Central Binomial Coefficients

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March 28, 2007

The largest coefficient of the polynomial $(1 + x)^n$ is [1]

$$A(n) = \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}.$$ 

It possesses recursion

$$\left[ \frac{n+1}{2} \right] A(n+1) = (n+1)A(n), \quad A(0) = 1$$

and asymptotics

$$A(n) \sim \sqrt{\frac{2}{\pi}} n^{-1/2} 2^n$$

as $n \to \infty$. Another interpretation of $A(n)$ is as the number of sign choices + and − such that

$$\pm 1 \pm 1 \pm 1 \pm \cdots \pm 1 = 0 \quad \text{if } n \text{ is even},$$

and

$$\pm 1 \pm 1 \pm 1 \pm \cdots \pm 1 = 1 \quad \text{if } n \text{ is odd}.$$ 

The latter is an especially attractive characterization of the $n^{\text{th}}$ central binomial coefficient.

Contrast this with the $n^{\text{th}}$ central trinomial coefficient, $B(n)$, defined to be the largest coefficient of the polynomial $(1 + x + x^2)^n$. There is no simple closed-form expression for $B(n)$ [2]. It possesses recursion

$$(n + 1)B(n + 1) = (2n + 1)B(n) + 3n B(n - 1), \quad B(0) = B(1) = 1$$

and asymptotics

$$B(n) \sim \sqrt{\frac{3}{4\pi}} n^{-1/2} 3^n.$$ 

Here, $B(n)$ can be interpreted as the number of solutions of

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \cdots + \varepsilon_n = 0$$

where each $\varepsilon_j \in \{-1, 0, 1\}$. Easy proofs of the asymptotics of $A(n)$ and $B(n)$ can be based on such additive representations, coupled with the Central Limit Theorem [3].

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0.1. Divisibility. Let $\omega(n, k)$ denote the number of distinct prime factors of $\binom{n}{k}$. Erdős [4, 5] proved that
\[ \omega(2n, n) \sim 2 \ln(2) \frac{n}{\ln(n)} \]
as $n \to \infty$ and wondered what else could be said about the prime factors. Let
\[ f(n) = \sum_{p \leq n, \ p | (2n, n)} \frac{1}{p}, \]
then [6, 7]
\[ c = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) = \sum_{k=2}^{\infty} \frac{\ln(k)}{2^k} = 0.5078339228..., \]
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (f(n) - c)^2 = 0. \]

These two facts together express that $f(n) \to c$ for almost all integers $n$, hence $\binom{2n}{n}$ is almost always divisible by high powers of small primes. Let $g(n)$ be the smallest odd prime factor of $\binom{2n}{n}$. Whether $f(n)$ or $g(n)$ are bounded remains an open question.

Sárközy [8] and others [9, 10, 11] proved that $\binom{2n}{n}$ is not square-free for any $n > 4$. The largest $n$ for which $\binom{2n}{n}$ is not divisible by $p^2$ for any odd prime $p$ is $n = 786$.

We turn attention to $\binom{n}{k}$, the $(k+1)^{st}$ element in the $n^{th}$ row of Pascal’s triangle. For each $k \geq 1$, the sequence of integers $n$ such that $\binom{n}{k}$ is square-free has asymptotic density $c_k$, where
\[ c_1 = \frac{6}{\pi^2} = 0.6079271018..., \quad c_2 = \frac{3}{4} \prod_{p \geq 3} \left( 1 - \frac{2}{p^2} \right) = 0.4839511484..., \]
(the latter is related to the Feller-Tornier constant [12]). More generally, write $k$ in base $p$:
\[ k = a_0 + a_1 p + a_2 p^2 + \cdots + a_\ell p^\ell, \quad 0 \leq a_j < p \text{ for all } 0 \leq j \leq \ell, \quad a_{\ell+1} = 0, \]
and define
\[ c_{k, p} = \begin{cases} \prod_{i=0}^{\ell} \left( 1 - \frac{a_i}{p} \right) \cdot \left( 1 + \sum_{j=0}^{\ell} \frac{a_j (p - 1 - a_{j+1})}{(p - a_j)(p - a_{j+1})} \right) & \text{if } p \leq k, \\ 1 - \frac{k}{p^2} & \text{if } p > k. \end{cases} \]
Then \( c_k \) is equal to \( \prod_p c_{k,p} \), where the product is taken over all primes \( p \). We have \( c_3 = 0.251..., c_4 = 0.360..., c_5 = 0.191..., c_6 = 0.189..., c_7 = 0.062... \) and

