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Lecture 9

9 Angular Momentum II

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9.1 Addition of Angular Momenta

Let us consider a system with a spin s and an orbital angular momentum l. Its arbitrary quantum state may be written in the basis

$$|l, m_l; s, m_s\rangle. \tag{1}$$

The system also can be characterized by a total angular momentum j, where the operator of the total angular momentum is defined as

$$\bar{J} = \bar{L} + \bar{S}. \tag{2}$$

For the operator of the total angular momentum we introduce a basis, where J_z and J^2 are diagonal. However, this basis differs from the basis given by Eq. (1). Our goal is to write a transformation between the two bases.

The states of the first basis are characterized by four numbers l, m_l, s, m_s. For the second basis we have only two numbers j and m_j. We need to find two more. Let us write commutation relations for $\bar{L}$, $\bar{S}$, $\bar{J}$, L^2, S^2 and J^2. From the previous two lectures we have that the set of commutators for components of an angular momentum can be written as a cross-product

$$\bar{L} \times \bar{L} = i\bar{L}, \tag{3}$$
$$\bar{S} \times \bar{S} = i\bar{S}, \tag{4}$$
$$\bar{J} \times \bar{J} = i\bar{J} \tag{5}$$

Operators $\bar{L}$ and $\bar{S}$ commute with each other because they are defined in orthogonal parts of the Hilbert space

$$[L_i, S_k] = 0. \tag{6}$$

For the squares of the angular momentum operators we have

$$[\bar{L}, L^2] = 0, \tag{7}$$
$$[\bar{S}, S^2] = 0, \tag{8}$$
$$[\bar{S}, L^2] = 0, \tag{9}$$
$$[\bar{L}, S^2] = 0. \tag{10}$$

We can show that J^2 does not commute with components of $\bar{L}$ and $\bar{S}$. It follows from the definition (2). However, J^2 commutes with S^2 and L^2. Thus, we can define a new basis as

$$|l, s; j, m_j\rangle. \tag{11}$$

Let us write a unitary transformation between the two bases as

$$|l, s; j, m_j\rangle = \sum_{m_s, m_l} C^{j,m_j}_{l,m_l;s,m_s} |l, m_l; s, m_s\rangle, \tag{12}$$

where the transformation matrix is written in terms of the *Clebsch-Gordan coefficients*

$$C^{j,m_j}_{l,m_l;s,m_s} = \langle l, m_l; s, m_s | j, m_j\rangle. \tag{13}$$

To simplify the notation we did not include s and l at the right hand side of Eq. (13). They are understood. There are a few different ways to denote the Clebsch-Gordan coefficients. However, all the notations include 6 parameters l, s, j, m_l, m_s, m_j.

The transformation matrix for the addition of angular momenta is unitary. Moreover, the Clebsch-Gordan coefficients are defined to be *real*. Therefore

$$\langle j, m_j | l, m_l; s, m_s\rangle = \langle l, m_l; s, m_s | j, m_j\rangle. \tag{14}$$

A general wavefunction $|\Psi\rangle$ written in the basis of decoupled angular momenta as

$$\langle l, m_l; s, m_s | \Psi\rangle, \tag{15}$$

can be transformed into the basis of a total angular momentum j as

$$\langle l, s; j, m_j | \Psi\rangle = \sum_{m_l, m_s} \langle j, m_j | l, m_l; s, m_s\rangle \langle l, m_l; s, m_s | \Psi\rangle. \tag{16}$$

Two arbitrary angular momenta j_1 and j_2 can be added into a total angular momentum j using the Clebsch-Gordan coefficients $\langle j_1, m_1; j_2, m_2 | j, m_j\rangle$. These coefficients are not equal to zero only if

$$m_1 + m_2 = m_j \tag{17}$$

and

$$|j_1 - j_2| \leq j \leq j_1 + j_2. \tag{18}$$

To show this let us continue with the previous example of adding l and s.

An arbitrary state of the uncoupled basis $|l, m_l; s, m_s\rangle$ is an eigenstate of J_z. It follows from the definition

$$J_z = L_z + S_z. \tag{19}$$

The corresponding eigenvalue is $m_j = m_l + m_s$, which is reflected in the property of the Clebsch-Gordan coefficients. However, this state is not an eigenstate of J^2.

Let us find possible values of j. For $m_l = l$ and $m_s = s$ we have only one state in the uncoupled basis $|l, l; s, s\rangle$. It should be transformed to a single state in the coupled basis $|l, s; j, m_j\rangle$, where $m_j = l + s$. From the properties of an angular momentum operator we have that $j_{max} = l + s$. The next lower value of m_j is equal to $l + s - 1$. Two state of the uncoupled basis may contribute to it. These are $|l, l-1; s, s\rangle$ and $|l, l; s, s-1\rangle$. The corresponding two state of the coupled basis are $|l, s; j', l+s-1\rangle$ and $|l, s; j'', l+s-1\rangle$. One of these states corresponds to the total angular momentum $j_{\max} = l + s$. The second one is $j_{\max-1} = l + s - 1$. We can continue this procedure until all the state are taken. The total number of states in the uncoupled basis is $(2l+1)(2s+1)$. Thus, we can write

$$\sum_{j_{\min}}^{j_{\max}} (2j+1) = (2l+1)(2s+1), \tag{20}$$

where $j_{\max} = l + s$ and we are looking for the value of $j_{\min}$. Using that the left hand side of Eq. (20) is a sum of an arithmetic progression we can get that

$$j_{\min} = |l - s|. \tag{21}$$

Exercise: Prove the above statement using Eq. (20).

