58, 59, 60, 61\]

\[
\frac{1}{x} \int_1^x \left( \sum_{m \leq y} \tau(m) \right)^2 \, dy \sim C_{\tau} \, x^{23/2}
\]
as $x \to \infty$, where \[62, 63\]

\[
C_{\tau} = \frac{1}{50\pi^2} \sum_{k=1}^\infty \frac{\tau(k)^2}{k^{3/2}} = \frac{1.5882400955...}{50\pi^2}.
\]

There are analogous formulas \[55, 64, 65, 66, 67, 68, 69\] for the error terms in the divisor and circle problems \[70\]:

\[
\frac{1}{x} \int_1^x \left( \sum_{m \leq y} d(m) - y \ln(y) - (2\gamma - 1)y \right)^2 \, dy \sim C_d \, x^{1/2},
\]

\[
\frac{1}{x} \int_1^x \left( \sum_{m \leq y} r(m) - \pi y \right)^2 \, dy \sim C_r \, x^{1/2}
\]

where

\[
C_d = \frac{1}{6\pi^2} \sum_{k=1}^\infty \frac{d(k)^2}{k^{3/2}} = \frac{\zeta(3/2)^4}{6\pi^2 \zeta(3)} = 0.6542839775...,
\]

\[
C_r = \frac{1}{3\pi^2} \sum_{k=1}^\infty \frac{r(k)^2}{k^{3/2}} = \frac{16\zeta(3/2)^2 \beta(3/2)^2}{3(1 + 2^{-3/2}) \pi^2 \zeta(3)} = 1.6939569917...
\]

and $\zeta(z) = L_1(z)$, $\beta(z) = L_{-1}(z)$ denote the Riemann zeta and Dirichlet beta functions, respectively \[71, 72\].

Returning finally to the problem of estimating $\tau(m)$ itself, we ask about the values of constants $c_+, c_-$ for which \[17\]

\[
0 < \limsup_{m \to \infty} m^{-1/2} \exp \left( \frac{-c_+ \ln(m)}{\ln(\ln(m))} \right) \tau(m) < \infty,
\]

\[
-\infty < \liminf_{m \to \infty} m^{-1/2} \exp \left( \frac{-c_- \ln(m)}{\ln(\ln(m))} \right) \tau(m) < 0.
\]

Is there a reason to doubt that $c_+ = c_-?$
0.3. Mahler’s Measure. Before beginning, we observe that the Laurent polynomial equation
\[ 1 + x + \frac{1}{x} + y + \frac{1}{y} = 0 \]
is isomorphic to the elliptic curve 15A8 via the change of coordinates [73, 74]
\[ (x, y) \mapsto \left( \frac{y}{x}, \frac{x^3 - y^2 - xy}{xy} \right). \]
Similarly, the equation
\[ 1 + x + \frac{1}{x} + y + \frac{1}{y} + xy + \frac{1}{xy} = 0 \]
is isomorphic to the curve 14A4, and the equation
\[ -1 + x + \frac{1}{x} + y + \frac{1}{y} + xy + \frac{1}{xy} = 0 \]
is isomorphic to the curve 30A1. Such representations of elliptic curves (as polynomials in \( x, x^{-1}, y, y^{-1} \)) are especially attractive when symmetric in \( x, y \) as shown.

The (logarithmic) Mahler measure of a Laurent polynomial \( P(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \) is defined to be
\[ m(P) = \int_0^1 \int_0^1 \cdots \int_0^1 \ln \left| P(e^{2\pi i \theta_1}, e^{2\pi i \theta_2}, \ldots, e^{2\pi i \theta_n}) \right| \, d\theta_1 d\theta_2 \cdots d\theta_n. \]
We studied \( \exp(m(P)) \) for univariate \( P \) in [75]; our focus here will be on the case \( n \geq 2 \). Smyth [76, 77] proved that
\[ m(1 + x_1 + x_2) = L_3'(-1) = \frac{3\sqrt{3}}{4\pi} L_{-3}(2) = 0.3230659472... \]
\[ = \ln(1.3813564445...), \]
\[ m(1 + x_1 + x_2 + x_3) = 14\zeta'(-2) = \frac{7}{2\pi^2} \zeta(3) = 0.4262783988... \]
\[ = \ln(1.5315470966...) \]
and Rodriguez-Villegas [78, 79, 80] conjectured that
\[ m(1 + x_1 + x_2 + x_3 + x_4) = -L_f'(-1) = \frac{675\sqrt{15}}{16\pi^5} L_f(4) = 0.5444125617..., \]
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$$m(1 + x_1 + x_2 + x_3 + x_4 + x_5) = -8L'_g(-1) = \frac{648}{\pi^6}L_g(5) = 0.6273170748...$$

