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Lectures 7-8

7 Angular Momentum I

Content:

- Definition of Orbital Angular Momentum.
- Commutation Rules.
- Matrix Representation of Angular Momentum.

7.1 Definition

Let us use the same definition as in classical mechanics

$$\bar{L} = \bar{r} \times \bar{p}, \tag{1}$$

where $\bar{r}$ and $\bar{p}$ are the position and the momentum operators respectively. The vector product (1) can be written as

$$\bar{L} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}. \tag{2}$$

In the spatial basis we need to replace the momentum operator as $\bar{p} = -i\hbar\bar{\nabla}$. The Cartesian components of the orbital angular momentum are

$$\begin{aligned} L_x &= -i\hbar\left(y \cdot \frac{\partial}{\partial z} - z \cdot \frac{\partial}{\partial y}\right), \\ L_y &= -i\hbar\left(z \cdot \frac{\partial}{\partial x} - x \cdot \frac{\partial}{\partial z}\right), \\ L_z &= -i\hbar\left(x \cdot \frac{\partial}{\partial y} - y \cdot \frac{\partial}{\partial x}\right). \end{aligned} \tag{3}$$

Exercise: How it can be written in the momentum basis?
We should introduce one more operator

$$L^2 = L_x^2 + L_y^2 + L_z^2. \tag{4}$$

The angular momentum defined by Eq. (1) has units of $\hbar$. Sometimes it is easier to operate with a unitless angular momentum assuming $\hbar = 1$. This is only a mathematical trick. A physical quantity should have units.

7.2 Commutation Rules

Commutators of an orbital angular momentum operator $\bar{L}$ with a position operator $\bar{r}$, a momentum operator $\bar{p}$ and itself may be written in a compact form

$$[L_j, A_k] = i\hbar\epsilon_{jkl}A_l, \tag{5}$$

where j, k, l are for the Cartesian components of the vectors, and the antisymmetric tensor ϵ_{jkl} is defined as

$$\epsilon_{jkl} = \begin{cases} 1 & j,k,l \quad \text{cyclic} \\ -1 & j,k,l \quad \text{anticyclic} \\ 0 & \quad\quad\;\; \text{otherwise} \end{cases}$$

Example 1:

$$[L_z, x] = [xp_y - yp_x, x] = xp_yx - yp_xx - xxp_y + xyp_x = x[p_y, x] + y[x, p_x] = i\hbar y. \tag{6}$$

Example 2:

$$[L_z, p_y] = [xp_y - yp_x, p_y] = \\ xp_yp_y - yp_xp_y - p_yxp_y + p_yyp_x = [p_y, y]p_x + [x, p_y]p_y = -i\hbar p_x. \tag{7}$$

One more useful commutator:

$$[L^2, L_i] = 0. \tag{8}$$

The vector of the angular momentum operator doesn't commute with itself, but it commutes with its absolute value. We can define a basis $|\psi\rangle$, where two of the operators L^2 and L_i, $i = x, y, z$ are simultaneously diagonal. Usually, the basis states are chosen to diagonalize L_z. In this case L_x and L_y are non-diagonal.

7.3 Matrix representation of the angular momentum operator

1. Let us introduce operator a mode general operator of angular momentum $\bar{J}$. It satisfies all the commutation rules given above, but not Eq. 1. For the sake of simplicity we assume that $\hbar = 1$.

2. We also define the rising J_+ and the lowering J_- operators as

$$J_\pm = J_x \pm J_y. \tag{9}$$

These operators are non-Hermitian

$$J_-^\dagger = J_+. \tag{10}$$

Components of the angular momentum operator in terms of $J_\pm$ are

$$J_x = \frac{1}{2}(J_+ + J_-), \tag{11}$$
$$J_y = \frac{-i}{2}(J_+ - J_-).$$

And the square of angular momentum is

$$J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+). \tag{12}$$

3. Commutation rules and useful relations:

$$[J^2, J_\pm] = 0, \tag{13}$$

$$[J_z, J_\pm] = \pm J_\pm, \tag{14}$$

$$[J_+, J_-] = 2J_z. \tag{15}$$

Combining Eq. (14) with Eq. (12) we get

$$J_+J_- = J^2 - J_z^2 + J_z, \tag{16}$$
$$J_-J_+ = J^2 - J_z^2 - J_z. \tag{17}$$

Operators $J_\pm$ are non-diagonal in the chosen basis. However, $J_\pm J_\mp$ are diagonal.

4. Let us define the basis states $|\mu, \nu\rangle$ that satisfy to two eigenvalue problems

$$J^2|\mu, \nu\rangle = \nu|\mu, \nu\rangle, \tag{18}$$
$$J_z|\mu, \nu\rangle = \mu|\mu, \nu\rangle. \tag{19}$$

We will construct the matrices of the operators J^2, J_z, and $J_\pm$ in this basis. Because $J_\pm$ commutes with J^2 we get

$$J^2 J_\pm|\mu, \nu\rangle = J_\pm J^2|\mu, \nu\rangle = \nu J_\pm|\mu, \nu\rangle \tag{20}$$

The state $J_\pm|\mu, \nu\rangle$ is an eigenstate of the operator J^2 with the same eigenvalue. Similarly, using the commutation relations for J_z and $J_\pm$ we get that the state $J_\pm|\mu, \nu\rangle$ is an eigenstate of the operator J_z with the eigenvalue $\mu \pm 1$ respectively. Thus, we can write that

$$J_\pm|\mu, \nu\rangle = \gamma_{\mu,\nu}|\mu \pm 1, \nu\rangle, \tag{21}$$

where $\gamma_{\mu,\nu}$ is a complex coefficient, which may depend on the eigenvalues of J^2 and J_z.

