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Lecture 14

14 Decoherence II

Decoherence can be defined as a decay of the off-diagonal elements of a density matrix due to entanglement of a quantum system with its environment. Here we should be careful because the initial density matrix may represent a statistical mixture of different states. Then the decoherence will be averaged over a distribution of initial states of a qubit or a bath. Frequently, this averaging is also included into the definition of decoherence. In this case the quantum evolution is mixed with the statistical averaging.

The decay rate for the off-diagonal elements of the density matrix in different bases can be different. One have to define a specific basis for the decoherence. This basis is defined by so called pointer states. The pointer states are the states that are minimally entangled with the environment during the evolution of the quantum system [1]. In general, these are dynamical states that evolve in time. However for QC, we are interested in systems that are weakly coupled to the environment. In this case we can use the energy basis. In the energy basis there exists both phase relaxation (T_2 processes) and energy relaxation (T_1 processes). Energy relaxation also results in the relaxation of the phase. Therefore, phase relaxation = T_1 processes + pure T_2 processes.

There is a requirements for the environment. It is more-or-less obvious that if we have just one qubit (system) coupled to one qubit (environment) the dynamics is reversible. It is difficult to call it decoherence. Usually, for the environment we make an assumption that the number of degrees of freedom is large and the splitting between the energy levels is small. Then the probability that the system will return to the initial state within some finite time period is very small (Poincaré cycle is very long).

We can define three different models for a qubit decoherence:

1. Decoherence due to coupling to the environment. The qubit interacts with some external degrees of freedom during its free evolution. This is standard system-bath dynamics. The Hamiltonian for this case is

$$H = H_s + H_b + V_{sb}, \tag{1}$$

where H_s is the qubit Hamiltonian, H_b is the Hamiltonian of the bath and V_{sb} is the qubit-bath interaction term. Within this model one can use different types of baths (bosons, spins) or qubit-bath interactions.

2. Decoherence due to noise in the control. In this case the coupling to the environment is through the field that controls qubit dynamics during a quantum operation. The Hamiltonian should be

$$H = H_s + V_c(s, X, t), \tag{2}$$

where $V_c(s, X, t)$ is an interaction of the qubit with the control field, which depends on time t, qubit state s and the noise variable X. Here I use a semiclassical description for the control field. However, one can use a completely quantum mechanical model as well[2].

3. Self-decoherence in a system of qubits. This is due to uncontrollable interaction between qubits in a multi-qubit system. It does not require coupling to an external environment. A simple example is a system of spin qubits coupled by the dipolar interaction. For each qubit the rest of the qubits are the environment. The Hamiltonian is

$$H = \sum_k H_k + \sum_{kl} V_{kl}, \tag{3}$$

where H_k is a single qubit Hamiltonian and V_{kl} an undesired interaction between qubits.

14.1 Initial state

Usually, in studies of decoherence the initial state is factorizable

$$\rho(0) = \rho_{\text{qubit}}(0) \otimes \rho_{\text{bath}}(0). \tag{4}$$

This assumption has some physical reasons. In many experiments the initial state of a qubit is prepared by external control. If the control pulse is strong enough (much stronger than the interaction of a qubit with the environment) and short, it destroys the coupling of the qubit with the bath. This is just an argument but not a rigorous proof. To have a more realistic initial state one should start with the initial state before the initialization pulse.

14.2 Dephasing and decoherence

The term *dephasing* can have different meanings. One is a loss of a phase of a single quantum system. Then, it is the same as decoherence. Frequently, it is used to characterize decay of coherent oscillations in an ensemble of quantum systems. For example, free induction decay in EPR (ESR) or NMR. In this case each single spin has its own precession frequency. The phase coherence between different spins is lost, but each spin precesses coherently. One can also call it ensemble decoherence.

14.3 Example II

The qubit is a single spin $S = 1/2$ interacting with a large system of spins $I_k = 1/2$ in an external magnetic field B_z. The coupling term has a particular form $V = \sum_k A_k S^z I_k^x$. This is a model Hamiltonian, though a similar Hamiltonian describes the effect called Electron Spin Echo Envelop Modulation (ESEEM) in pulsed paramagnetic spectroscopy [3]. Moreover, I guess that this term can determine spin decoherence in large QDs at low magnetic fields.

The Hamiltonian is

$$H = \omega_0 S^z + \sum_k A_k S^z I_k^x + \sum_k \omega_k I_k^z. \tag{5}$$

Initially, the spin S is prepared in the superposition state $|s\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, which corresponds to the polarization vector $\mathbf{P} = (1, 0, 0)$ directed along the x-axis. The initial state of the spin system I_k is $\rho_I(0) = |j_1, j_2, ..., j_n\rangle\langle j_1, j_2, ..., j_n|$, where each j_k is $\uparrow$ or $\downarrow$.

The reduced qubit density matrix at time t is

$$\rho_S(t) = \mathrm{Tr}_I\{e^{-iHt}\rho_0 e^{iHt}\}. \tag{6}$$

One can put the Hamiltonian (5) into Eq. (6). The diagonal elements of the qubit density matrix do not evolve in time because the bath-coupling does not flip the spin S. For the off-diagonal elements we can write

$$\rho_{\uparrow\downarrow}(t) = \rho_{\uparrow\downarrow}(0)e^{-i\omega_0 t}\mathrm{Tr}_I\{e^{-i\sum_k(A_k/2 I_k^x + \omega_k I_k^z)}\rho_I(0)e^{-i\sum_k(A_k/2 I_k^x - \omega_k I_k^z)}\}. \tag{7}$$

You can see that in Eq. (7) the part under the trace, which describes qubit decoherence, can be written as a product of single spin operators. Because the initial state of the system I is also factorizable we can easily simplify Eq. (7) to the form

$$\rho_{\uparrow\downarrow}(t) = \rho_{\uparrow\downarrow}(0)e^{-i\omega_0 t}\prod_k\{1 - \frac{2A_k^2}{\Omega_k^2}\sin^2(\frac{\Omega_k t}{2})\}, \tag{8}$$

or

$$\rho_{\uparrow\downarrow}(t) = \rho_{\uparrow\downarrow}(0)e^{-i\omega_0 t}e^{\sum_k \ln\{1 - \frac{2A_k^2}{\Omega_k^2}\sin^2(\frac{\Omega_k t}{2})\}}. \tag{9}$$

Up to this point the derivation was exact. Now we make an approximation. On short timescales the latter equation can be written as

$$\rho_{\uparrow\downarrow}(t) = \rho_{\uparrow\downarrow}(0)e^{-i\omega_0 t}e^{-\sum_k \frac{A_k^2 t^2}{2}\mathrm{sinc}^2(\frac{\Omega_k t}{2})}, \tag{10}$$

which shows that at short timescales ($t < \Omega_k$) the qubit coherence decays as e^{-t^2}. This result is different from the example in the previous lecture. The intermediate timescale dynamics is not universal.

Another interesting example of non-exponential decoherence can be found in Ref. [4].

References

[1] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).

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[4] N. G. van Kampen, J. Stat. Phys. **78**, 299 (1995).