\[
0 < c_k = \exp \left[ -(\alpha + o(1))\sqrt{k}/\ln(k) \right]
\]
as \( k \to \infty \), where

\[
\alpha = \sum_{j=1}^{\infty} \frac{1}{j(1+j)} \int_0^\infty \{x\}^j x^{-3/2} dx
\]

\[
= \sum_{j=1}^{\infty} \binom{2j}{j} \zeta(j+1/2) \frac{1}{2^{2j-1}} \left(1 - j \sum_{i>j} \frac{1}{i^2}\right)
\]

\[
= 1.825108....
\]

Integrals involving \( \{x\} = x - \lfloor x \rfloor \) as such also appear in [13, 14]. It follows that there are \( \sim \tau N \) square-free binomial coefficients \( \binom{n}{k} \) with \( 0 \leq k < n \leq N \), where

\[
\tau = 2 \sum_{k=0}^{\infty} c_k = 2(5.3275...) = 10.655....
\]

In words, each row of Pascal’s triangle possesses approximately \( 10^2 \) square-free entries (on average).

### 0.2. Relevant Sums

Let \( \varphi \) denote the Golden mean \( (1 + \sqrt{5})/2 \). We have [15, 16, 17, 18, 19]

\[
\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{1}{3} + \frac{2\sqrt{3}\pi}{27}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\binom{2n}{n}} = \frac{1}{5} + \frac{4\sqrt{5}\ln(\varphi)}{25},
\]

\[
\sum_{n=1}^{\infty} \frac{n}{\binom{2n}{n}} = \frac{2}{3} + \frac{2\sqrt{3}\pi}{27}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{\binom{2n}{n}} = \frac{6}{25} + \frac{4\sqrt{5}\ln(\varphi)}{125},
\]

\[
\sum_{n=1}^{\infty} \frac{n^2}{\binom{2n}{n}} = \frac{4}{3} + \frac{10\sqrt{3}\pi}{81}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{\binom{2n}{n}} = \frac{4}{25} - \frac{4\sqrt{5}\ln(\varphi)}{125},
\]

\[
\sum_{n=1}^{\infty} \frac{n^3}{\binom{2n}{n}} = \frac{10}{3} + \frac{74\sqrt{3}\pi}{243}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^3}{\binom{2n}{n}} = -\frac{2}{125} - \frac{28\sqrt{5}\ln(\varphi)}{625}
\]

and, more generally [20],

\[
\sum_{n=1}^{\infty} \frac{n^k}{\binom{2n}{n}} = p_k + q_k \sqrt{3}\pi, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^k}{\binom{2n}{n}} = r_k + s_k \sqrt{5}\ln(\varphi)
\]
for appropriate rationals $p_k, q_k, r_k, s_k$. Let $L_D$ denote the Dirichlet L-series with character $(D/\cdot)$ and $\text{Li}_k$ denote the $k^{\text{th}}$ polylogarithm function. The following are more difficult [15, 16, 17, 18, 19]:

$$
\sum_{n=1}^{\infty} \frac{1}{(2n)^n} = \frac{\sqrt{3\pi}}{9}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)^n} = \frac{2\sqrt{5}\ln(\varphi)}{5},
$$

$$
\sum_{n=1}^{\infty} \frac{1}{(2n)^{n^2}} = \frac{\pi^2}{18}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)^{n^2}} = 2\ln(\varphi)^2,
$$

