Blind Deconvolution via Sequential Imputations

Jun S. Liu and Rong Chen*

The sequential imputation procedure is applied to adaptively and sequentially reconstruct discrete input signals that are blurred by an unknown linear moving average channel and contaminated by additive Gaussian noises, a problem known as blind deconvolution in digital communication. A rejuvenation procedure for improving the efficiency of sequential imputation is introduced and theoretically justified. The proposed method does not require the channel to be nonminimum phase and can be used in real-time signal restoration. Two simulated systems are studied to illustrate the proposed method. Our result shows that the ideas of multiple imputations and flexible simulation techniques are as powerful in engineering as in survey sampling.

KEY WORDS: Bayesian model; Communication; Gibbs sampling; Importance sampling; Predictive distribution; Signal transmission.

1. INTRODUCTION

Signal transmissions and various associated adaptive filtering procedures have been the major focus of communication techniques for more than 50 years. In this article we consider the following linearly degraded moving average system commonly seen in digital communication:

\[ y_t = \sum_{i=0}^q h_i x_{t-i} + e_t, \quad t = 1, 2, \ldots, \tag{1} \]

where \( e_t \sim N(0, \sigma^2) \) and the \( y_t \) are sequentially observed output signals. The unobserved input signal \( x_t \) is assumed to be discrete valued with known levels \( s_1, \ldots, s_m \). The coefficients of the blurring mechanism, \( h_0, \ldots, h_q \), are assumed unknown (i.e., blind). The main objective is to recover the input signals, \( x_t, t = 1, 2, \ldots, \) and the coefficients, \( h_i, i = 0, \ldots, q \).

There are two requirements for a desirable solution to this problem. The first is, of course, the blindness: The second is the adaptivity; that is, the deconvolution procedure must be sequential and adaptive. With these features, the system can recover itself while receiving and processing the transmission and has the ability to recover after a rapid system coefficient change. This combination of blindness and adaptivity creates the challenge.

In this article we propose an adaptive blind deconvolution algorithm based on a Bayesian formulation of the problem and on the sequential imputation method developed by Kong, Liu, and Wong (1994). The main idea is to treat the input signals \( \{x_t\} \)'s as missing data and to sequentially impute multiples of them based on currently observed output signals \( \{y_t\} \)'s. For each imputed signal sequence, its importance weight is computed according to its relative "predictability" of the (future) observed blurred signals. The imputed sequences with their importance weights provide Bayesian estimates of the input signal sequence. The coefficients for the linear system can be estimated based on these estimates. The algorithm is easy to implement, and the system is not required to be minimum/nonminimum phase.

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Because the procedure is developed under a Bayesian framework, prior information can be easily incorporated.

Adaptive blind deconvolution has important applications in many fields such as digital communications, seismology, and underwater acoustics. Linear restoration procedures are widely used for recovering input signals when a training sequence of signals is available for an initial estimation of the system coefficients. But training sequences are not always available. For instance, transmission monitoring and digital microwave radio link modems are usually required to start up or restart their adaptive equalizer without a training sequence. In a multipoint network, if a channel from the master transmitter to one of its tributary stations goes down after the initial training period, then the channel must be recovered without training so as to avoid interrupting the communication between the master transmitter and other stations.

Blind deconvolution problems that do not involve additive noises have been addressed by several researchers. Most of the existing algorithms belong to one of two groups. Sato (1975), Godard (1980), and Donoho (1981) minimized nonconvex cost functions that differ from the classical mean squared error criterion. Another group of algorithms uses higher-order cumulants, spectrum, or correlation structure to identify a nonminimum phase system (Chen 1990; Giannakis and Mendel 1989; Hatzinakos and Nikias 1991; Li and Rosenblatt 1982). Some of the above approaches can be extended to cases with additive Gaussian noise. Li (1992, 1993) used inverse filtering procedures; Chen and Li (1993) derived a Gibbs sampling method for signal recovery in system (1).

The rest of the article is organized as follows. Section 2 gives a brief overview of sequential imputation. Section 3 provides computational details for implementing the method in blind deconvolution. Sections 4 and 5 contain some modifications of the plain sequential imputation method making it more suitable for the blind deconvolution problem. In particular, Section 4 describes and studies a procedure called rejuvenation; and Section 5 contains some other modifications. Finally, Section 6 includes two simulation studies to illustrate the properties of the procedures.

2. THE SEQUENTIAL IMPUTATION

We begin with a brief description of sequential imputation in a general Bayesian framework. Let \( \theta \) be the parameter of
interest and let $z$ denote the (complete) data set. The posterior distribution $\pi(\theta | z)$ is assumed to be simple. In many practical situations, however, $z$ is only partially observed. In this case we partition $z$ as $(x, y)$, where $y$ denotes the observed part and $x$ represents the missing part. Now suppose that $x$ and $y$ can each be further decomposed into $n$ corresponding components so that

$$z = (z_1, z_2, \ldots, z_n) = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

where $z_i = (x_i, y_i)$ for $i = 1, \ldots, n$. Typically the $z_i$ are iid or follow some linear time series models given $\theta$.

