Variants of Brownian Motion

Steven Finch

June 12, 2004

We defined standard Brownian motion \( \{ W_t : t \geq 0 \} \) in [1]. An alternative characterization of the Wiener process involves the limit of random walks. Let \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \) be a sequence of independent identically distributed random variables, each possessing mean 0 and variance 1. Let

\[
S_0 = 0, \quad S_1 = \varepsilon_1, \quad S_2 = \varepsilon_1 + \varepsilon_2, \quad \ldots, \quad S_n = \sum_{k=1}^{n} \varepsilon_k.
\]

Then the random walk \( \{ S_k \}_{k=1}^{n} \) approaches Brownian motion on the unit interval in the sense that

\[
\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \to W_t, \quad 0 \leq t \leq 1
\]

as \( n \to \infty \), via the functional central limit theorem of Donsker [2, 3]. We are interested in the \( L^p \)-norm of Brownian motion

\[
\|W\|_p = \begin{cases} 
\left( \int_0^1 |W_t|^p \, dt \right)^{1/p} & \text{if } 0 < p < \infty, \\
\max_{0 \leq t \leq 1} |W_t| & \text{if } p = \infty
\end{cases}
\]

for a number of reasons [4, 5]. Note that \( \|W\|_p \) is itself a random variable. A distributional statement about \( \|W\|_p \), hence translates into an asymptotic distributional statement about the \( l_p \)-norm of the random walk:

\[
P\left( \|W\|_p \leq x \right) = \begin{cases} 
\lim_{n \to \infty} P \left( \sum_{k=1}^{n} \left| S_k \right|^p \leq n^{\frac{1}{2} + \frac{1}{p}} x \right) & \text{if } 0 < p < \infty, \\
\lim_{n \to \infty} P \left( \max \left\{ |S_1|, |S_2|, \ldots, |S_n| \right\} \leq n^{1/2} x \right) & \text{if } p = \infty.
\end{cases}
\]

In the following sections, we will discuss the cases \( p = \infty, 1 \) and 2 for several variants of Brownian motion. Corresponding problems for all other values of \( p > 0 \) remain unsolved.

\(^0\)Copyright \( \odot \) 2004 by Steven R. Finch. All rights reserved.
Some preliminary definitions include
\[
\delta_m = \frac{\Gamma(m + \frac{1}{2})}{\sqrt{\pi m!}} = \begin{cases} 
1 \cdot 3 \cdot 5 \cdots (2m - 1) & \text{if } m \geq 1, \\
\frac{2}{1} \cdot 4 \cdot 6 \cdots (2m) & \text{if } m = 0,
\end{cases}
\]

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \, dt = 1 - \text{erfc}(x),
\]

\[
\text{Ai}(x) = \begin{cases} 
\frac{1}{3}(-x)^{1/2} \left[ J_{- \frac{1}{3}} \left( \frac{2}{3}(-x)^{3/2} \right) + J_{\frac{1}{3}} \left( \frac{2}{3}(-x)^{3/2} \right) \right] & \text{if } x < 0, \\
\frac{1}{3}x^{1/2} \left[ I_{- \frac{1}{3}} \left( \frac{2}{3}x^{3/2} \right) - I_{\frac{1}{3}} \left( \frac{2}{3}x^{3/2} \right) \right] & \text{if } x \geq 0,
\end{cases}
\]

\[
K_{\frac{1}{4}}(x) = \frac{\pi}{\sqrt{2}} \left[ I_{- \frac{1}{4}}(x) - I_{\frac{1}{4}}(x) \right]
\]

where \(J_\nu(x)\) and \(I_\nu(x)\) are the well-known Bessel functions. Also, for \(x > 0\) and \(0 < a < b\), let

\[
U(a, b, x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tx} t^{a-1} (1 + t)^{b-a-1} \, dt.
\]

This is called the confluent hypergeometric function of the second kind (in contrast to [1]). Finally, define the Riemann xi function

\[
\xi(z) = \frac{1}{2}z(z - 1)\pi^{-z/2} \Gamma(\frac{1}{2}z) \zeta(z), \quad \text{Re}(z) > 1,
\]

which serves as a tantalizing link between Brownian motion and number theory [6]. This can be analytically continued to an entire function via functional equation \(\xi(z) = \xi(1 - z)\).