5. Let us impose the condition that the norm of the eigenvector is not negative

$$\langle \mu, \nu | \mu, \nu \rangle \geq 0. \tag{22}$$

It has been shown above that if we apply the operator J_+ to the state $|\mu, \nu\rangle$ the resulting state should be proportional to $|\mu + 1, \nu\rangle$. Its norm also should be not negative. This condition can be written as

$$\langle \mu, \nu | J_- J_+ | \mu, \nu \rangle \geq 0. \tag{23}$$

We can substitute the $J_- J_+$ operator with Eq. 17 and obtain the relation for eigenvalues of J^2 and J_z

$$\nu - \mu(\mu - 1) \geq 0, \tag{24}$$

which set up the upper boundary on the value of μ

$$\mu_{\max} \leq -\frac{1}{2} + \sqrt{\frac{1}{4} + \nu}. \tag{25}$$

Let assume that $\mu_{\max} = j$, then $\nu = j(j+1)$. Actually, from the the norm inequality we can get only a weaker condition $\nu \geq j(j+1)$. To get the equality we need to use the definition of J^2, see Eq. (4) and matrices of all the components of $\bar{J}$. Another way to get ν is using the relation

$$\langle \mu_{\max}, \nu | J_- J_+ | \mu_{\max}, \nu \rangle = 0. \tag{26}$$

This is true because $J_+|\mu_{\max}, \nu\rangle = 0$. Then, we can substitute $J_- J_+$ with Eq. (17) and get $\nu = j(j+1)$. Applying J_+ to the state with a maximal value of μ we should get 0. But we can apply J_-. If we apply it k-times the final state will be proportional to $|j - k, \nu\rangle$. This state also should satisfy the condition of Eq. 22. In the way similar to the discussed above we can get that

$$k_{\max} = 2j, \tag{27}$$

where $k_{\max}$ is a maximal number of J_- operators that we can apply to the state $|j - k, \nu\rangle$.
Excercise: Derive the lower boundary for μ.

In Eq. (27) $k_m ax$ is an integer number. Therefore, j can be

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, ... \tag{28}$$

and for each j the value of m can be

$$m = -j, -j+1, ..., j-1, j \tag{29}$$

Now, for each j we can construct matrices of J^2 and J_z.

$$\langle m', j'|J_z|m, j\rangle = m\delta_{jj'}\delta_{mm'} \tag{30}$$

and

$$\langle m', j'|J^2|m, j\rangle = j(j+1)\delta_{jj'}\delta_{mm'} \tag{31}$$

For example for $j = 3/2$

$$J_z = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix} \tag{32}$$

and

$$J^2 = \frac{3}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{33}$$

For the *orbital* angular momentum operator the the values of m can be integer only. This can be shown using a spatial representation of $\bar{L}$. Half-integer values may appear when we study spin.

For the operators $J_\pm$ we know that

$$J_+|m, j\rangle = \lambda_{m,j}|m+1, j\rangle \tag{34}$$

or

$$\langle m+1, j|J_+|m, j\rangle = \lambda_{m,j} \tag{35}$$

In a most general case the coefficient $\lambda_{m,j}$ can be a complex number. If we take a Hermitian conjugate of the last equation we get

$$J_-|m+1, j\rangle = \lambda^*_{m,j}|m, j\rangle \tag{36}$$

Now, we can calculate matrix elements in the following way

$$J_-J_+|m, j\rangle = |\lambda_{m,j}|^2|m, j\rangle = (J^2 - J_z^2 - J_z)|m, j\rangle, \tag{37}$$

or

$$\lambda_{m,j} = \sqrt{j(j+1) - m(m+1)} = \sqrt{(j-m)(j+m+1)} \tag{38}$$

The matrix elements of the operators $J_\pm$ are defined by the following relations

$$\langle m', j'|J_+|m, j\rangle = \sqrt{(j-m)(j+m+1)}\delta_{j'j}\delta_{m'm+1} \tag{39}$$

and

$$\langle m', j'|J_-|m, j\rangle = \sqrt{(j-m+1)(j+m)}\delta_{j'j}\delta_{m'm-1}. \tag{40}$$

For example, for $j = 3/2$

$$J_+ = J_-^\dagger = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{41}$$

At this point we can come back and prove that the eigenvalue of the operator J^2 is $j(j+1)$. We just need to write it in terms of the components of the angular momentum.

Exercise 1: Show that $\nu = j(j+1)$ *(*$j = 3/2$*) using the matrix form for* J_+*,* J_-*,* J_z*, and* J^2*.*

Exercise 2: Write matrices $J_{x,y,z,}$*,* $J_\pm$*,* J^2 *for* $j = 2$*.*

7.4 Spin

Now, it is easy to introduce a spin $\bar{S}$ as an intrinsic angular momentum of a particle. It possesses all the properties of J. Its basis states may be written as $|m_s, s\rangle$. Because the spin of a particle does not depend on its orbital motion for a given type of particles $s = \mathrm{const}$. For electrons, protons and neutrons $s = 1/2$.

References

[1] W. H. Louisell, *Quantum statistical properties of radiation.* (Wiley, New York, 1990), Part II, Chapters 2.6.

[2] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, New York, 1994), Chapters 3.5-3.6.