If $N$ copies (ideally, independent ones), $x(1), x(2), \ldots, x(N)$, can be drawn from the conditional distribution $\pi(x | y)$, then, because $\pi(\theta | y) = \int \pi(\theta | x, y) \pi(x | y) dx$, the posterior distribution $\pi(\theta | y)$ can be approximated by

$$\sum_{j=1}^N \pi(\theta | z(j))/N,$$

where $z(j)$ is the augmented complete data set $\{x(j), y\}$. But drawing from $\pi(x | y)$ directly is usually difficult. The Gibbs sampler (see Gelman and Smith 1990 for details) evolves a Markov chain to generate dependent samples $x(j), j = 1, \ldots, N,$ from the target distribution $\pi(x | y)$. Sequential imputation, however, achieves a similar goal by imputing the $x$ sequentially and then adjusting them by using importance weights. In general, sequential imputation involves two steps:

**Step A.** Draw $x^*_t$ from $\pi(x_t | y_t)$, and then for $t = 2, \ldots, n$, draw $x^*_t$ recursively from the conditional distribution

$$\pi(x^*_t | x^*_t, y_1, \ldots, x^*_{t-1}, y_t).$$

Note that the unobserved signal $x_t$ is imputed by drawing $x^*_t$ from its predictive distribution based on the current observations and imputations.

**Step B.** For $t = 1, \ldots, n - 1$, compute $\pi(y_{t+1} | x^*_1, y_1, \ldots, x^*_t, y_t)$. The importance weight of the augmented sequence is computed as

$$w = \pi(y_t) \prod_{t=1}^{n-1} \pi(y_{t+1} | x^*_1, y_1, \ldots, x^*_t, y_t).$$

Here the predictive probability of the next observation $y_{t+1}$ is computed and accumulated to credit the imputed sequence.

Suppose that $A$ and $B$ are done repeatedly and independently $N$ times and let the imputed samples be denoted by $x^*(1), \ldots, x^*(N)$, and the respective weights by $w(1), \ldots, w(N)$, where $x^*(j) = (x^*_1(j), \ldots, x^*_n(j))$ for $j = 1, \ldots, N$. The posterior $\pi(\theta | y)$ is estimated by the weighted mixture

$$\hat{\pi}(\theta | y) = \sum_{j=1}^N w(j) \pi(\theta | z^*(j)),$$

where $w(j) = w(j)/\sum_{k=1}^N w(k)$ and $z^*(j) = (x^*(j), y)$ for $j = 1, \ldots, N$, and the expected values of the "missing part" $x$ is estimated by the weighted average

$$\hat{x} = \sum_{j=1}^N w(j) x^*(j).$$

The maximum a posteriori estimator can be obtained similarly.

The method can be regarded as a special importance sampling scheme. To see this, note that each imputed sequence $x^*(j)$ is drawn from the "trial distribution" $\pi^*(x | y)$ as described in Step A instead of the ideal distribution $\pi(x | y)$. The weight computed in Step B is equivalent to the importance sampling weight $\pi^*(x | y)/\pi(x | y)$ associated with the scheme (see Kong et al. 1994 for more details). Geweke (1989) also provided some general discussions of the use of importance sampling in Bayesian computation.

### 3. SEQUENTIAL IMPUTATION ALGORITHM FOR BLIND DECONVOLUTION

#### 3.1 Model Setting and Priors

For the system defined by (1), the likelihood function, up to time $n$, can be written as

$$L(X_n, H, \sigma^2 | Y_n),$$

$$\propto \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^n \left(y_i - \sum_{j=1}^q h_i x_{i-j}\right)^2\right],$$

where $Y_n = (y_1, \ldots, y_n)^T$, $X_n = (x_{1-q}, \ldots, x_n)^T$, and $H = (h_0, \ldots, h_q)^T$ are column vectors.

Taking a Bayesian viewpoint, we a priori assume that $H \sim N(0, \Sigma)$, where $H$ is a $(q + 1)$-dimensional column vector and $\Sigma$ is a $(q + 1) \times (q + 1)$ positive definite matrix. Furthermore, because the $x_i$ are iid from a discrete distribution with probability vector $(\theta_1, \ldots, \theta_m)$ on the $m$ possible states $(s_1, \ldots, s_m)$, where $\theta_i > 0$ and $\theta_1 + \cdots + \theta_m = 1$, we put a Dirichlet prior on $\Theta$, i.e., $\Theta = (\theta_1, \ldots, \theta_m) \sim$ Dirichlet$(\delta_1, \ldots, \delta_m)$ when $\Theta$ is unknown.

Ideally, the posterior distribution of the linear coefficients can be obtained by "summing out" those $x$'s in (5). But the impracticability of this can be easily seen. Furthermore, there are no simple conditional independence structures to facilitate a Kalman filtering-type algorithm. The Gibbs sampler, a Monte Carlo method for overcoming this difficulty, has been implemented by Chen and Li (1993) for this problem. But the Gibbs sampler is not an adaptive procedure and has difficulty dealing with sequentially observed data. With new data coming in, the whole computation must be repeated to incorporate new information. Sequential imputation provides a more efficient alternative. To present our result, we introduce the following notation:

- $n$ is the total number of observed signals, $m$ is the number of states of input signals, and $q + 1$ is the number of moving average lags.
- The observed and the input signals, up to time $t$, are denoted by $Y_t = (y_1, \ldots, y_t)^T$ and $X_t = (x_{1-q}, \ldots, x_t)^T$.
- $x_t = (x_t, x_{t-1}, \ldots, x_{t-q})^T$ is a $(q + 1)$-dimensional column vector.
- $X_t = (x_1, \ldots, x_t)^T = (x_1, x_0, \ldots, x_{t-q})^T = (x_{t-q}, \ldots, x_1)^T$.
- $n = \sum_{t=1}^T \delta_t(x_t)$, where $\delta_t(x_t) = 1$ if $x = s_t$ and zero as calculated. The correct solution is $\hat{x}$, which is estimated by the weighted average $\hat{x} = \frac{\sum w(j) x^*(j)}{\sum w(j)}$.
otherwise. In other words, \( n_u \) is the number of times of \( x_j = s_j \) for \( j = 1, \ldots, t \).