0.1. Bridge. A Brownian bridge \(\{X_t : 0 \leq t \leq 1\}\) has the same distribution as \(\{W_t : 0 \leq t \leq 1\}\), conditioned on \(W_1 = 0\). The maximum of \(|X_t|\) turns out to be closely allied with the Kolmogorov-Smirnov goodness-of-fit test [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]:

\[
P(\|X\|_\infty \leq x) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 x^2} = \frac{\sqrt{2\pi}}{x} \sum_{k=0}^{\infty} e^{-\pi^2 (2k+1)^2/(8x^2)}
\]

(and the right-hand equality follows via Poisson summation). This distribution has moments

\[
E(\|X\|_\infty) = \sqrt{\frac{\pi}{2} \ln(2)}, \quad E(\|X\|_\infty^2) = \frac{\pi^2}{12}
\]
and median 0.8275735551. It also satisfies [17, 18]

$$E(\|X\|_\infty^z) = 2^{1-\frac{1-z}{z}} \left(\frac{\pi}{2}\right)^{z/2} \xi(z)$$

for all complex $z$.

Takács [19, 20], building on Cifarelli [21], Shepp [22], Rice [23] and Johnson & Killeen [24], computed that

$$P(\|X\|_1 \leq x) = \frac{\sqrt{\pi}}{18^{1/6}x} \sum_{j=1}^{\infty} e^{-u_j} u_j^{-1/3} \text{Ai}\left((3u_j/2)^{2/3}\right)$$

for $x > 0$, where $u_j = (a_j^{3/2}/(27x^2))$ and $0 < a_1 < a_2 < \ldots$ are the zeroes [25] of $\text{Ai}'(-x)$. This distribution has moments

$$E(\|X\|_1) = \frac{1}{4} \sqrt{\frac{\pi}{2}}, \quad E(\|X\|_1^2) = \frac{7}{60}$$

and median 0.2817802658. . .

Anderson & Darling [26, 27, 28, 29], building on Smirnov [30], obtained that

$$P(\|X\|_2^2 \leq x) = \frac{1}{\pi \sqrt{x}} \sum_{j=0}^{\infty} \sqrt{4j + 1} e^{-(4j+1)^2/(16x)} \delta_j \frac{K_{1/4}\left((4j+1)^2/(16x)\right)}{},$$

which has moments

$$E(\|X\|_2^2) = \frac{1}{6}, \quad E(\|X\|_2^4) = \frac{1}{20}$$

and median 0.1188795509. . . The $L_2$-norm, squared, of $X_t$ turns out to be closely allied with the Cramér-von Mises goodness-of-fit test [31, 32, 33].

0.2. Excursion. A Brownian excursion $\{Y_t : 0 \leq t \leq 1\}$ has the same distribution as $\{W_t : 0 \leq t \leq 1\}$, conditioned on $W_t > 0$ for all $0 < t < 1$ and $W_1 = 0$.

Chung [34, 35], Kennedy [36] and Durrett & Iglehart [37, 38] showed that

$$P(\|Y\|_\infty \leq x) = \sum_{k=-\infty}^{\infty} (1 - 4k^2 x^2) e^{-2k^2 x^2} = \frac{\sqrt{2\pi x^{5/2}}}{x^3} \sum_{k=1}^{\infty} k^2 e^{-\pi^2 k^2/(2x^2)},$$

which has moments

$$E(\|Y\|_\infty) = \sqrt{\frac{\pi}{2}}, \quad E(\|Y\|_\infty^2) = \frac{\pi^2}{6}.$$
median 1.2234880197..., and also satisfies [17, 18]
\[ E(\|Y\|_\infty^z) = 2 \left( \frac{\pi}{2} \right)^{z/2} \xi(z) \]
for all complex \( z \).

