A one-dimensional contact process is a continuous-time Markov process on the lattice \( \mathbb{Z} \) of integers. The state at time \( t \) is given by a set \( \eta_t \subseteq \mathbb{Z} \) of the lattice sites which we visualize as being occupied by particles. The system evolves as follows:

- if \( x \in \eta_t \), then \( x \) becomes vacant at rate 1
- if \( x \notin \eta_t \), then \( x \) becomes occupied at rate \( f(N_x) \)

where \( N_x = |\eta_t \cap \{x-1, x+1\}| \) is the number of nearest-neighbor sites that are occupied,

\[
f(N) = \begin{cases} 
0 & \text{if } N = 0, \\
\lambda & \text{if } N = 1, \\
2\lambda & \text{if } N = 2
\end{cases}
\]

and \( \lambda > 0 \) is a fixed parameter. This process is a simple model of the spread of an infectious disease [1, 2, 3, 4, 5]. An individual at \( x \in \mathbb{Z} \) is infected if \( x \in \eta_t \) and healthy if \( x \notin \eta_t \). Healthy individuals become infected at a rate which is proportional to the number of infected neighbors. Infected individuals recover at rate 1.

As \( \lambda \) increases from zero, the contact process undergoes an extinction–survival phase transition. There is a unique critical threshold \( \lambda_c \) such that \( \lambda < \lambda_c \) implies \( \eta_t = \emptyset \) for large \( t \) almost surely, whereas \( \lambda > \lambda_c \) implies \( \eta_t \neq \emptyset \) for all \( t \) almost surely. The best rigorous bounds for \( \lambda_c \) are \( 1.5517 < \lambda_c < 1.9412 \) [6, 7, 8, 9, 10, 11]; the best non-rigorous numerical estimate is

\[
\lambda_c = 1.64892... = \frac{1}{2}(3.29784...) = \frac{1}{2} \cdot \frac{1}{0.30322...} = \frac{1}{0.60645...}
\]

obtained via numerical means/simulation [12, 13, 14, 15, 16, 17] and via lengthy series expansions [18, 19, 20].

One variation on the preceding is to replace \( f(N_x) \) by \( g(N_x) \), where

\[
g(N) = \begin{cases} 
0 & \text{if } N = 0, \\
\lambda & \text{if } N = 1, \\
\lambda & \text{if } N = 2
\end{cases}
\]
and $\lambda_c$ here is $1.74173... = 1/0.57414...$. Another variation is to replace $f(N_x)$ by $h(N_x)$, where

$$h(N) = \begin{cases} 
0 & \text{if } N = 0, \\
\lambda/4 & \text{if } N = 1, \\
\lambda & \text{if } N = 2 
\end{cases}$$

and $\lambda_c$ here is $6.17066 = 1/0.16205....$ No closed-form expressions are known for any of these critical thresholds [21, 22, 23, 24, 25].

Such models are often referred to as *interacting particle systems* or asynchronously-updated *probabilistic cellular automata*. Our opening example ($f$) is often called the *basic* contact process and is clearly connected to epidemiology and ecology [26, 27, 28]. In statistical physics, it is closely related to Schlögl’s first model of an autocatalytic chemical reaction, to directed percolation in two dimensions, and to Reggeon field theory. The other examples are associated with the poisoning of a catalytic surface ($g$) and the testing of an order-parameter exponent universality conjecture ($h$). To describe the latter idea – that a certain exponent $\beta = 0.277...$ is valid for a wide class of nonequilibrium systems with phase transition – would take us too far afield [19, 25, 29, 30, 31].

0.1. Implementation. The following discussion is based on what is called the *graphical representation* of the basic contact process [32, 33, 34]. Let $M$ be a large positive integer. For every integer $1 \leq x \leq M$, let $\{t^x_n : n \geq 1\}$ be the arrival times of a Poisson process with rate 1. For every integer $1 \leq x \leq M - 1$, let $\{u^x_n : n \geq 1\}$ be the arrival times of a Poisson process with rate $\lambda$. Likewise, for every integer $2 \leq x \leq M$, let $\{v^x_n : n \geq 1\}$ be the arrival times of a Poisson process with rate $\lambda$. To generate times $v^x_n$ up to a large value $N$, for example, simply generate a single random integer $K$ via Poisson($\lambda N$), then generate $K$ uniform $\mathbb{[0, N]}$ random values and sorted in increasing order [35]. Of course, $K$ will usually be different for each $x$.