- Function \( \phi(\cdot; \cdot) \) is the density function of the standard normal distribution.
- Throughout the article, \( \pi(\cdot) \) and \( \pi(\cdot | \cdot) \) are used to denote the densities or conditional densities.

### 3.2 Two Steps of the Sequential Imputation

For simplicity, assume that the system variance \( \sigma^2 \) is known. Experience shows that an approximate knowledge of \( \sigma^2 \) suffices. To initiate, presignals \( x_{1:q_1}, \ldots, x_0 \) are imputed from the prior distribution. Namely, a \( \Theta \) is drawn from its prior distribution Dirichlet(\( \theta_1, \ldots, \theta_m \)), and then \( x_{1:q_1}, \ldots, x_0 \) are drawn independently from the multinomial distribution with parameter \( \Theta \). The procedure is repeated \( N \) times to start the \( N \) imputation sequences. As illustrated in Section 2, two steps (A and B) are needed to implement the method. When a new signal \( y_{t+1} \) comes in, however, the two steps are usually done simultaneously with a reverse order in programming:

**Step B.** Compute the predictive probability of the observed value \( y_{t+1} \) and update the weight

\[
\tilde{w}_{t+1} = \tilde{w}_t \cdot \pi(y_{t+1} | x_t, Y_t).
\]

**Step A.** Impute the unobserved value \( x_{t+1} \) by a draw from the distribution

\[
\pi(x_{t+1} | X_t, Y_t) = \frac{1}{\phi_y(Y_{t+1} | \mu_{t+1}, \sigma_{t+1}^2)} \left( \frac{\gamma_{t+1} - \mu_{t+1}}{\sigma_{t+1}^2} \right) (n_q + b_t),
\]

and drawing \( x_{t+1} \) from the following updated discrete distribution

\[
\pi(x_{t+1} = s_j | X_t, Y_{t+1}) \propto \frac{1}{\sigma_{t+1}^2} \phi_y(Y_{t+1} | \mu_{t+1}, \sigma_{t+1}^2) (n_q + b_t),
\]

\( t = 1, \ldots, m \).

Furthermore, \( H_t \) and \( \Sigma_t \) enjoy nice recursive updating formulas that greatly reduce the computational burden. Precisely, after \( x_{t+1} \) has been imputed, the inverse of the posterior covariance matrix of \( H_t \) can be updated by

\[
\Sigma_t^{-1} = \Sigma_t^{-1} + \frac{1}{\sigma_t^2} X_{t+1} X_{t+1}^T.
\]

Let \( \mu_{t+1} \) and \( \sigma_{t+1}^2 \) be the quantities computed by (6) and (7) for the imputed \( x_{t+1} \). It is easily seen that

\[
\Sigma_t = \frac{1}{\sigma_t^2} X_{t+1} X_{t+1}^T \Sigma_t^{-1} X_{t+1} \sigma_t^2
\]

\( t = 1, \ldots, T \).

For the posterior mean of \( H_t \) at time \( t + 1 \),

\[
H_{t+1} = \Sigma_t^{-1} \left( \Sigma_t^{-1} H_t + \frac{1}{\sigma_t^2} y_{t+1} x_{t+1} \right)
\]

\( t = 1, \ldots, T \).

The sequential imputation algorithm for blind deconvolution can be summarized as follows:

**Step 0.** To initiate sequential imputations, draw \( \Theta(1), \ldots, \Theta(N) \) from the prior distribution Dirichlet(\( b_1, \ldots, b_m \)), and for each \( \Theta(j), j = 1, \ldots, N \), impute \( x_0(j) \) to \( x_{1:q_1}(j), \ldots, x_0(j) \) with each \( x_t(j) \) iid drawn from the distribution \( \Theta(j) \) on \( s_1, \ldots, s_m \). Set \( \tilde{w}_0(j) = 1 \), for \( j = 1, \ldots, N \).

**Step 1.** Repeat the following two procedures for every new observation starting from \( t = 0 \):

- Compute \( \pi(y_{t+1} | X_t(j), Y_t) \) by (8), let \( w_{t+1}(j) = \tilde{w}_t(j) \pi(y_{t+1} | X_t(j), Y_t) \), and impute \( x_{t+1}(j) \) by a draw from \( \pi(x_{t+1} | X_t(j), Y_{t+1}) \) computed by (9), for \( j = 1, \ldots, N \).
- For \( j = 1, \ldots, N \), update \( H_t(j) \) and \( \Sigma_t(j), j = 1, \ldots, N \), by (11) and (10), in which \( x_{t+1} \) is replaced by \( x_{t+1}(j) \).

