Electrical Capacitance

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We mentioned logarithmic capacity or transfinite diameter in [1]. Given a compact set \( \mathcal{F} \) in \( \mathbb{R}^2 \), the measure

\[
\gamma_0(A) = \lim_{n \to \infty} \max_{\xi_1, \ldots, \xi_n \in A} \left( \prod_{j < k} |\xi_j - \xi_k| \right)^{2/n(n-1)},
\]

is invariant under rigid motions and continuous, but fails to be additive since \( \gamma_0(A) = \gamma_0(\partial A) \) [2, 3, 4]. The unit interval has logarithmic capacity 1/4; the unit disk, square and equilateral triangle have logarithmic capacities

\[
1, \quad \frac{1}{4\pi^{3/2}} \Gamma\left(\frac{1}{4}\right)^2 = 0.5901702995\ldots, \quad \frac{\sqrt{3}}{8\pi^2} \Gamma\left(\frac{1}{3}\right)^3 = 0.4217539346\ldots
\]

respectively. Discussion of the geometric mean (of all pairs of points) often seems to be restricted to planar sets; we now turn to the harmonic mean and subsequently to the arithmetic mean.

Given a compact set \( A \) in \( \mathbb{R}^3 \), define [5, 6]

\[
\gamma_{-1}(A) = \lim_{n \to \infty} \max_{\xi_1, \ldots, \xi_n \in A} \left( \frac{2}{n(n-1)} \sum_{j < k} \frac{1}{|\xi_j - \xi_k|} \right)^{-1}
\]

to be the Newtonian capacity or electrical capacitance or generalized transfinite diameter of order \(-1\). This is also the reciprocal of what is known as the optimal Riesz \(1\)-energy [7]. The unit interval and unit circle both have electrical capacitance 0; one way to see the latter is to notice the inequality [8]

\[
\sum_{j < k} \frac{1}{|\xi_j - \xi_k|} \geq \frac{n}{4} \sum_{\ell=1}^{n-1} \csc \left( \frac{\ell\pi}{n} \right)
\]

(for which equality holds when \( \xi_1, \ldots, \xi_n \) are \( n^{th} \) roots of unity). The unit disk has capacitance \( 2/\pi \) [9] If \( A \) is the closure of a bounded, open, connected set in \( \mathbb{R}^3 \), then

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$\gamma_{-1}(A) = \gamma_{-1}(\partial A)$ \cite{10}. The unit ball (and hence the unit sphere) has capacitance 1. Another way to see this is to invoke a formula for $s$-energy of the $d$-sphere \cite{11} with $s = 1, d = 2$.

Interesting constants arise here. For example, let $A$ be the solid formed by revolving a disk of radius 1 about a tangent line (a “torus without hole”). It follows that \cite{12}

$$\gamma_{-1}(A) = \frac{4}{\pi} \int_0^\infty \frac{1}{I_0(t)^2} dt = 4 \cdot (0.4353450662\ldots)$$

where $I_0(t)$ is the zeroth modified Bessel function. More generally, consider the surface formed by revolving an arc of a circle about its chord (a “spindle”). A definite integral involving Legendre functions of complex degree, parametrized by the included angle, is found \cite{13}. As another example, consider the (disconnected) set consisting of two congruent parallel line segments. Its capacitance is obtained via a transcendental equation that involves elliptic integrals \cite{14, 15, 16}. See \cite{10, 17, 18, 19, 20} for more examples.

Seemingly simple sets present formidably difficult challenges \cite{21}. The unit cube $C$ has attracted enormous attention \cite{22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41} and the best numerical estimate is \cite{2, 9, 42}

$$\gamma_{-1}(C) = 0.6606781540\ldots = \frac{1}{2}(1.3213563081\ldots).$$

A conjectured exact expression for $\gamma_{-1}(C)$ in \cite{43, 44} is evidently incorrect. For the unit square $S$ and the unit equilateral triangle $T$, we have less precision:

$$\gamma_{-1}(S) = 0.366789\ldots = \frac{1}{2}(0.733579\ldots) = \frac{2}{\pi}(0.576151\ldots),$$

$$\gamma_{-1}(T) = 0.2508\ldots = \frac{2}{\pi}(0.3940\ldots).$$

It would be good someday to see improvements of these estimates, as well as $0.3565\ldots = (1.7465\ldots)/\sqrt{24}$ for the unit regular tetrahedron. We wonder if formulation in \cite{45, 46} might assist in accomplishing this.

The preceding results are dimensionless, of course. Certain authors chose to express their estimates in the following manner:

$$\gamma_{-1}(C) \approx \frac{1}{4\pi \varepsilon_0} (73.51036),$$

$$\gamma_{-1}(S) \approx \frac{1}{4\pi \varepsilon_0} (40.811) \approx \frac{1}{\sqrt{2}} \frac{1}{4\pi \varepsilon_0} (57.715),$$

$$\gamma_{-1}(T) \approx \frac{1}{\pi \varepsilon_0} (31.518).$$
Electrical Capacitance

\[
\gamma_{1}(T) \approx \frac{1}{4\pi \varepsilon_{0}}(27.91) \approx \frac{1}{\sqrt{3}4\pi \varepsilon_{0}}(48.33)
\]

where \(4\pi \varepsilon_{0} \approx 111.265006\) picofarads/meter and \(\varepsilon_{0}\) is the permittivity constant of free space. Such decisions are a little unfortunate for us, since the value of \(\varepsilon_{0}\) is based on physical experimentation and thus the normalization has changed somewhat with the passage of time.

Moving back to geometry, define the \textbf{generalized transfinite diameter} of order 1 or \textbf{optimal Riesz \((-1)\)-energy}

\[
\gamma_{1}(A) = \lim_{n \to \infty} \max_{\varepsilon_{1}, \ldots, \varepsilon_{n} \in A} \left( \frac{2}{n(n - 1)} \sum_{j < k} |\xi_{j} - \xi_{k}| \right)
\]

where \(A\) is a compact set in \(\mathbb{R}^{3}\) \([5, 7]\). For lack of a convenient phrase (“sums of distances” is vague), we call \(\gamma_{1}(A)\) the \textbf{Euclidean capacity} of \(A\). The unit interval has Euclidean capacity \(1/2\). The unit disk (and hence the unit circle) has Euclidean capacity \(4/\pi\); notice the inequality \([8]\)

\[
\sum_{j < k} |\xi_{j} - \xi_{k}| \leq n \cot \left( \frac{\pi}{2n} \right)
\]

(for which equality holds when \(\xi_{1}, \ldots, \xi_{n}\) are \(n^{th}\) roots of unity). The unit ball (and hence the unit sphere) has Euclidean capacity \(4/3\); set \(s = -1, d = 2\) in the formula for \(s\)-energy of the \(d\)-sphere \([11]\). We wrote \(2/3\) in \([47]\) since sums were divided by \(n^{2}\) rather than \(2/(n(n - 1))\). Higher order asymptotics for the latter are conjectured in \([48]\).

It is remarkable that no numerical results for Euclidean capacity (akin to those for Newtonian capacity) of the unit cube, square, equilateral triangle or regular tetrahedron appear yet to exist. A starting point for a literature search might be \([49, 50, 51, 52, 53, 54]\).

\textbf{References}