$$
\sum_{n=1}^{\infty} \frac{1}{(2n)^{n^3}} = \frac{\sqrt{3\pi}}{2} L_{-3}(2) - \frac{4\zeta(3)}{3}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)^{n^3}} = \frac{2\zeta(3)}{5},
$$

$$
\sum_{n=1}^{\infty} \frac{1}{(2n)^{n^4}} = \frac{17\pi^4}{3240},
$$

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)^{n^4}} = 8\text{Li}_4\left(\frac{1}{\varphi}\right) + 8\ln(\varphi)\text{Li}_3\left(\frac{1}{\varphi}\right) - \frac{1}{2}\text{Li}_4\left(\frac{1}{\varphi^2}\right)
+ \frac{7\pi^2\ln(\varphi)^2}{15} - \frac{13\ln(\varphi)^4}{6} - \frac{4\zeta(3)\ln(\varphi)}{5} - \frac{7\pi^4}{90},
$$

$$
\sum_{n=1}^{\infty} \frac{1}{(2n)^{n^5}} = \frac{9\sqrt{3\pi}}{8} L_{-3}(4) + \frac{\pi^2\zeta(3)}{9} - \frac{19\zeta(5)}{3},
$$

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)^{n^5}} = \frac{5}{2}\text{Li}_5\left(\frac{1}{\varphi^2}\right) + 5\ln(\varphi)\text{Li}_4\left(\frac{1}{\varphi^2}\right)
+ 4\zeta(3)\ln(\varphi)^2 - \frac{4\pi^2\ln(\varphi)^3}{9} + \frac{4\ln(\varphi)^5}{3} - 2\zeta(5).
$$

We wonder whether the last two alternating series possess expressions involving L-series values rather than polylogarithmic values. Let $G = L_{-4}(2)$ denote Catalan’s constant. Other series include

$$
\sum_{n=0}^{\infty} \frac{1}{(2n)(2n+1)} = \frac{2\sqrt{3\pi}}{9}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)(2n+1)} = \frac{4\sqrt{5}\ln(\varphi)}{5},
$$

$$
\sum_{n=0}^{\infty} \frac{1}{(2n)(2n+1)^2} = \frac{8G}{3} - \frac{\pi\ln(2+\sqrt{3})}{3}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)(2n+1)^2} = \frac{\pi^2}{6} - 3\ln(\varphi)^2.
$$
and
\[
\sum_{n=0}^{\infty} \frac{2^n}{\binom{2n}{n}(2n+1)} = \frac{\pi}{2}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n}(2n+1)} = \frac{2}{\sqrt{3}} \ln \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right),
\]
\[
\sum_{n=0}^{\infty} \frac{2^n}{(2n+1)^2} = 2L_8(2) - \frac{\sqrt{2} \pi}{4} \ln(1 + \sqrt{2}),
\]
\[
\sum_{n=0}^{\infty} \frac{2^{2n}}{\binom{2n}{n}(2n+1)^2} = 2G, \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{\binom{2n}{n}(2n+1)^2} = \frac{\pi^2}{8} - \frac{1}{2} \ln(1 + \sqrt{2})^2,
\]

but similar expressions for
\[
\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^3}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)^3}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n}(2n+1)^2}
\]
remain open (as far as is known).

Batir [21, 22] recently proved that
\[
\sum_{n=1}^{\infty} \frac{2^{4n}}{(\binom{2n}{n})^2 n^3} = 8\pi G - 14\zeta(3), \quad \sum_{n=0}^{\infty} \frac{2^{4n+2}}{(\binom{2n}{n})^2 (2n+1)^3} = 14\zeta(3) - 4\pi G
\]
and also derived a complicated formula for \(\sum_{n=1}^{\infty} 1/\binom{3n}{n}\). We will barely mention cases for which \(\binom{2n}{n}\) is in the numerator, for example [15, 17, 23],
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{2n}(2n+1)} = \frac{\pi}{2},
\]
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{2n}(2n+1)^2} = \frac{\pi \ln(2)}{2}, \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^2} = \frac{\sqrt{2}}{8} (\pi \ln(2) + 4G),
\]
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^2} = \frac{3\sqrt{3}}{4} L_3(2),
\]
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^3} = \frac{7\pi^3}{216}, \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{4n}(2n+1)^4} = \frac{\pi \zeta(3)}{12} + \frac{27\sqrt{3}}{64} L_3(4),
\]
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{2}{2^{4n}(2n+1)} = \frac{4G}{\pi}.
\]