**Step 2.** For a fixed number of observed signals \( y_1, \ldots, y_n \), the final importance weights \( \tilde{w}(j) \) is equal to \( w_n(j) \), for \( j = 1, \ldots, N \). An estimate of the posterior conditional expectation of the input signal \( x_t \) is then

\[
\hat{x}_t = \frac{\tilde{w}(1)x_t(1) + \cdots + \tilde{w}(N)x_t(N)}{\tilde{w}(1) + \cdots + \tilde{w}(N)},
\]

\( t = 1 - q, \ldots, n \).
Similarly, for \( i = 1, \ldots, m \), the marginal posterior probability can be approximated by

\[
P(x_i = s_i | Y_n) \approx \frac{w_i}{{\sum}_{j=1}^{m} w_j + \cdots + w(N)} I(x_i = s_i)
\]

where \( I(\cdot) \) is the indicator function. Therefore, an alternative estimate of the input signal \( x_i \) is the maximum a posteriori (MAP) estimate \( \hat{x}_i = s_i \); that is, a maximizer of \( P(x_i = s_i) \).

The posterior mean vector and covariance matrix of \( H \) are provided similarly by the weighted averages of \( H_n(j), j = 1, \ldots, N \), and \( \Sigma_0(j), j = 1, \ldots, N \).

**Remarks.**

1. It is noted that the proposed sequential imputation procedure can be easily modified to accommodate complex variables, which are common in digital communication.

2. In many practical situations, the mechanism that generates the unobserved input signals is known. This information should be used. For example, if it is known that the \( x \) are iid uniformly distributed on \( [s_1, \ldots, s_m] \), then formulas (8) and (9) become

\[
\pi(y_{t+1} | x_t, Y_t) = \frac{1}{m} \sum_{j=1}^{m} \phi \left( \frac{y_{t+1} - \mu_{t+1}(j)}{\sigma_{t+1}(j)} \right)
\]

and

\[
\pi(x_{t+1} = s_i | X_t, Y_{t-1}) \approx \frac{1}{\sigma_{t+1}(i)} \phi \left( \frac{y_{t+1} - \mu_{t+1}(i)}{\sigma_{t+1}(i)} \right).
\]

If the signals are transmitted according to a Markov transition function, then the predictive distribution of \( y_{t+1} \) and the sampling distribution of \( x_{t+1} \) can be revised accordingly.

**4. REJUVENATION PROCEDURE**

**4.1 Rejuvenated Sequential Imputation**

The importance sampling weight \( w(j) \) evaluates the quality of an imputed signal. A relatively small importance weight for the imputed sequence implies that it is drawn far from the main body of the posterior, and the resulting imputation will have a small effect in the final estimation. Such an imputed sequence is said to be "ineffective." If there are too many ineffective imputations, then the Monte Carlo scheme will be inefficient. This can be detected by observing a large coefficient of variation in the importance weights (defined later). Kong et al. (1994) showed that the importance weight resulting from sequential imputation forms a martingale sequence. As more and more data are processed, the coefficient of variation of the weights increases; that is, the number of ineffective imputations increases.

To cope with this difficulty, we introduce a restarting strategy, called rejuvenation.

Let the current \( N \) independent imputations be \( \{X_1(1), \ldots, X_N(N)\} \), with weights \( \{w_1, \ldots, w(N)\} \). To rejuvenate sequential imputations at time \( t + 1 \) (after observing \( y_{t+1} \)), we first compute the weight \( w_{t+1}(j) \) for \( j = 1, \ldots, N \) and generate \( N \) iid random numbers, \( j_1, \ldots, j_N \), from the multinomial distribution \( P(j = j) \propto w_{t+1}(j) \) for \( j = 1, \ldots, N \). Then the \( N \) original imputations are replaced by \( X_{t+1}(j) = X_k(j) \) for \( k = 1, \ldots, N \), and the weights are replaced by \( w_{t+1}(k) = 1/N \); that is, identical weights. After the replacement, the \( x_{t+1}(j) \) are imputed according to \( \pi(x_{t+1} | X_{t+1}(j)), Y_{t+1} \). The intuition is that the "good" imputations with large weights have high probability to be resampled, whereas the ineffective imputations have high probability to be discarded. When \( N \) is large, it is easy to show that \( X_{t+1}(j) \) approximately follows \( \pi(X_{t+1} | Y_{t+1}) \), which is "closer" to the target distribution \( \pi(X_{t+1} | Y_{t}) \) than \( \pi(X_{t+1} | Y_{t}) \).

Rejuvenation can be done at any stage of the process. But rejuvenating too often adds computational burden, and information may be lost in the process. On the other hand, rejuvenating too rarely may result in loss of efficiency. We found from simulation results (shown in Sec. 6) that rejuvenation is important, but the threshold chosen is not crucial.

To study the actual benefit of rejuvenation, we need to quantify the efficiency of an importance sampling scheme. Geweke (1989) provided such a discussion when a particular function of interest is considered. Here, because we consider an importance sampling scheme that is used to estimate many quantities of interest, we adopt a "rule of thumb" as discussed below.

