The Law of Aggregate Demand and Welfare in the Two-Sided Matching Market*

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Abstract

In the college admission problem, we show the student-optimal stable matching is weakly Pareto optimal for students if colleges' preferences satisfy substitutability and the law of aggregate demand. We also show both of these properties are important for the result. *Journal of Economic Literature Classification Numbers*: C71, C78, D71, D78, J44.

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1 Introduction

The theory of two-sided matching considers matching between two types of agents, for example colleges and students. A classical result states that the student-optimal stable matching is weakly Pareto optimal for students if preferences of colleges are responsive.

We show that the above welfare conclusion holds more generally. More specifically, if preferences of colleges satisfy substitutability and the law of aggregate demand (Hatfield and Milgrom 2005), then the student-optimal stable matching is weakly Pareto optimal for students.

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Then we investigate how important substitutability and the law of aggregated demand are for the result to hold. We find that even if a college's prefereces violate the law of aggregate demand, there is an instance such that weak Pareto optimality of the student optimal stable matching can be guaranteed for any preferences of students and other colleges that satisfy substitutability and the law of aggregate demand. However violation of the law of aggregate demand does let us show a weaker converse result. If a college's preference relation violates the law of aggregate demand, then there is a preference profile of students and other colleges with singleton preferences under which there is an individually rational matching that all students weakly prefer and all but certain students strictly prefer to the student optimal stable matching. Taken together, we conclude that the law of aggregate demand is important for the welfare properties of a stable matching.

The weak Pareto optimality of the student optimal stable matching has been obtained under more restrictive assumptions in the literature. Roth (1982) shows the result when colleges have responsive preferences. Martinez, Masso, Neme, and Oviedo (2004) show the result when colleges have substitutable and q-separable preferences. The result seems to have been unknown under affirmative action constraints, such as those studied by Abdulkadiroğlu (2005). The class of substitutable preferences with the law of aggregate demand subsumes all these domains. Under substitutability and the law of aggregate demand, Hatfield and Kojima (2007a) show that the student optimal stable mechanism is group strategy proof for students, and apply this result to obtain an alternative proof of the weak Pareto optimality. On the other hand, results in the other directions that investigate if these conditions are minimal sufficient conditions for the welfare property are new to the best of our knowledge.

The above conclusions are especially interesting in the context of school choice (Abdulkadiroğlu and Sönmez 2003). Focusing on welfare of students is relevant in the school choice setting since colleges are regarded as objects to be assigned rather than agents. Moreover extending the domain of preferences is important, since school preferences often violate assumptions such as responsiveness and q-separability under which the conclusion has been known in the literature. For instance, some of the public schools in the New York City are required to admit a certain proportion of students from each of high, middle and low test score populations. Such requirements violate the above simple conditions, but still satisfy substitutability and the law of aggregate demand. Our analysis gives some justification for the use of the student-optimal stable mechanism originally advocated by Balinski and

Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003).[1]

Finally, the current paper contributes to the growing literature on matching with contracts. Hatfield and Milgrom (2005) present a new framework that generalizes models of matching and auction. They show that the law of aggregate demand they introduce and substitutability are crucial for some of the key results in the matching literature. The current paper gives another instance in which the law of aggregate demand plays a key role in matching theory.

2 Model

A market is tuple $\Gamma = (S, C, (\succeq_i)_{i \in S \cup C})$. S and C are finite and disjoint sets of students and colleges. For each student $s \in S$, $\succeq_s$ is a strict preference relation over C and being unmatched (being unmatched is denoted by $\emptyset$). For each college, $\succeq_c$ is a strict preference relation over the set of subsets of students. If $j \succ_i \emptyset$, then j is said to be **acceptable** to i. Given college c and a set of students $S' \subseteq S$, define $Ch_c(S')$ to be a set such that $Ch_c(S') \subseteq S'$ and $Ch_c(S') \succeq_c S''$ for any $S'' \subseteq S'$. In words, $Ch_c(S')$ is the set of students who c chooses from set S'.

A **matching** μ is a mapping from $C \cup S$ to $C \cup S \cup \{\emptyset\}$ such that (i) $\mu(s) \in C \cup \{\emptyset\}$ for every s, (ii) $\mu(c) \subseteq S$, and (iii) $\mu(s) = c$ if and only if $s \in \mu(c)$. For any pair of matchings μ and μ' and for any $i \in S \cup C$, we write $\mu \succeq_i \mu'$ if and only if $\mu(i) \succeq_i \mu'(i)$. Given a matching μ, we say that it is **blocked** by (s, c) if $c \succ_s \mu(s)$ and $s \in Ch_c(\mu(c) \cup s)$. A matching μ is **individually rational** if $\mu(s) \succeq_s \emptyset$ for each $s \in S$ and $\mu(c) = Ch_c(\mu(c))$ for each $c \in C$. A matching μ is **stable** if it is individually rational and is not blocked.