Takács [19, 39], building on Getoor & Sharpe [40], Darling [41], Louchard [42, 43] and Groenboom [44], obtained that
\[ P(\|Y\|_1 \leq x) = \sqrt{\frac{2}{x}} \sum_{j=1}^{\infty} e^{-v_j} v_j^{2/3} U \left( \frac{1}{6}, \frac{4}{3}, v_j \right) \]
for \( x > 0 \), where \( v_j = 2a_j^3/(27x^2) \) and \( 0 < a_1 < a_2 < \ldots \) are the zeroes [25] of \( \text{Ai}(-x) \).

This distribution has moments
\[ E(\|Y\|_1) = \sqrt{\frac{\pi}{8}}, \quad E(\|Y\|_1^2) = \frac{5}{12} \]
and median 0.6070363869....

The \( L_2 \) case seems to be open for Brownian excursion.

0.3. Meander. A Brownian meander \( \{Z_t : 0 \leq t \leq 1\} \) has the same distribution as \( \{W_t : 0 \leq t \leq 1\} \), conditioned on \( W_t > 0 \) for all \( 0 < t < 1 \). Note that \( Z_1 \) need not be zero.

Durrett & Iglehart [37, 38] computed that
\[ P(\|Z\|_\infty \leq x) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-k^2x^2/2} = \frac{2^{3/2}\sqrt{\pi}}{x} \sum_{k=0}^{\infty} e^{-\pi^2(2k+1)^2/(2x^2)}. \]

Observe that the distribution of \( \|Z\|_\infty \) is the same as the distribution of \( 2\|X\|_\infty \).
Hence it has moments
\[ E(\|Z\|_\infty) = \sqrt{2\pi \ln(2)}, \quad E(\|Z\|_\infty^2) = \frac{\pi^2}{3} \]
and median 1.6551471103...

Takács [45] proved that
\[ P(\|Z\|_1 \leq x) = \frac{\sqrt{\pi}}{18^{1/6}x} \sum_{j=1}^{\infty} b_j e^{-\tilde{v}_j} \tilde{v}_j^{-1/3} \text{Ai} \left( (3\tilde{v}_j/2)^{2/3} \right) \]
for \( x > 0 \), where \( \tilde{v}_j = v_j/2 \) and \( v_j, a_j \) are as before, and where
\[ b_j = \frac{a_j}{3 \text{Ai}(-a_j)} \left( 1 + 3 \int_0^{a_j} \text{Ai}(-s) \, ds \right). \]
This distribution has moments
\[ E(\|Z\|_1) = \frac{3}{4} \sqrt{\frac{\pi}{2}} \quad E(\|Z\|^2) = \frac{59}{60} \]
and median 0.8900420723....

The \( L_2 \) case seems to be open for Brownian meander.

0.4. Motion. We return to Brownian motion. Erdős & Kac [46, 47, 48, 49] computed that
\[
P(\|W\|_\infty \leq x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} e^{-\pi^2(2k+1)^2/(8x^2)}
\]
\[= \frac{1}{2} \sum_{k=-\infty}^{\infty} (-1)^k \left[ \text{erf} \left( \frac{(2k+1)x}{\sqrt{2}} \right) - \text{erf} \left( \frac{(2k-1)x}{\sqrt{2}} \right) \right]
\]
\[= -1 + \sum_{k=-\infty}^{\infty} \left[ \text{erf} \left( \frac{(4k+1)x}{\sqrt{2}} \right) - \text{erf} \left( \frac{(4k-1)x}{\sqrt{2}} \right) \right],
\]
which has moments
\[ E(\|W\|_\infty) = \sqrt{\frac{\pi}{2}}, \quad E(\|W\|^2_\infty) = 2G \]
and median 1.1489732581.... This is a remarkable appearance of Catalan’s constant \( G! \)