Suppose that in an importance sampling scheme, \( \pi(X) \) is the target distribution and \( p(X) \) is the trial distribution from which we draw independent samples. The importance weight is defined as \( W(X) = \pi(X)/p(X) \), in which \( X \) is drawn from \( p(X) \). In the following discussion, we use \( E_q \) and \( var_q \) to indicate the expectation and variance taken under \( p \), and use \( E_p \) and \( var_p \) to indicate those taken under \( \pi \). Then \( var_p \{W(X)\} \) can be viewed as a measure of the chi-squared distance between \( \pi \) and \( p \). The larger the value of \( var_p \{W(X)\} \), the less efficient the corresponding importance sampling scheme. This variance cannot be computed directly in practice, however. Now suppose that \( w(1), \ldots, w(N) \) are the normalized importance weights (i.e., \( \sum w(j) = 1 \)) in an important sampling scheme; then the coefficient of variation, \( C(w) \), is computed as

\[
C^2(w) = \frac{1}{N} \sum_{j=1}^{N} \left( \frac{Nw(j) - 1}{N} \right)^2.
\]

When \( N \) is large, \( C^2(w) \) is a reasonable approximation of \( var_p \{W(X)\} \) because of the fact that \( E_p \{W(X)\} = 1 \). In the discussion of Section 4.2, we use \( C^2(w) \) and \( var_p(W) \) interchangeably.

A heuristic measure of efficiency for the sampling scheme is expressed by the "rule of thumb":

**Effective sample size (ESS)**

\[
= \frac{\text{total Monte Carlo sample size (N)}}{1 + C^2(w)}.
\]

The intuition of this "rule" is as follows. Suppose that \( N \) samples have been drawn, each associated with a weight \( w(k) \) that is either zero or \( 1/M \), where \( M \) is the number of nonzero weights. Then \( C^2(w) = N/M - 1 \), or \( M = N/(1 + C^2(w)) \). On the other hand, the effective sample size is the number
of samples with nonzero weights; that is, \( \text{ESS} = M = N/(1 + C^2(w)) \). Therefore, a large \( C^2(w) \) indicates that the effective sample size is small and memory and computation are being wasted by carrying too many ineffective sequences.

### 4.2 Losses and Gains From Rejuvenation

In this section the losses and gains from the rejuvenation procedure are studied. To simplify the presentation, we study a case where only two observations are involved. Its analysis will shed some light on the general situation of sequential imputation.

Suppose that \( V_1 \) and \( V_2 \) are incomplete observations with missing parts \( U_1 \) and \( U_2 \). We are interested in estimating \( \mu_1 = E_x\{g_1(U_1) \mid V_1, V_2\} \) and \( \mu_2 = E_x\{g_2(U_2) \mid V_1, V_2\} \) by Monte Carlo integration, where \( g_1 \) and \( g_2 \) are some general functions. The plain sequential imputation can be applied to this problem and involves two stages. First, \( N \) iid samples \( U_1(1), \ldots, U_1(N) \) are drawn from \( \pi(U_1 \mid V_1) \). Note that the importance weights at time \( t = 1 \) are \( w_1(j) = \pi(V_1) \propto 1 \). Second, the new weights \( w_2(j) \propto \pi(V_2 \mid U_1(j), V_1) \) are computed and renormalized, and at \( t = 1 \), and \( U_1(j) \) is independently drawn from \( \pi(U_1(j) \mid V_1, V_2) \) for \( j = 1, \ldots, N \). Note that \( w_2(j) \) can be regarded both as the importance weight for \( U_1(j) \) and as the importance weight for \( (U_1(j), U_2(j)) \) jointly. In the rejuvenation procedure we resample with replacement from the \( U_1(j) \) with the probability \( w_2(j) \) to obtain \( U_1^*(1), \ldots, U_1^*(N) \), and draw \( U_2^*(j) \) from \( \pi(U_2 \mid U_1^*(j), V_1, V_2) \) for \( j = 1, \ldots, N \), each associated with equal weight.

We argue that rejuvenation loses efficiency in estimating \( \mu_1 \) but gains efficiency in estimating \( \mu_2 \) in many situations, especially when \( C^2(w_2) \) is large, in which case the loss in the first stage is comparatively small.

#### 4.2.1 Losses From Rejuvenation

An estimate of \( \mu_1 \) from sequential imputation is

\[
\hat{\mu}_1 = \frac{1}{N} \sum_{j=1}^{N} w_2(j) g_1(U_1(j)),
\]

where \( w_2(j) \) are the normalized importance weights. Another estimate can be constructed after rejuvenation; that is,

\[
\tilde{\mu}_1 = \frac{1}{N} \sum_{j=1}^{N} g_1(U_1^*(j)).
\]

To simplify the notation, we use \( \pi \) to denote the true posterior marginal distribution \( \pi(U_1 \mid V_1, V_2) \) and let \( \rho \) be the trial distribution \( \pi(U_1 \mid V_1) \), from which we draw samples \( U_1(j) \). Let \( W(U_1) = \pi(U_1) / \rho(U_1) \); then when \( N \) is large, the normalized weight \( w_2 \) is an approximation of \( W/N \).