where $f, g$ are the cusp forms defined at the end of [0.1]. Deninger [81] conjectured that

$$m \left( 1 + x + \frac{1}{x} + y + \frac{1}{y} \right) = L'_{15A}(0) = \frac{15}{4\pi^2}L_{15A}(2) = 0.2513304337...$$

$$= \ln(1.2857348642...)$$

and Boyd [74] conjectured that

$$m \left( 1 + x + \frac{1}{x} + y + \frac{1}{y} + xy + \frac{1}{xy} \right) = L'_{14A}(0) = \frac{7}{2\pi^2}L_{14A}(2) = 0.2274812230...$$

$$= \ln(1.2554338662...)$$

The latter is the smallest known measure of bivariate polynomials; the former is the second-smallest known. Both conjectures can be rephrased in completely explicit terms [74]: If

$$\sum_{n=1}^{\infty} a_n q^n = q \prod_{k=1}^{\infty} \left( 1 - q^k \right) \left( 1 - q^{3k} \right) \left( 1 - q^{5k} \right) \left( 1 - q^{15k} \right),$$

$$\sum_{n=1}^{\infty} b_n q^n = q \prod_{k=1}^{\infty} \left( 1 - q^k \right) \left( 1 - q^{2k} \right) \left( 1 - q^{7k} \right) \left( 1 - q^{14k} \right)$$

then

$$\int_0^{2\pi} \int_0^{2\pi} \ln |1 + 2\cos(s) + 2\cos(t)| \, ds \, dt = 15 \sum_{j=1}^{\infty} \frac{a_j}{j^2},$$

$$\int_0^{2\pi} \int_0^{2\pi} \ln |1 + 2\cos(s) + 2\cos(t) + 2\cos(s + t)| \, ds \, dt = 14 \sum_{j=1}^{\infty} \frac{b_j}{j^2}.$$ 

These integrals bear some resemblance to certain constants in [82]. Trivariate analogs of these two examples are [83, 84, 85]

$$m \left( 1 + x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} \right) = 0.3703929298... = \ln(1.4483035845...),$$

$$m \left( 1 + x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + xy + \frac{1}{xy} + yz + \frac{1}{yz} + xyz + \frac{1}{xyz} \right) = 0.4798982839...$$
but no relation to special L-series values has yet been proposed. Other variations include [74, 85]

\[ m \left( -1 + x + \frac{1}{x} + y + \frac{1}{y} + xy + \frac{1}{xy} \right) = L'_{30A}(0) = \frac{15}{2\pi^2}L_{30A}(2) = 0.6168709387... , \]

\[ m \left( -1 + x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + xy + \frac{1}{xy} + yz + \frac{1}{yz} + xyz + \frac{1}{xyz} \right) = 0.8157244463... . \]

The third-smallest known measure of bivariate polynomials is [74, 84, 86]

\[ m \left( -1 + x + \frac{1}{x} - y - \frac{1}{y} + x^2y^2 + \frac{1}{x^2y^2} \right) = 0.2693386412... = \ln(1.3090983806...) \]

and the fourth-smallest known is [74, 84, 87]

\[ m \left( 1 + x^2 + \frac{1}{x^2} + y^2 + \frac{1}{y^2} + xy + \frac{1}{xy} + x^2y^2 + \frac{1}{x^2y^2} + \frac{1}{x} + \frac{1}{y} \right) = 0.2743632972... \]

\[ = \ln(1.3156927029...). \]

We emphasize that, of all the \( m(P) \) formulas exhibited here, only Smyth’s results are rigorously proved.

0.4. Klein’s Modular Invariant. The only modular form \( f : \mathbb{H} \to \mathbb{C} \) of weight 0 is a constant. (Assume, as at the beginning, that \( f \) is of level 1 and has trivial character.) What happens if we weaken our hypotheses on \( f \)? A modular function \( f \) is an \( SL_2(\mathbb{Z}) \)-invariant meromorphic function on \( \mathbb{H} \) whose Fourier series \( f(z) = \sum_{n=-\infty}^{\infty} \gamma_n q^n \) has at most finitely many \( \gamma_n \neq 0 \) for \( n < 0 \). The set of modular functions can be proved to be a field, \( \mathbb{C}(j) \), generated by Klein’s \( j \)-invariant or Hauptmodul [1, 88, 89, 90, 91, 92]