Takács [50, 51], building on Kac [52] and Schwinger [53], found that
\[
P(\|W\|_1 \leq y) = \sqrt{\frac{3}{\pi}} \int_0^y \frac{1}{x} \sum_{j=1}^{\infty} c_j e^{-\tilde{u}_j \tilde{u}_j^{2/3}} U \left( \frac{1}{3}, \frac{4}{3}, \tilde{u}_j \right) dx
\]
for \( y > 0 \), where \( \tilde{u}_j = 2u_j \) and \( u_j, a'_j \) are as before, and where
\[
c_j = \frac{1}{3a'_j Ai(-a'_j)} \left( 1 + 3 \int_0^{a'_j} Ai(-s) ds \right).
\]
This distribution has moments
\[ E(\|W\|_1) = \frac{4}{3} \sqrt{\frac{1}{2\pi}}, \quad E(\|W\|^2_1) = \frac{3}{8} \]
Variants of Brownian Motion

and median 0.4510953819... We wonder whether the integral for \( P(\|W\|_1 \leq y) \) can be termwise integrated.

Cameron & Martin [46, 54, 55, 56, 57] proved that

\[
P(\|W\|_2^2 \leq x) = \sqrt{2} \sum_{j=0}^{\infty} (-1)^j \delta_j \text{erfc}\left(\frac{4j+1}{2\sqrt{2x}}\right)
\]

which has moments

\[
E(\|W\|_2^2) = \frac{1}{2}, \quad E(\|W\|_4^2) = \frac{7}{12},
\]

median 0.2904760595... and Laplace transform

\[
E(\exp(-\lambda \|W\|_2^2)) = \sqrt{\sec\left(\sqrt{-2\lambda}\right)}.
\]

We close with several unanswered questions. Define the positive part of \( W_t \) to be \( W_t^+ = \max\{W_t, 0\} \). Perman & Wellner [58, 59] studied the 1-norm of \( W_t^+ \) and found the following double Laplace transform:

\[
\int_0^\infty e^{-\mu \lambda} E\left\{ \exp\left(-\sqrt{2} \lambda^{3/2} \|W^+\|_1\right) \right\} d\lambda = \frac{\mu^{-1/2} \text{Ai}(\mu) + \frac{1}{3} - \int_0^\mu \text{Ai}(s) ds}{\sqrt{\mu} \text{Ai}(\mu) - \text{Ai}'(\mu)}
\]

as well as moments:

\[
E(\|W^+\|_1) = \frac{2}{3} \frac{1}{\sqrt{2\pi}}, \quad E\left(\|W^+\|_1^2\right) = \frac{17}{96}.
\]

Does an explicit formula for \( P(\|W^+\|_1 \leq x) \) exist? What can be said for other values of \( p > 0 \)?

Brownian motion with drift (of linear type \( W_t + \alpha t \) or parabolic type \( W_t - \beta t^2 \)) would be interesting to report on later [60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73]. Of all possible issues, we examine just two. When analyzing \( W_t + \alpha t \) for \( \alpha > 0 \), is the formula [63]

\[
\int_0^{\pi/2} \frac{\exp(-x \cot(x)) \sin(x)}{1 + \exp(-\pi \cot(x))} dx = \int_0^{\infty} \left[ \frac{1}{2} - \exp(-y \coth(y)) \sinh(y) \right] dy
\]

valid? Numerics suggest yes; a rigorous proof would be good to see someday. The expected maximum value of \( W_t - (1/2)t^2 \) is [72]

\[
\int_{-\infty}^{\infty} \frac{2^{-1/3}}{2\pi i} \frac{z}{\text{Ai}(iz)^2} dz = 0.9961930199...
\]

(among several integral expressions) and we wonder if similar formulas exist for higher moments.
References