By the variance decomposition, we have

\[
\text{var}_{\rho}(\tilde{\mu}_1) = \text{var}_{\rho}\{E(\tilde{\mu}_1 \mid U_1(1), \ldots, U_1(N))\}
\]

\[+ \ E_{\rho}\{\text{var}(\tilde{\mu}_1 \mid U_1(1), \ldots, U_1(N))\},
\]

where the conditional expectation and variance are taken with respect to the random resampling. The first term on the right side is equal to \( \text{var}_{\rho}(\tilde{\mu}_1) \), whereas the second term is equal to

\[
\frac{1}{N} E_{\rho}\left[ \sum_{j=1}^{N} w_2(j) \{g_1(U_1(j)) - \hat{\mu}_1\}^2 \right],
\]

which is the extra variation introduced by the resampling. But the key quantity here is the ratio of this extra variation to the intrinsic variation \( \text{var}_{\rho}(\hat{\mu}_1) \). It is seen that

\[
\frac{\text{var}_{\rho}(\tilde{\mu}_1)}{\text{var}_{\rho}(\hat{\mu}_1)} = 1 + \frac{1}{N} \frac{E_{\rho}\{W[g_1(U_1) - \hat{\mu}_1]^2\} - \text{var}_{\rho}(\hat{\mu}_1)}{\text{var}_{\rho}(\hat{\mu}_1)}
\]

\[= \frac{N}{N - 1} + \frac{E_{\rho}\{W[g_1(U_1) - \mu_1]^2\}}{\text{var}_{\rho}(W)\{W_{\rho}(U_1)\}}.
\]

Now that \( E_{\rho}\{W[g_1(U_1) - \mu_1]^2\} = \text{var}_{\rho}(g_1(U_1)) \), and by an argument of Liu (1992),

\[
\text{var}_{\rho}(W_{\rho}(U_1)) = \text{var}_{\rho}(g_1(U_1)) \{1 + \var_{\rho}(W) + r \}. \tag{15}
\]

The term \( r \) tends to be small when the function \( g \) is relatively flat but is not guaranteed to be small in general. A nice consequence of the approximation (15) is the following "rule of thumb" for a reference of efficiency:

\[
\frac{\text{var}_{\rho}(\tilde{\mu}_1)}{\text{var}_{\rho}(\hat{\mu}_1)} \approx \frac{N - 1}{N} + \frac{1}{1 + \text{var}_{\rho}(W)}. \tag{16}
\]

When \( W = 1 \) (i.e., the trial density is the same as the target density), the foregoing formula coincides with the familiar result for resampling at random with replacement. Because \( N \) is usually large, we regard \( \text{var}_{\rho}(W) \) as equivalent to \( C^2(W) \). This formula implies that when the coefficient of variation in the importance weights is large, rejuvenation does not sacrifice much.

#### 4.2.2 Gains From Resampling

A natural estimate of \( \mu_2 \) is

\[
\tilde{\mu}_2 = \sum_{j=1}^{N} w_2(j) g_2(U_2(j)).
\]

The conditional variance of this estimator, given the first-stage imputation \( U_1(j) \) for \( j = 1, \ldots, N \), is

\[
\text{var}(\tilde{\mu}_2) = \sum_{j=1}^{N} w_2^2(j) \sigma^2(j), \tag{17}
\]

where \( \sigma^2(j) = \text{var}_{\rho}\{g_2(U_2(j) \mid U_1(j), V_1, V_2)\} \). Also let \( \bar{\rho}(j) = E_{\rho}\{g_2(U_2(j) \mid U_1(j), V_1, V_2)\} \).

Suppose that we resample with replacement to obtain \( U_1^*(1), \ldots, U_1^*(N) \), among which there are \( M_1 U_1(1){\zeta}_1, \ldots, M_N U_1(N){\zeta}_N \) such that \( M_1 + \cdots + M_N = N \). Then another estimator is

\[
\tilde{\mu}_2 = \frac{1}{N} \sum_{j=1}^{N} g_2(U_2^*(j)),
\]

and its variance, conditional on the first-stage sampling \( U_1(1), \ldots, U_1(N) \), is computed by the variance decomposition formula as
\[
\text{var}(\tilde{\mu}_2) = \text{var} \left( \frac{1}{N} \sum_{j=1}^{N} M_j \mu(j) \right) + \frac{1}{N^2} E \left( \sum_{j=1}^{N} M_j \sigma^2(j) \right)
\]
\[
= \frac{1}{N} \left( \sum_{j=1}^{N} w_2(j) \mu(j) \right)^2 - \left( \sum_{j=1}^{N} w_2(j) \mu(j) \right)^2
\]
\[
+ \frac{1}{N} \sum_{j=t}^{N} w_2(j) \sigma^2(j).
\]

It is not easy to compare (17) and (18) in general. Here we compare \(\text{var}(\tilde{\mu}_1)\) and \(\text{var}(\tilde{\mu}_2)\) in several extreme cases.

1. When all the weights \(w_2(j)\) are equal, \(w_2(j) = 1/N\), then \(V_2\) is conditionally independent of \(U_1\) given \(V_1\), and it is easily seen that \(\text{var}(\tilde{\mu}_1) \leq \text{var}(\tilde{\mu}_2)\).

2. When the \(\mu(j)\) and the \(\sigma(j)\) are all the same (in this case, \(U_2\) is conditionally independent of \(U_1\) given \(V_1\) and \(V_2\)), it can be shown that \(\text{var}(\tilde{\mu}_1) \geq \text{var}(\tilde{\mu}_2)\) by the Cauchy–Schwarz inequality.