Similar expressions for
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^3}, \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2^{3n}(2n+1)^4}, \quad \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n}{2^{4n}(2n+1)}
\]
remain open (as far as is known). Techniques in [24] might be helpful in evaluating sums as these.
0.3. Addendum. Let $\chi_k$ denote the $k^{th}$ Legendre chi function:

$$\chi_k(x) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)^k} = \frac{1}{2} (\text{Li}_k(x) - \text{Li}_k(-x)) = i \text{Ti}_k(-ix).$$

Gosper [25] evaluated one of the preceding open sums:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n}(2n+1)^2} = \frac{\pi^2 - 2 \text{arccosh}(7) \text{arccoth}(\sqrt{2 + \sqrt{3}}) - 8 \chi_2 \left(\sqrt{2 - \sqrt{3}}\right)}{2\sqrt{2}}$$

which is apparently new. The other sums all possess multiple $\text{Li}_k$ terms, for example,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{2n}{n}(2n+1)^3} = \frac{19}{3} \ln(\varphi)^3 - 2 \ln(2) \ln(\varphi)^2 - 4 \ln(\varphi) \text{Li}_2\left(\frac{\varphi}{2}\right)$$

$$-4 \ln(2)^2 \ln(\varphi) + \frac{7\pi^2 \ln(\varphi)}{10} - \text{Li}_3\left(\frac{1}{2} - \frac{\varphi}{4}\right)$$

$$+4 \text{Li}_3\left(1 - \frac{\varphi}{2}\right) - 3 \text{Li}_3\left(\frac{1}{\varphi}\right) + 4 \text{Li}_3\left(\frac{1}{2\varphi}\right) + \frac{3\zeta(3)}{10}$$

and further simplification seems to be impossible.

Deninger’s conjecture [26]

$$\sum_{n=0}^{\infty} \frac{(2n)^2}{\binom{2n}{n} (2n+1)} = \frac{15}{\pi^2} L_{15A}(2)$$

was recently proved by Rogers & Zudilin [27], where $L_{15A}$ is the L-series for the elliptic curve isogeny class 15A. See [28] for a sampling of other conjectures.

Asymptotic results for middle Stirling numbers are more complicated than those for central binomial coefficients. Let $s_{2n,n}$ denote the number of permutations on $2n$ symbols possessing exactly $n$ cycles; let $S_{2n,n}$ denote the number of partitions of a $(2n)$-element set possessing exactly $n$ blocks. We have [29, 30, 31, 32]

$$\frac{n!}{(2n)!} s_{2n,n} \sim \kappa_1 \frac{\lambda_1^n}{\sqrt{n}}, \quad \frac{n!}{(2n)!} S_{2n,n} \sim \kappa_2 \frac{\lambda_2^n}{\sqrt{n}}$$

where

$$\lambda_1 = \frac{\xi}{[1 - \exp(-\xi)]^2} = 2.4554074822..., \quad \lambda_2 = \frac{\exp(\eta) - 1}{\eta^2} = 1.5441386523...$$
and $\xi$, $\eta$ are unique positive solutions of the equations

$$\frac{\exp(\xi) - 1}{\xi} = 2, \quad \frac{\eta}{1 - \exp(-\eta)} = 2.$$ 

The latter is a Lambert $W$ function value: $\eta = 2 + W(-2e^{-2}) = 1.5936242600...$ [33] while the former satisfies $2\xi/(2\xi + 1) = 0.7153318629...$. Generalizations of such results appear in [34, 35, 36, 37].

**References**


Central Binomial Coefficients