For each college $c \in C$, its preference relation $\succeq_c$ is **substitutable** if $Ch_c(S') \cap S'' \subseteq Ch_c(S'')$ for any $S'' \subseteq S' \subseteq S$ (Kelso and Crawford 1982). That is, a student who is chosen from a larger set of students is always chosen from a smaller one. Preference relation $\succeq_c$ satisfies the **law of aggregate demand** if $|Ch_c(S')| \geq |Ch_c(S'')|$ for any $S'' \subseteq S' \subseteq S$ (Hatfield and Milgrom 2005). That is, c chooses a larger number of students from a larger set of students.

Gale and Shapley (1962) propose the following **student optimal stable mechanism** (SOSM).

[1]The student-optimal stable matching is, on the other hand, not necessarily strictly Pareto optimal for students. For discussion on this tradeoff between stability and Pareto optimality in the school choice context, see Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu, Pathak, and Roth (2005) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005).

Step 1: Each student applies to her first choice college. Denote by $A^1(c)$ the set of students who apply to college c at this step. College c holds all the students in $B^1(c) \equiv Ch_c(A^1(c))$ and rejects everyone else.

Step $t \geq 2$: Each student who was rejected in Step (t-1) applies to her next highest choice. Denote by $A^t(c)$ the set of students who apply to college c at this step. College c holds all the students in $B^t(c) \equiv Ch_c(A^t(c) \cup B^{t-1}(c))$ and rejects everyone else.

The algorithm terminates in a finite number T of steps, either when every student is matched to a college or every unmatched student has been rejected by every acceptable college. The final assignment is given by $\mu(c) = B^T(c)$ for every $c \in C$. When colleges have substitutable preferences, the resulting matching is stable (see Roth and Sotomayor (1990)).

3 Result

Theorem 1. *Suppose that preferences of every college satisfy substitutability and the law of aggregate demand. Then μ_S is weakly Pareto optimal for students. That is, there exists no individually rational matching μ such that $\mu \succ_s \mu_S$ for every $s \in S$.*

To prove Theorem 1, we assume on the contrary that there exists an individually rational matching μ such that $\mu \succ_s \mu_S$ for every $s \in S$.

Claim 1. *$\mu(s) \in C$ for every $s \in S$. That is, every student is matched at μ.*

Proof. Since μ_S is individually rational, $\mu_S(s) \succeq_s \emptyset$ for every $s \in S$. By assumption on μ, we have $\mu(s) \succ_s \mu_S(s) \succeq_s \emptyset$. In particular $\mu(s) \neq \emptyset$, which implies $\mu(s) \in C$. □

Claim 2. *Suppose that preferences of every college satisfy substitutability and the law of aggregate demand.*

(1) *$|\mu(c)| = |\mu_S(c)|$ for every $c \in C$.*

(2) *$\mu_S(s) \in C$ for every $s \in S$. That is, every student is matched at μ_S.*

Proof. Since $\mu \succ_s \mu_S$ for every $s \in S$, c rejected every $s \in \mu(c)$ under SOSM. Since a choice function is path-independent when preferences satisfy substitutability,[2] we have that $\mu_S(c) = Ch_c(\mu(c) \cup \mu_S(c))$. Since $\succeq_c$ satisfies

[2] The choice function of c is **path-independent**, i.e. $Ch_c(Ch_c(S') \cup S'') = Ch_c(S' \cup S'')$ for any $S', S'' \subseteq S$, when preference of c is substitutable.

the law of aggregate demand and $\mu(c)$ is individually rational for c, we have that $|\mu_S(c)| = |Ch_c(\mu(c) \cup \mu_S(c))| \geq |Ch_c(\mu(c))| = |\mu(c)|$. By Claim 1, we have

$$|S| = \sum_{c \in C} |\mu(c)| \leq \sum_{c \in C} |\mu_S(c)| \leq |S|. \qquad (1)$$

The above inequality implies $|\mu(c)| = |\mu_S(c)|$ for every c, showing Part (1). Part (2) follows from inequalities in (1) since they imply $|S| = \sum_{c \in C} |\mu_S(c)|$. □