3. When the weights \(w_2(j)\) are either zero or a constant, and there are \(K < N\) nonzero weights, each being \(1/K\), assume that the \(\mu(j)\) corresponding to the nonzero \(w_2(j)\) are close to each other, so that the first term of (18) can be neglected (e.g., when \(U_2\) takes binary value). Then \(\text{var}(\tilde{\mu}_2) = \sum_j \sigma^2(j)/K^2\) and \(\text{var}(\tilde{\mu}_2) = \sum_j \sigma^2(j)/NK\), implying that the rejuvenated scheme is about \((1 + C^2(W))\) times more efficient than the standard one.

More generally, because
\[
\sum_{j=1}^{N} w_2(j) \mu(j) \bigg( \sum_{j=1}^{N} w_2(j) \mu(j) \bigg)^{-1}
\]
and
\[
\sum_{j=1}^{N} w_2(j) \sigma^2(j) = E \text{var} \{g_2(U_2) \mid U_1, V_1, V_2 \mid V_1, V_2\},
\]
we have that \(\text{var}(\tilde{\mu}_2) \approx \text{var} \{g_2(U_2) \mid U_1, V_1, V_2 \}/N\) for the rejuvenated second-stage sampling. Similarly, the second-stage sampling variance for \(\tilde{\mu}_2\) is
\[
\text{var}(\tilde{\mu}_2) = \frac{1}{N} E [W \text{var} \{g_2(U_2) \mid U_1, V_1, V_2 \} \mid V_1, V_2],
\]
where \(W = \pi(U_1 \mid V_1, V_2)/\pi(U_1 \mid V_1)\). Therefore, comparing the two schemes amounts to comparing
\[
E [W \text{var} \{g_2(U_2) \mid U_1, V_1, V_2 \} \mid V_1, V_2]
\]
and
\[
\text{var} \{g_2(U_2) \mid V_1, V_2\}.
\]
Which one is larger depends on the problem at hand. A rule of thumb, however, is that the latter quantity is usually smaller when \(C^2(W)\) is large and \(W\) is positively correlated with the conditional variance \(\text{var} \{g_2(U_2) \mid U_1, V_1, V_2\}\).

We also note that the potential gains are not limited to \(U_2\). If there are more observations coming in, then all the future imputations and estimations will gain efficiency from rejuvenation. More precisely, after rejuvenation, each of the \(U_{\tilde{2}}(j)\) follows approximately the distribution \(\pi(U_1 \mid V_1, V_2)\). When a new observation \(V_2\) is considered, the trial density \(\pi(U_1, U_2 \mid V_1, V_2)\) associated with the rejuvenated procedure is closer to the target density \(\pi(U_1, U_2 \mid V_1, V_2)\) than is the trial density \(\pi(U_1 \mid V_1)\pi(U_2 \mid U_1, V_1, V_2)\) induced by the plain sequential imputation. In addition, if the expectation of a function of all the imputed \(U_2\)'s, say \(E_\theta(g_1, \ldots, g_\kappa)\), in a Bayesian problem is estimated by using the imputations \(U_i(j), \ldots, U_n(j)\), then rejuvenation becomes a good strategy when \(C^2(W)\) is large, because the loss is minimal and the potential gain is large.

**Remark.** An alternative method of rejuvenation is to select the rejuvenated imputation sequences by deterministic plus-residual sampling. Specifically, let the current \(N\) independent imputations be \(\{X_1(1), \ldots, X_N(N)\}\). To rejuvenate after observing \(X_1\), we compute the normalized weights \(\{w_{\text{rev}}(1), \ldots, w_{\text{rev}}(N)\}\). Let \(n_{\text{rev}} = \lfloor Nw_{\text{rev}}(i) \rfloor\) the greatest integer less than or equal to \(Nw_{\text{rev}}(i)\). Then \(n_i\) copies of \(X_i(i)\) are retained after rejuvenation. This is done for \(i = 1, \ldots, N\) to obtain \(X_{\text{rev}}(i)\), \(n_i\) rejuvenated imputations. The remaining \(N - \sum_{i=1}^{N} n_{\text{rev}}\) rejuvenated imputations are selected by sampling from \(\{X_1(1), \ldots, X_N(N)\}\), with replacement with probabilities proportional to \(w_{\text{rev}}(i) - n_{\text{rev}}(i)/N\). This scheme can effectively bring down the extra variation introduced by resampling. For example, in the ideal case when all the \(Nw_{\text{rev}}(i)\) are integers, there is no loss from rejuvenation in the first stage; that is, \(\mu_1, \mu_2\) are identical. On the other hand, the first part of formula (18) is reduced to zero. By writing \(w_2(j) = n_j/N\), we immediately observe \(\text{var}(\tilde{\mu}_2) \approx \text{var}(\tilde{\mu}_2)\), which implies that in this case the rejuvenation is always beneficial. A theoretical analysis for the general case is difficult, however.

### 5. OTHER MODIFICATIONS

#### 5.1 Early-Stage Imputation With Gibbs Sampling

Sequential imputation is most useful when the data are collected sequentially. But a practical drawback is that a large \(N\) is required when the system is large; that is, when \(m\) and \(q\) are large. This occurs because the early imputations are primarily based on our priors and are very likely to lose predictability for the future observations. To retain a reasonable number of effective imputations while information accumulates, one would have to start with a large number of imputation sequences. An alternative is to observe several output values from sequence before starting sequential imputation. With enough information, “good” starting imputations for the unobserved signal can be found, so that \(N\) can be kept small. In other words, instead of starting sequential imputation early on, we observe the first batch of signals \(y_1, \ldots, y_n\) and try to impute \(X_{\text{rev}}(j), \ldots, X_{\text{rev}}(j)\) from the partial posterior distribution \(\pi(X_{\text{rev}}(j) \mid Y_{\text{rev}})\). The Gibbs sampler is a natural choice for this purpose.