\[ j(z) = \frac{1}{Q} (1 + 256Q)^3 = \frac{1}{R} (1 + 250R + 3125R^2)^3 = \sum_{m=-1}^{\infty} c(m)q^m \]

where

\[ Q = q \prod_{n=1}^{\infty} \left( \frac{1 - q^{2n}}{1 - q^n} \right)^{24} = \frac{\Delta(2z)}{\Delta(z)}, \]

\[ R = q \prod_{n=1}^{\infty} \left( \frac{1 - q^{5n}}{1 - q^n} \right)^{6} = \left( \frac{\Delta(5z)}{\Delta(z)} \right)^{1/4} \]

and \( c(-1) = 1, c(0) = 744, c(1) = 196884, c(2) = 21493760, \ldots \) Moreover, \( j \) is the unique modular function having a simple pole with residue 1 at \( q = 0 \). Closed-form
expressions and asymptotics for $c(m)$ are known [93, 94, 95], akin to those for the number $p(m)$ of partitions of $m$ [96]. Special values include

$$j(i) = 12^3, \quad j \left( (1 + i\sqrt{3})/2 \right) = 0, \quad j \left( (1 + i\sqrt{163})/2 \right) = (-640320)^3;$$

the latter, plus the fact that $j(z) \approx q^{-1} + 744$, is responsible for the surprising consequence that $e^{\pi\sqrt{163}}$ misses being an integer by less than $10^{-12}$. More special values include

$$j \left( (1 + i\sqrt{15})/2 \right) = x, \quad j \left( (1 + i\sqrt{23})/2 \right) = y$$

where $x, y$ have minimal polynomials $x^2 + 191025x - 121287375$ and $y^3 + 3491750y^2 - 5151296875y + 12771880859375$, respectively. (The class numbers $h_{-1} = h_{-3} = h_{-163} = 1$, $h_{-15} = 2$ and $h_{-23} = 3$ play a role here [44].) Schneider [97] proved that, if $j(\zeta)$ is algebraic, then $\zeta$ is algebraic if and only if $\zeta$ is imaginary quadratic.

It is also known that, if $\tau \in \mathbb{Q}$ is algebraic and $0 < |\tau| < 1$, then $j(\tau)$ is transcendental [98, 99, 100]. A connection between sporadic simple group theory and modular functions (on $\Gamma_0(N)$ and extensions) is beyond the scope of our study [101, 102, 103].

0.5. Addendum. The constants $A$ and $B$, associated with $|\tau(m)|$ and $\tau(m)^4$, were estimated to be 0.0996 and 0.0026 respectively by Fel [104]. Rogers & Zudilin [105, 106] proved Deninger’s conductor 15 conjecture and Brunault [107] & Mellit [108] proved Boyd’s conductor 14 conjecture. We await word on Rodriguez-Villegas’ two conjectures involving $L_f(4)$ and $L_q(5)$.

Here is a seemingly unrelated calculus problem. Let $f(x) = (\pi/4 - x) \ln(g(x))$ be integrable on $[0, \pi/4]$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} \left( 1 - \frac{2k}{n} \right) \ln \left[ g \left( \frac{\pi k}{2n} \right) \right] = \frac{8}{\pi^2} \lim_{n \to \infty} \sum_{k=1}^{\lfloor n/2 \rfloor} f \left( \frac{\pi k}{2n} \right) \left( \frac{\pi(k + 1)}{2n} - \frac{\pi k}{2n} \right)$$

$$= \frac{8}{\pi^2} \int_0^{\pi/4} f(x) \, dx$$

(a limit of Riemann sums). As a simple example,

$$\lim_{n \to \infty} \prod_{k=1}^{\lfloor n/2 \rfloor} \left( \frac{n}{2k} \right)^{\frac{1}{n} \left( 1 - \frac{2k}{n} \right)} = e^\frac{3}{\pi}$$

after setting $g(x) = \pi/(4x)$ and exponentiating. As a more complicated example,

$$\lim_{n \to \infty} \prod_{k=1}^{\lfloor n/2 \rfloor} \cot \left( \frac{\pi k}{2n} \right)^{\frac{1}{n} \left( 1 - \frac{2k}{n} \right)} = e^{\frac{\tau(1)}{2\pi}} = \exp(0.4262783988...)$$

$$= \sqrt{2} \exp(0.0797048085...)$$
after setting $g(x) = \cot(x)$. The latter appears in the asymptotics of what is called the Atiyah determinant from quantum physics [109].

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