There are recursive methods to calculate the Clebsch-Gordan coefficients. For an example see Ref. [1], Section 3.7. However, all of them are tabulated and can be easily found on Internet. Sometimes for addition of angular momenta instead of the Clebsch-Gordan coefficients Wigner's 3-j symbols

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m_j \end{pmatrix}. \tag{22}$$

This is just another form to represent the transformation matrix. The relation between the Clebsch-Gordan coefficients and 3-j symbols is

$$\langle j_1, m_1; j_2, m_2 | j, m_j \rangle = (-1)^{j_1 - j_2 + m_j} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m_j \end{pmatrix}. \tag{23}$$

9.2 Example: two particles with $j = 1/2$

We have two particles with $j_1 = 1/2$ and $j_2 = 1/2$. Possible eigenvalues of the operators J_z^1 and J_z^2 are $m_1 = \pm 1/2$ and $m_2 = \pm 1/2$. *For the particular case $j = 1/2$ the two states of its projection $m_j = \pm 1/2$ also can be denoted as $\uparrow$ and $\downarrow$.* In the coupled basis the possible values of the total angular momentum are $j = 1, 0$ and its projections are $m_j = \pm 1, 0$.

The four basis states of the uncoupled basis are: $|m_1 = +1/2, m_2 = +1/2\rangle$, $|m_1 = +1/2, m_2 = -1/2\rangle$, $|m_1 = -1/2, m_2 = +1/2\rangle$ and $|m_1 = -1/2, m_2 = -1/2\rangle$, where we use only projections $m_{1,2}$ to denote the states and $j_{1,2}$ are understood. The four states of the coupled basis are: $|j = 1, m_j = 1\rangle$, $|j = 1, m_j = 0\rangle$, $|j = 1, m_j = -1\rangle$ and $|j = 0, m_j = 0\rangle$. Similar to the previous section we start with the maximal value of $m_j = 1$. The corresponding state of the coupled basis is $|j = 1, m_j = 1\rangle$. Only one state of the uncoupled basis $|m_1 = +1/2, m_2 = +1/2\rangle$ can contribute to it. Therefore,

$$|j = 1, m_j = 1\rangle = |m_1 = +1/2, m_2 = +1/2\rangle. \tag{24}$$

Similarly, we get that

$$|j = 1, m_j = -1\rangle = |m_1 = -1/2, m_2 = -1/2\rangle. \tag{25}$$

For the projection of the total angular momentum $m_j = 0$ we have two states. To write them in terms of uncoupled states we can use the following procedure. Let us apply J_- to the state $|j = 1, m_j = 1\rangle$. In the uncoupled basis $J_- = J_-^1 + J_-^2$. We know from the previous lecture that non-zero matrix elements are

$$\langle j, m-1|J_-|j, m\rangle = \sqrt{j(j+1) - (m-1)m}. \tag{26}$$

Therefore, we can write

$$\begin{gathered} J_-|j = 1, m_j = 1\rangle = \sqrt{2}|j = 1, m_j = 0\rangle = \\ |m_1 = +1/2, m_2 = -1/2\rangle + |m_1 = -1/2, m_2 = +1/2\rangle. \end{gathered} \tag{27}$$

It follows that

$$|j = 1, m_j = 0\rangle = \frac{1}{\sqrt{2}}(|m_1 = +1/2, m_2 = -1/2\rangle + |m_1 = -1/2, m_2 = +1/2\rangle). \tag{28}$$

The other state $|j = 0, m_j = 0\rangle$ we can get from the orthogonality condition. It is

$$|j = 0, m_j = 0\rangle = \frac{1}{\sqrt{2}}(|m_1 = +1/2, m_2 = -1/2\rangle - |m_1 = -1/2, m_2 = +1/2\rangle). \tag{29}$$

As the result we have three states corresponding to $j = 1$, *triplet* states. And one state with $j = 0$, a *singlet* state. They have different symmetry properties. For example, under rotation of the coordinate system the triplet states transform between each other, while the singlet state remains the same.

Let us consider nuclear spins of protons in a hydrogen molecule H_2. Each proton has a spin $s_p = 1/2$. The total spin of the two nuclei can be $j = 0$ – para-hydrogen and $j = 1$ – ortho-hydrogen. The latter state is three-fold degenerate. The difference between these two types of hydrogen shows up, for example, in rotational spectroscopy. Transitions between rotational sublevels of ortho-hydrogen are 3 times more intense than of para-hydrogen.

References

[1] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, New York, 1994), Chapter 3.7.

[2] R. Shankar, *Principles of Quantum Mechanics* (2nd ed., Plenum Press, New York, 1994), Chapters 15.1-15.2.