Two-Colorings of Positive Integers

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Let \( f : \{1, 2, 3, \ldots \} \to \{-1, 1\} \) be an arbitrary function. Given a threshold \( M > 0 \), we ask two questions:

- Do there exist integers \( a > 0, b \geq 0, \ell > 0 \) such that
  \[
  |f(a + b) + f(2a + b) + f(3a + b) + \cdots + f(\ell a + b)| > M?
  \]

- Do there exist integers \( a > 0, \ell > 0 \) such that
  \[
  |f(a) + f(2a) + f(3a) + \cdots + f(\ell a)| > M?
  \]

The answer to the first question is yes. In words, every two-coloring of the positive integers has unbounded discrepancy, taken over the family of arithmetic progressions. Restricting attention to the subset \{1, 2, 3, \ldots, n\}, we have [1, 2, 3, 4]

\[
c n^{1/4} \leq P(n) = \min_f \max_{a,b,\ell} \left| \sum_{k=1}^{\ell} f(k a + b) \right| \leq C n^{1/4}
\]

for all \( n \), with constants \( c \geq 1/20 \) and \( C < \infty \). The lower bound on \( c \) is improved to 1/7 in [5]; no finite upper bound on \( C \) is apparently known. It is natural to wonder about the numerical values of

\[
\liminf_{n \to \infty} n^{-1/4} P(n), \quad \limsup_{n \to \infty} n^{-1/4} P(n).
\]

The second question, due to Erdős [6, 7, 8] and Chudakov [9, 10], remains open. It is remarkable that, upon mere constraint to homogeneity \((b = 0)\), the problem becomes unsolved! If we expand the family under consideration, more can be said. For almost all real numbers \( \alpha \geq 1 \), there exists \( \ell > 0 \) such that [11, 12, 13]

\[
|f \left( \lfloor \alpha \rfloor \right) + f \left( \lfloor 2\alpha \rfloor \right) + f \left( \lfloor 3\alpha \rfloor \right) + \cdots + f \left( \lfloor \ell \alpha \rfloor \right)| > M.
\]

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Such quasi-arithmetic progressions collapse to homogeneous arithmetic progressions when $\alpha$ is an integer. Even though the set $S$ of counterexamples $\alpha$ has measure zero, it is not known whether $S$ avoids all integers.

We also examine the expression $\[14, 15]\[14, 15]Q(n) = \min_f \max_{a, b, \ell} \left| \sum_{k=1}^{\ell} f(k+a)f(k+b) \right|$ and wonder about the numerical values of $\liminf_{n \to \infty} n^{-1/2}Q(n), \quad \limsup_{n \to \infty} n^{-1/2}Q(n)$.

0.1. Addendum. If $f$ is random (independently taking the values $\pm 1$ with probability $1/2$ at each integer $1 \leq k \leq n$), then an asymptotic statement can be made about the mean: $[16]E (|f(1) + f(2) + f(3) + \cdots + f(n)|) \sim \sqrt{\frac{2n}{\pi}}$ as $n \to \infty$, and likewise

$$E (|f(1)f(1+b) + f(2)f(2+b) + f(3)f(3+b) + \cdots + f(n-b)f(n)|) \sim \sqrt{\frac{2n}{\pi}}$$

for fixed $b \geq 1$. A proof of the latter can be based on $[17]$; the order of $Q(n)$ is $n^{1/2}$ (in agreement) whereas the order of $P(n)$ is only $n^{1/4}$ (in disagreement).

From $[18, 19]$, we learned of the Polymath wiki – which documents massively collaborative online mathematical projects – and which includes work on problems given here $[20]$.

Nikolov & Talwar $[21]$, building on Alon & Kalai $[22]$, have shown that the following statement is true for infinitely many positive integers $n$. There is a set $W \subseteq \{1, \ldots, n\}$ of square-free integers such that, for any $f : W \to \{-1, 1\}$, there exists a positive integer $a$ so that

$$\left| \sum_{w \in W, a \mid w} f(w) \right| = n^{1/O(\ln(\ln(n)))}$$

as $n \to \infty$. (If we were permitted to define $f = 0$ outside of $W$, then the Erdős-Chudakov problem would be solved. The values of $f$, however, are restricted to $\pm 1$, disallowing such a construction.)
Konev & Lisitsa [23, 24], assisted by computer, exhibited a length 1160 sequence whose discrepancy is bounded by \( M = 2 \), but proved that such cannot be true for any sequence of length \( \geq 1161 \). Hence the Erdős-Chudakov conjecture (for infinite sequences) is true for \( M = 2 \). They also exhibited a length 13000 sequence whose discrepancy is bounded by \( M = 3 \). Gowers’ survey [25] contains additional discussion.

Tao [26] evidently has the final word on this subject: every two-coloring of the positive integers has unbounded discrepancy, taken over the family of homogeneous arithmetic progressions. The existence of near-counterexamples (four are given in [26]) serve to isolate the key difficulty of the problem. The argument also applies to functions \( f \) taking values in the unit sphere of a real or complex Hilbert space.

REFERENCES


