Pattern-Avoiding Permutations

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Let \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_m \) be a permutation on \( \{1, 2, \ldots, m\} \). Define a pattern \( \tilde{\sigma} \) to be the string \( \sigma_1 \varepsilon_1 \sigma_2 \varepsilon_2 \cdots \varepsilon_{m-1} \sigma_m \), where each \( \varepsilon_j \) is either the dash symbol \(-\) or the empty string. For example,

\[
1\-3\-2, \quad 1\-32, \quad 132
\]

are three distinct patterns. The first is known as a classical pattern (dashes in all \( m-1 \) slots); the third is also known as a consecutive pattern (no dashes in any slots). Some authors call \( \tilde{\sigma} \) a “generalized pattern” and use the word “pattern” exclusively for what we call “classical patterns”.

Let \( \tau = \tau_1 \tau_2 \cdots \tau_n \) be a permutation on \( \{1, 2, \ldots, n\} \), where \( n \geq m \). We say that \( \tau \) contains \( \tilde{\sigma} \) if there exist \( 1 \leq i_1 < i_2 < \ldots < i_m \leq n \) such that

- for each \( 1 \leq j \leq m-1 \), if \( \varepsilon_j \) is empty, then \( i_{j+1} = i_j + 1 \);
- for all \( 1 \leq k \leq m, 1 \leq l \leq m \), we have \( \tau_{i_k} < \tau_{i_l} \) if and only if \( \sigma_k < \sigma_l \).

The string \( \tau_{i_1} \tau_{i_2} \cdots \tau_{i_m} \) is called an occurrence of \( \tilde{\sigma} \) in \( \tau \). If \( \tau \) does not contain \( \tilde{\sigma} \), then we say \( \tau \) avoids \( \tilde{\sigma} \) or that \( \tau \) is \( \tilde{\sigma} \)-avoiding. For example,

\[
24531 \text{ contains } 1\-3\-2
\]

because 253 has the same relative order as 132, but

\[
42351 \text{ avoids } 1\-3\-2.
\]

As another example,

\[
6725341 \text{ contains } 4132
\]

because 7253 has the same relative order as 4132 and consists of four consecutive elements, but

\[
41352 \text{ avoids } 4132.
\]

As a final example,

\[
3542716 \text{ contains } 12\-4\-3
\]
because 3576 has the same relative order as 1243 and its first two elements are consecutive, but

$$3542716$$ avoids 12-43.

Define $$\alpha_n(\tilde{\sigma})$$ to be the number of $$n$$-symbol, $$\tilde{\sigma}$$-avoiding permutations. We naturally wish to understand the rate of growth of $$\alpha_n(\tilde{\sigma})$$ with increasing $$n$$.

### 0.1. Classical Patterns.

The Stanley-Wilf conjecture, proved by Marcus & Tardos [1], was rephrased by Arratia [2] as follows:

$$L(\tilde{\sigma}) = \lim_{n \to \infty} (\alpha_n(\sigma_1\sigma_2\cdots\sigma_m))^{1/n}$$

exists and is finite. We have [3, 4, 5, 6, 7]

$$L(\tilde{\sigma}) = 4 \quad \text{when } m = 3,$$

$$L(1-2\cdots-m) = (m - 1)^2 \quad \text{for all } m \geq 2,$$

$$L(1-3-4-2) = 8,$$

$$L(1-2-4-5-3) = \left(1 + \sqrt{8}\right)^2 = 9 + 4\sqrt{2}.$$  

A conjecture that $$L(\tilde{\sigma}) \leq (m - 1)^2$$ has been disproved [8]:

$$9.47 \leq L(1-3-2-4) \leq 288$$

and hence the maximum limiting value (as a function of $$m$$) remains open. Also, we wonder if $$L(\tilde{\sigma})$$ is always necessarily an algebraic number.

### 0.2. Consecutive Patterns.

Elizalde & Noy [9, 10] examined the cases $$m = 3$$ and $$m = 4$$. The quantities $$\alpha_n(123)$$ and $$\alpha_n(132)$$ satisfy

$$\alpha_n(123) \sim \gamma_1 \cdot \rho_1^n \cdot n!, \quad \alpha_n(132) \sim \gamma_2 \cdot \rho_2^n \cdot n!$$

where

$$\rho_1 = 3\sqrt{3}/(2\pi) = 0.8269933431..., \quad \gamma_1 = \exp\left(\pi/(3\sqrt{3})\right) = 1.8305194665..., \quad \rho_2 = 1/\xi = 0.7839769312..., \quad \gamma_2 = \exp(\xi^2/2) = 2.2558142944...$$

and $$\xi = 1.2755477364...$$ is the unique positive solution of

$$\int_{0}^{x} \exp(-t^2/2) \, dt = 1, \quad \text{that is, } \sqrt{\frac{\pi}{2}} \text{erf}\left(\frac{x}{\sqrt{2}}\right) = 1.$$
The quantities $\alpha_n(1342)$, $\alpha_n(1234)$ and $\alpha_n(1243)$ satisfy

