Aissen’s Convex Set Function

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Let $D$ be a bounded open convex set in the plane and let $C$ denote the boundary of $D$. For each $p \in D$ and $q \in C$, let $h_{pq}$ be the Euclidean distance from $p$ to the support line (tangent line) to $D$ at $q$. Let $ds_q$ denote the line element at $q$. It is known that [1, 2]

$$\text{arclength of } C = \int_C ds_q,$$

$$\text{area of } D = \frac{1}{2} \int_C h_{pq} ds_q \quad \text{(independent of } p),$$

$$r(D) = \text{inradius of } D = \max_{p \in D} \min_{q \in C} h_{pq}$$

where $r$ is the radius of the largest disk contained by $D$ [3]. The boundary of such a disk is called an incircle; its center is called an incenter. Aissen [1, 2] studied the function

$$B(D) = \min_{p \in D} \int_C h_{pq}^{-1} ds_q$$

and deduced that the optimizing point $p$ corresponds to an incenter of $D$ if $D$ is a triangle, parallelogram, regular polygon or ellipse. (We are careful to say “an incenter” rather than “the incenter”; a suitably elongated parallelogram has infinitely incircles, all of the same radius. In contrast, the incenter for an arbitrary triangle is unique.) This is a remarkable feature of $B$. It is natural to wonder whether the same is true for an arbitrary convex set.

The simplest counterexample is a trapezoid with vertices $(\pm 1, 1), (\pm 3, -1)$, for which the optimizing point $p$ has $x$-coordinate 0 (by symmetry) but $y$-coordinate $> 0$. More generally, examine the trapezoid with vertices $(\pm (\sqrt{2} - 1 + t), 1), (\pm (\sqrt{2} + 1 + t), -1)$ where $t \geq 0$ is fixed. The integral within $B$ becomes a sum of four ratios:

$$2 \left( \frac{\sqrt{2} - 1 + t}{1 - y} + \frac{\sqrt{2} + 1 + t}{1 + y} + \frac{2}{\sqrt{2} + t + x - y} + \frac{2}{\sqrt{2} + t - x - y} \right)$$

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each of the form sidelength/distance. As an instance, the rightmost side has equation
\[ v - \frac{1}{\sqrt{2}} = -u + \left( \frac{1}{\sqrt{2}} + t \right) \]
in the uv-plane, that is, \( u + v - \sqrt{2} - t = 0 \). The distance from the point \( (x, y) \) to the line is
\[ \frac{|x + y - \sqrt{2} - t|}{\sqrt{1^2 + 1^2}} = \frac{\sqrt{2} + t - x - y}{\sqrt{2}} \]
and the sidelength is \( \sqrt{2^2 + 2^2} = 2\sqrt{2} \). Forming a ratio gives the final term in the sum. Differentiating the sum with respect to \( x \), we see that \( x = 0 \) is necessary for minimization. The derivative with respect to \( y \) is more complicated. In the special case \( t = 0 \), each of the trapezoidal sides is tangent to the unit circle, thus \( y = 0 \).

Another counterexample – the half-disk \( 0 \leq v \leq \sqrt{1 - u^2} \) – comes from [1, 2]. Again \( x = 0 \) follows by symmetry. The integral within \( B \) here becomes
\[ \frac{2}{y} + \frac{2\arcsin(y) + \pi}{\sqrt{1 - y^2}} \]
and is minimized when \( y = 0.5432763603... > 1/2 \). The value of \( B \) itself is 8.7915361561... Such values play a role in estimating hard physical quantities like torsional rigidity \( P \) in terms of area \( A \) [4]. For the half-disk, \( P \) turns out to be known exactly and the lower bound [5]
\[ 0.2975567820... = \frac{\pi}{2} - \frac{4}{\pi} = P \geq A^2 B^{-1} = \frac{(\pi/2)^2}{8.7915361561...} \approx 0.280 \]
is excellent.

Returning to geometry, let \( d_{pq} \) simply be the Euclidean distance from \( p \) to \( q \). Clearly
\[ R(D) = \text{circumradius of } D = \min_{p \in D} \max_{q \in C} d_{pq} \]
where \( R \) is the radius of the smallest disk containing \( D \) [3]. The boundary of such a disk is called a circumcircle; its center is called a circumcenter. The circumcenter for an arbitrary convex set is unique. We wonder if a “dual” to Aissen’s function can be defined and what its interplay with the circumcenter for various \( D \) might be.
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References