Proof of Theorem 1. Consider the last step of SOSM. At that step there exists a college c and a nonempty set of students S' such that all the students in S' apply to c. Since every student is matched at μ_S by Claim 2 (2), c accepts all the students in S' and no student is newly rejected (if this is not the case, either some $s \in S$ is unmatched or SOSM does not terminate, either of which is a contradiction.) Since $\mu \succ_s \mu_S$ for every s, every student in $\mu(c)$ has already applied to c and was rejected by c by the beginning of the last step of SOSM. Since preference of c satisfies the law of aggregate demand and μ is individually rational, c is matched to at least $|\mu(c)|$ students at the beginning of the last step of SOSM. Since c accepts all the students in S' and rejects no student at the last step of SOSM, we have $|\mu_S(c)| \geq |\mu(c)| + |S'| > |\mu(c)|$, which contradicts Claim 2 (1). □

A natural question is whether substitutability and the law of aggregate demand are minimal sufficient conditions for the above result. One way to formalize the question is the following: Suppose that preferences of one college violate either substitutability or the law of aggregate demand, are there preferences of students and other colleges satisfying substitutability and the law of aggregate demand, under which the student optimal stable matching is not weakly Pareto optimal?[3] Sönmez and Ünver (2003) show that if there is a college whose preferences violate substitutability, then there is a preference profile of other students and colleges under which a stable matching does not even exist.[4] Can a similar conclusion be derived about the law of aggregate demand? The answer turns out to be no, that is, sometimes the weak Pareto optimality can be established even without the law of aggregate demand. Suppose that $S = \{s_1, s_2, s_3, s_4\}$ and $Ch_{c_1}(S') =$

[3] We are grateful to the referee for asking the question as well as suggesting the formalization of the question we pursue here.

[4] This result is generalized by Hatfield and Milgrom (2005) beyond simple matching markets (the generalization contains an error: see Hatfield and Kojima (2007b) for correction.)

$\{s_3, s_4\}$ if $\{s_3, s_4\} \subseteq S'$ and $Ch_{c_1}(S') = S'$ otherwise. It is easy to see that preferences of c_1 are substitutable, but the law of aggregate demand is violated since $Ch_{c_1}(\{s_1, s_2, s_3\}) = \{s_1, s_2, s_3\}$ and $Ch_{c_1}(S) = \{s_3, s_4\}$.

Proposition 1. *Suppose $S = \{s_1, s_2, s_3, s_4\}$ and $Ch_{c_1}(S') = \{s_3, s_4\}$ if $\{s_3, s_4\} \subseteq S'$ and $Ch_{c_1}(S') = S'$ otherwise, and preferences of all other colleges satisfy substitutability and the law of aggregate demand. Then the student optimal stable matching μ_S is weakly Pareto optimal.*

Proof. There are two possible cases.

Case 1. Suppose that not both s_3 and s_4 apply to c_1 under SOSM. In such a case, μ_S is the same as a student optimal stable matching in an alternative problem in which $Ch_{c_1}(S') = S'$ for every $S' \subseteq S$ and preferences of all other agents are unchanged. Since the alternative problem satisfies conditions for Theorem 1, μ_S is weakly Pareto optimal for students.

Case 2. Suppose that both s_3 and s_4 apply to c_1 under SOSM. Then, by definition of $Ch_{c_1}(\cdot)$, we have $\mu_S(c_1) = \{s_3, s_4\}$. For contradiction, suppose that there exists a matching μ such that $\mu(s) \succ_s \mu_S(s)$ for every $s \in S$. By Claim 1, every student is matched at μ (note that Claim 1 holds even if the law of aggregate demand is not satisfied.) Since every $c \neq c_1$ satisfies the law of aggregate demand, we have $|\mu(c)| \leq |\mu_S(c)|$ for every $c \neq c_1$. Since $\mu(s_3) \neq \mu_S(s_3) = c_1$ and $\mu(s_4) \neq \mu_S(s_4) = c_1$ by assumption of μ, $\mu(c_1) \subseteq \{s_1, s_2\}$ and hence $|\mu(c_1)| \leq 2 = |\mu_S(c_1)|$. Therefore inequality (1) is satisfied, and conclusions of Claim 2 hold.