Let

$$W = \bigcup_{1 \leq x \leq M} \{t^x_n : n \geq 1\} \cup \bigcup_{1 \leq x \leq M - 1} \{u^x_n : n \geq 1\} \cup \bigcup_{2 \leq x \leq M} \{v^x_n : n \geq 1\}$$

be sorted in increasing order, keeping track for each value the corresponding site $x$ and whether it arose as a $t$, $u$ or $v$. The event that two values coincide exactly has probability zero. The list $W$ captures all changes occurring on the finite lattice $[1, M]$ over the finite time interval $[0, N]$. 
Without loss of generality, assume $M$ is divisible by 3. Figures 1 and 2 are constructed with initial state taken to be the binary $M$-vector

$$\xi_0 = \begin{pmatrix} 0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0 \end{pmatrix}_{M/3}$$

which serves as an indicator for the set $\eta_0$. Now select the first element $w$ in the list $W$. If $w$ arose as a $t$, then place a 0 at $x$ (there is a death at $x$ if $x$ is occupied). If $w$ arose as a $u$ and if there is a 1 at $x$, then place a 1 at $x+1$ (there is a birth at $x+1$ if $x$ is occupied and $x+1$ is vacant). If $w$ arose as a $v$ and if there is a 1 at $x$, then place a 1 at $x-1$ (there is a birth at $x-1$ if $x$ is occupied and $x-1$ is vacant). This gives $\xi_w$ and hence $\eta_w$. Now select the second element in $W$ and continue similarly until either the list is exhausted or all $\xi_w$ are 0s (the vacuum state is absorbing). Figures 1 and 2 exhibit only a subsample of states, one per unit time. The vertical axis is space ($1 \leq x \leq M$) and the horizontal axis is time ($0 \leq t \leq N$).

In closing, we mention rigorous bounds $0.3597 < \lambda_c < 0.79$ for the contact process in two dimensions [4, 36, 37], as well as a non-rigorous estimate $\lambda_c \approx 0.412$ [10, 12]. Every lattice site here has four nearest neighbors, complicating the analysis. We hope to revisit calculations [37] of the upper bound 0.79 here, as well as to examine more carefully the series expansions [38] giving precise results earlier in one dimension.
0.2. Discrete Time Analog. An exceedingly simple model, described in [26], deserves further study. The time interval $[0, N]$ from earlier is replaced by $\{0, 1, \ldots, N\}$; we need “collision rules” to decide the outcome when several events occur simultaneously in space and time.

For every integer $1 \leq x \leq M$, let $\{t_n^x : n \geq 1\}$ be the arrival times of a Bernoulli process with rate $\gamma$. Hence each $t_n^x$ corresponds to a biased coin toss yielding heads. For every integer $1 \leq x \leq M - 1$, let $\{u_n^x : n \geq 1\}$ correspond to the coin tosses yielding tails. Likewise, for every integer $2 \leq x \leq M$, let $\{v_n^x : n \geq 1\}$ correspond to the coin tosses yielding tails. Note that only one Bernoulli process is involved here for each $x$, not three independent Poisson processes as before.

Form the multilist $W$ as before – many coincident values appear here unlike before – keeping track for each value the corresponding site $x$ and whether it arose as a $t$, $u$ or $v$. Take the initial state $\xi_0$ as before. Select all the elements $w$ in the multilist $W$ equal to 1. First, for each $w = 1$ arising as a $t$, assign a 0 at $x$ (there is a death at $x$ if $x$ is occupied). This gives a provisional state, called $\xi_1$, and we make a copy, called $\xi'_1$, on which further changes are written. Second, for each $w = 1$ arising as a $u$, if there is a 1 at $x$, then assign a 1 at $x' + 1$ (there is a birth at $x' + 1$ if $x$ is occupied and $x' + 1$ is vacant). Third, for each $w = 1$ arising as a $v$, if there is a 1 at $x$, then assign a 1 at $x' - 1$ (there is a birth at $x' - 1$ if $x$ is occupied and $x' - 1$ is vacant). Finally, overwrite $\xi_1$ by $\xi'_1$. Now continue with all elements $w$ in $W$ equal to 2, assign deaths followed by births, and so forth.

Durrett & Levin [26] estimated the critical threshold $\gamma_c$ to be approximately 0.47 for large $M$ and $N$. A more accurate estimate is highly desirable!
0.3. Oriented or Directed Percolation. The graphs of one-dimensional discrete-time contact processes bear resemblance to two-dimensional percolation [39]. More precisely, they are similar to the oriented or directed case of percolation in which fluid must flow either north or east [2, 40, 41, 42, 43]. For both bonds and sites, there exist critical probabilities $p_{cb}$ and $p_{cs}$ below which all clusters are finite and above which an infinite cluster must exist. No closed-form expressions are known in this case (unlike ordinary percolation). Without giving any details, we have rigorous bounds on bond critical probability [37, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53]

$$0.6383 \leq p_{cb} \leq 2/3;$$

rigorous bounds on site critical probability [37, 46, 47, 49, 50, 52, 54]

$$0.6977 \leq p_{cs} \leq 0.7491;$$

and numerical estimates [29, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65]

$$p_{cb} = 0.64470018... = 1 - 0.35529982...,$$

$$p_{cs} = 0.7054852... = 1 - 0.2945148...$$

for the square lattice. Different probabilities apply for the triangular and hexagonal (honeycomb) lattices in $\mathbb{R}^2$ as well as for the cubic lattice in $\mathbb{R}^3$. A percolation-theoretic analog of the connective constant for self-avoiding walks [66] is investigated in [67].

REFERENCES


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