$$
\alpha_n(1342) \sim \gamma_1 \cdot \rho_1^n \cdot n!, \quad 
\alpha_n(1234) \sim \gamma_2 \cdot \rho_2^n \cdot n!, \quad 
\alpha_n(1243) \sim \gamma_3 \cdot \rho_3^n \cdot n!
$$

where

$$
\rho_1 = 1/\xi = 0.9546118344..., \quad \gamma_1 = 1.8305194..., \\
\rho_2 = 1/\eta = 0.9630055289..., \quad \gamma_2 = 2.2558142..., \\
\rho_3 = 1/\zeta = 0.9528914198..., \quad \gamma_3 = 1.6043282...;
$$

$\xi$, $\eta$ and $\zeta$ are the smallest positive solutions of

$$
\int_0^x \exp(-t^3/6) \, dt = 1, \quad \cos(y) - \sin(y) + \exp(-y) = 0,
$$

$$
3^{1/2} \int_0^z \text{Ai}(-s) \, ds + \int_0^z \text{Bi}(-s) \, ds = \frac{3^{1/3} \Gamma(1/3)}{\pi},
$$

respectively, where $\text{Ai}(t)$ and $\text{Bi}(t)$ are the Airy functions [11].

0.3. Other Results. Elizalde [12, 13] proved that

$$
\lim_{n \to \infty} \left( \frac{\alpha_n(1-23-4)}{n!} \right)^{1/n} = 0
$$

and believed that the same applies to $\alpha_n(12-34)$, although a proof is not yet known. Ehrenborg, Kitaev & Perry [14] gave more detailed asymptotic expansions for $\alpha_n(123)$ and $\alpha_n(132)$; a similar “translation” of combinatorics into operator eigenvalue analysis was explored in [15]. The field is wide open for research.

Let us focus on classical patterns in the following. Define $\sigma \leq \tau$ if $\tau$ contains $\sigma$. A permutation class $C$ is a set of permutations such that, if $\tau \in C$ and $\sigma \leq \tau$, then $\sigma \in C$. Let $C_n$ denote the permutations in $C$ of length $n$. If $C = \{\text{all permutations}\}$, then $|C_n| = n!$; such behavior is regarded as degenerate and this case is excluded from now on. The Marcus-Tardos theorem implies that, for nondegenerate $C$,

$$
L(C) = \limsup_{n \to \infty} |C_n|^{1/n} < \infty.
$$

Consider the set $R$ of all growth rates $L(C)$ and the derived set $R'$ of all accumulation points of $R$. Vatter [16] proved that

$$
\inf \{ r \in R : r > 2 \} = 2.0659948920...
which is the unique positive zero of \(1 + 2x + x^2 + x^3 - x^4\), and
\[
\inf \{s : s \text{ is an accumulation point of } R'\} = 2.2055694304...
\]
which is the unique positive zero of \(1 + 2x^2 - x^3\). Albert & Linton [17] proved that \(R\) is uncountable and thus contains transcendental numbers. Vatter [18] subsequently proved that
\[
\inf \{t : R \text{ contains the interval } (t, \infty)\} \leq 2.481728574...
\]
which is the unique positive zero of \(-1 - 2x - 2x^2 - 2x^4 + x^5\) and conjectured that \(\leq\) can be replaced by \(=\). The question of whether \(\limsup\) in the definition of \(L(C)\) can be replaced by \(\lim\) is also unanswered.

### 0.4. Addendum.
With regard to classical patterns, the upper bound on \(L(1-3-2-4)\) has been improved to \(7 + 4\sqrt{3} < 13.93\) [19, 20]. With regard to consecutive patterns, a permutation \(\tau\) is **nonoverlapping** if it contains no permutation \(\sigma\) such that two copies of \(\sigma\) overlap in more than one entry [21]. For example, \(\tau = 214365\) contains both 2143 and 4365, both which follow the same pattern and overlap in two entries, hence \(\tau\) is overlapping. Bóna [22] examined the probability \(p_n\) that a randomly selected \(n\)-permutation is nonoverlapping, showed that \(\{p_n\}_{n=2}^{\infty}\) is strictly decreasing, and computed \(\lim_{n \to \infty} p_n = 0.36409...\).

From the fact that \(\alpha_n(123) > \alpha_n(132)\) and \(\alpha_n(1234) > \alpha_n(1342) > \alpha_n(1243)\) for suitably large \(n\), it is natural to speculate that \(\alpha_n(123...m)\) is asymptotically larger than \(\alpha_n(\sigma)\) for any other \(m\)-permutation \(\sigma\) (except \(m(m-1)...21\), which is equivalent by symmetry). This conjecture is now a theorem, due to Elizalde [23].

### References