Consider the last step of SOSM. At that step there exists a college c and a nonempty set of students S' such that all the students in S' apply to c. If $c \neq c_1$, an identical argument as in the proof of Theorem 1 leads to contradiction. Suppose $c = c_1$. By the above argument, at least two students have applied to c_1 by the beginning of the last step of SOSM (this is because $|\mu(c_1)| = 2$ by the above argument, and both students in $\mu(c_1)$ are rejected under SOSM by assumption that $\mu(s) \succ_s \mu_S(s)$ for every s). By definition of $Ch_{c_1}(\cdot)$, this implies that c_1 is matched to at least two students at the beginning of the last step of SOSM. Since c_1 accepts all the students in S' and rejects no student at the last step of SOSM, we have $|\mu_S(c_1)| \geq 2 + |S'| > 2 = |\mu(c_1)|$, which contradicts Claim 2 (1). ☐

Proposition 1 demonstrates that the law of aggregate demand is not a minimal sufficient condition for the welfare conclusion to hold. Despite this conclusion, the next theorem shows that there is a sense in which the law of aggregate demand is still important for the weak Pareto optimality.

Theorem 2. *Fix S and college c_1. Suppose that preferences of c_1 are substitutable and there exist $S' \subseteq S$ and $s_1 \in S$ such that*

$$|Ch_{c_1}(S' \cup s_1)| < |Ch_{c_1}(S')|.$$

Then there exist colleges other than c_1 with singleton preferences and student preferences such that there exists an individually rational matching μ with $\mu(s) \succeq_s \mu_S(s)$ for every $s \in S$ and $\mu(s) \succ_s \mu_S(s)$ for every $s \notin Ch_{c_1}(S' \cup s_1) \setminus s_1$.[5]

Proof. Assume without loss of generality that $Ch_{c_1}(S') = S'$. By definition, there are at least two students s_2 and s_3 such that $s_2, s_3 \in Ch_{c_1}(S')$ and $s_2, s_3 \notin Ch_{c_1}(S' \cup s_1)$. Label students in $S \setminus (S' \cup s_1)$ by $s_4, s_5, \ldots, s_n$ for some n (note that n may be strictly smaller than $|S|$). Let there be $n-2$ colleges, $\{c_2, \ldots, c_{n-1}\}$, other than c_1. Preferences of these agents are given by[6]

$$\begin{aligned}
c_2 &: s_2, s_3, s_1, \\
c_m &: s_m, s_{m+1}, && m \in \{3, \ldots, n-1\}, \\
s_1 &: c_2, c_1, \\
s_2 &: c_1, c_2, \\
s_3 &: c_2, c_1, c_3, \\
s_m &: c_{m-1}, c_m, && m \in \{4, \ldots, n-1\}, \\
s_n &: c_{n-1}, \\
s &: c_1, && s \in S' \setminus \{s_2, s_3\}.
\end{aligned}$$

Conducting the SOSM, we find that the student optimal stable matching μ_S is given by

$$\begin{aligned}
\mu_S(s) &= c_1 && s \in Ch_{c_1}(S' \cup s_1), \\
\mu_S(s_m) &= c_m, && m \in \{2, 3, \ldots, n-1\}, \\
\mu_S(s) &= \emptyset, && \text{otherwise.}
\end{aligned}$$

Consider the matching μ defined by

$$\begin{aligned}
\mu(s) &= c_1 && s \in S', \\
\mu(s_1) &= c_2, \\
\mu(s_m) &= c_{m-1}, && m \in \{4, \ldots, n\}.
\end{aligned}$$

[5] Note that we allow varying the number of colleges as long as c_1 is included, while the set of students S is given.

[6] The notation is read as: c_2 chooses one student who appears first on the list and rejects everyone else (if any). Notation is defined similarly for other agents.

It can be readily seen that μ is individually rational, $\mu(s) \succeq_s \mu_S(s)$ for every $s \in S$ and $\mu(s) \succ_s \mu_S(s)$ for every $s \notin Ch_{c_1}(S' \cup s_1) \setminus s_1$, completing the proof. □

A straightforward corollary of the above result gives a condition under which violation of the weak Pareto optimality can be obtained.

Corollary 1. *Fix S and college c_1. Suppose that preferences of c_1 are substitutable and there exist $s_1, s_2, s_3 \in S$ such that*

$$Ch_{c_1}(\{s_2, s_3\}) = \{s_2, s_3\}, \quad Ch_{c_1}(\{s_1, s_2, s_3\}) = \{s_1\}.$$

Then there exist colleges other than c_1 with singleton preferences and student preferences such that μ_S is not weakly Pareto optimal.

Proof. The above conditions imply that preferences of c_1 violate the law of aggregate demand. Applying Theorem 2 for $S' = \{s_2, s_3\}$ we complete the proof since $Ch_{c_1}(S' \cup s_1) \setminus s_1 = \emptyset$. □

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