If \( n \) is a positive integer, let \( \sigma(n) \) denote the sum of all positive divisors of \( n \) that are strictly less than \( n \). Then \( n \) is said to be **perfect** or **1-sociable** if \( \sigma(n) = n \). We mentioned perfect numbers in [1], asking whether infinitely many exist, but did not report their reciprocal sum [2].

\[
\frac{1}{6} + \frac{1}{28} + \frac{1}{496} + \frac{1}{8128} + \frac{1}{33550336} + \frac{1}{8589869056} + \cdots = 0.2045201428\ldots
\]

This constant can, in fact, be rigorously calculated to 149 digits (and probably much higher accuracy if needed).

Define \( \sigma_k(n) \) to be the \( k \)th iterate of \( \sigma \) with starting value \( n \). The integer \( n \) is **amicable** or **2-sociable** if \( \sigma^2(n) = n \) but \( \sigma(n) \neq n \). Such phrasing is based on older terminology [3]: two distinct integers \( \mu, \nu \) are said to form an “amicable pair” if \( \sigma(\mu) = \nu \) and \( \sigma(\nu) = \mu \). The (infinite?) sequence of amicable numbers possesses zero asymptotic density [4] and, further, has reciprocal sum [5, 6]

\[
\frac{1}{240} + \frac{1}{284} + \frac{1}{1184} + \frac{1}{1210} + \frac{1}{2620} + \frac{1}{2924} + \frac{1}{5020} + \frac{1}{6232} + \frac{1}{6368} + \cdots = 0.01198415\ldots
\]

In contrast with the preceding, **none** of the digits are provably correct. The best rigorous upper bound for this constant is almost \( 10^9 \); deeper understanding of the behavior of amicable numbers will be required to improve upon this poor estimate.

Fix \( k \geq 3 \). An integer \( n \) is **k-sociable** if \( \sigma^k(n) = n \) but \( \sigma^\ell(n) \neq n \) for all \( 1 \leq \ell < k \). No examples of 3-sociable numbers are known [7, 8]; the first example for \( 4 \leq k < 28 \) is the 5-cycle \( \{12496, 14288, 15472, 14536, 14264\} \) and the next example is the 4-cycle \( \{1264460, 1547860, 1727636, 1305184\} \). Let \( S_k \) denote the sequence of all \( k \)-sociable numbers and \( S \) be the union of \( S_k \) over all \( k \). It is conjectured that the (infinite?) sequence \( S \) possesses zero asymptotic density and progress toward confirming this appears in [9]. No one is ready to compute the reciprocal sum of \( S \); a proof of convergence would seem to be faraway.

As an aside, we mention the sequence of **prime-indexed primes**, which is clearly infinite and has reciprocal sum [10]

\[
\frac{1}{3} + \frac{1}{5} + \frac{1}{11} + \frac{1}{17} + \frac{1}{31} + \frac{1}{41} + \frac{1}{59} + \frac{1}{67} + \frac{1}{83} + \frac{1}{109} + \frac{1}{127} + \frac{1}{157} + \frac{1}{179} + \frac{1}{191} + \cdots = 1.0432015\ldots
\]
Again, this is conjectural only. The best rigorous lower/upper bounds for this constant are 1.04299 and 1.04365 [2]. Such bounds are tighter than those (1.83408 and 2.34676) for the reciprocal sum of twin primes [11].

Our main interest is in the “aliquot sequence” \( \{s^k(n)\}_{k=1}^{\infty} \), where we assume WLOG that \( n \) is even. For example, if \( n = 12 \), the sequence \( \{16, 15, 9, 4, 3, 1\} \) is finite (terminates at 1). From earlier, we know that infinite cyclic behavior is possible. Does an infinite unbounded aliquot sequence exist? On the one hand, starting with \( n = 276 \), extensive computation has yielded 1769 terms with no end in sight [12, 13, 14, 15, 16]; probabilistic arguments in [17, 18], based on the arithmetic mean of \( s(2n)/(2n) \), also support a belief that most sequences grow without bound.

On the other hand, the geometric mean of \( s(2n)/(2n) \):

\[
\left( \prod_{n=1}^{N} \frac{s(2n)}{2n} \right)^{1/N} = \exp \left( \frac{1}{N} \sum_{n=1}^{N} \ln \left( \frac{s(2n)}{2n} \right) \right)
\]

(which seems a more appropriate tool than a simple average) predicts the opposite. Bosma & Kane [19] proved that

\[
\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \ln \left( \frac{s(2n)}{2n} \right) = 2\alpha(2) + \sum_{p \geq 3} \alpha(p) - \sum_{j \geq 1} \left( \frac{2\beta_j(2) - 1}{\prod_{p \geq 3} \beta_j(p)} \right) \frac{1}{j}
\]

\[
= -0.0332594808... < 0
\]

which implies that the geometric mean \( \mu = \exp(\lambda) = 0.9672875344... < 1 \). The indicated numerical estimates are due to Sebah [20]. Sums and products over \( p \) are restricted to primes; further,

\[
\alpha(p) = \left( 1 - \frac{1}{p} \right) \sum_{m=1}^{\infty} \frac{1}{p^m} \ln \left( \frac{p^{m+1} - 1}{p^m(p-1)} \right),
\]

\[
\beta_j(p) = \left( 1 - \frac{1}{p} \right) \sum_{m=0}^{\infty} \frac{1}{p^m} \left( \frac{p^{m+1} - 1}{p^m(p-1)} \right)^{-j}.
\]

The fact that \( \mu < 1 \) suggests that aliquot sequences tend to decrease ultimately, evidence in favor of the Catalan-Dickson conjecture. It would be good to compute other related constants, appearing in [21], to similar levels of precision.

From [1, 22], the probability that \( s(n) \) exceeds \( n \), for arbitrary \( n \), is

\[
\lim_{n \to \infty} \frac{1}{n} \left\{ i \leq n : \frac{s(i)}{i} > 1 \right\} = 0.2476...
\]

(what was called \( A(2) \)). The fact that these odds are significantly less than \( 1/2 \) again suggests that unboundedness is a rare event, if it occurs at all.
References