More precisely, Gibbs sampling iterations are used to process the first batch of signals \(y_1, \ldots, y_n\) to produce \(N\) imputed complete data \(\{X_{\text{rev}}(j), Y_{\text{rev}}\}, j = 1, \ldots, N\), each as-
associated with equal weight $1/N$. (See Chen and Li 1993 for
details on implementing the Gibbs sampler.) When new sig-
nals $y_{n+1}, y_{n+2}, \ldots$ are collected, $x_{n+1}(j), x_{n+2}(j), \ldots, j = 1, \ldots, N$, are imputed by using the sequential imputation
method described in Section 3.2, treating the estimated sys-
tem variance as known.

Our experiences show that this approach works well when
the system variance is unknown or the number of possible
levels $m$ of the input signal is large. In addition, we observe
that the combination strategy is able, with the help of impor-
tance weight, to discriminate good imputations from bad
ones resulting from the Gibbs sampler and, consequently,
without being trapped in a local mode of the posterior distri-
bution. Chen and Li (1993) reported that the Gibbs sampler
is often trapped in a shift mode. A further implication of
this observation is that in our use of the Gibbs sampler, the
global convergence is not required for each independent
Gibbs sampling chain.

5.2 Time Delay Output in Real Time Restoration

It is most desirable to estimate the input signals by using
the final weighted average as indicated in (12) or (13).
To carry out the real time signal restoration, however, the $x_i$
must be estimated during the process. One obvious solu-
tion to this problem is to instantaneously estimate the sig-
nals by computing the weighted average of the $x_i(j)$ using
the current weights $w_i(j)$ for time $t$. But this is usually
very inaccurate at early stages of processing. A better alter-
native is to allow for a $d$-lag delay, because future obser-
vations $y_{t+1}, \ldots, y_{t+d}$ contain substantial information
about the current signal $x_t$. Note that in general, $\pi(X_t|Y_{t+d})$
is closer to the target distribution $\pi(X_t|Y_t)$ than $\pi(X_t|Y_t)$
is. Specifically, if we are allowed to have $d$-lag delay, the
signal $x_t$ is estimated using $x_{i}(j)$ with importance weights
at time $t + d$. Thus at time $t + d$, the posterior expectation
of the input signal $x_t$ ($d$-lag backwards) is estimated by
using the current importance weights $w_{i+d}(j), j = 1, \ldots, N$;
that is, $\hat{x}_t = \left( \sum_{j=1}^{N} w_{i+d}(j)x_{i}(j) \right) / W_{i+d}$, where $W_{i+d} = \sum_{j=1}^{N} w_{i+d}(j)$. The MAP estimate $\hat{x}_t$ is approximated by
$\hat{x}_t$, a maximizer of the approximate marginal posterior
probability function $\pi(x_t = \hat{x}_t|Y_{t+d})$ that can be com-
puted by (13) with the $w(j)$ replaced by the $w_{i+d}(j)$. The
delay time was chosen to be $q + 1$ in our examples that
follow.
Table 1. Comparing Different Thresholds for Rejuvenation

<table>
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<th>4</th>
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<td>3</td>
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<td>6</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>5</td>
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<tr>
<td>Final</td>
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<td>13</td>
<td>14</td>
<td>17</td>
<td>22</td>
<td>31</td>
<td>44</td>
<td>1,000</td>
</tr>
</tbody>
</table>

6. SIMULATION STUDY

6.1 Direct Application of Sequential Imputation

This example is taken from Chen and Li (1993), who assumed the states of signals to be 0, 1, and 3. The blurring equation is

$$y_i = x_i + 0.8x_{i-1} - 0.4x_{i-2} + e_i.$$  \hspace{1cm} (19)

The input signals are iid uniform from \{0, 1, 3\}. The standard deviation of the white noise, which is roughly 3, was chosen to make the signal-to-noise ratio fixed at 15 dB.

One thousand observations were simulated from this system. We carried out the sequential imputation procedure using \(N = 1,000\) imputation sequences. The prior distribution for the system coefficient \(\mathbf{H}\) was assumed to be a product of independent \(N(0, 1,000)\), and the distribution of the \(x_i\) were assumed known. The threshold for rejuvenation was set to \(ESS = 3\), where \(ESS\) is defined in (14). For real time signal restoration, the delay time was chosen to be 3. We used the MAP estimate for the input signals. Note also that the weighted average estimate \(\hat{x}_i\) in (12) works well in most situations, because the posterior probability \(\pi(x_i = \hat{x}_i | \gamma)\) often concentrates on one value.

A signal is said to be misclassified if the estimated signal differs from the true signal. We tested our algorithm on the first 50, 100, 200, 500, and 1,000 of the simulated observations. The real time restoration misclassified 3, 3, 5, 7, and 12 signals, whereas the estimation using the final weights computed by (13) misclassified 0, 0, 3, 5, and 12 signals. Figure 1a displays the first 200 observations of the output series. Figure 1b shows a real time restoration for the first 200 observations. The dots are the true signals, and the lines are the estimates. Figure 1c illustrates the estimates using the final weights.

To study the effect of rejuvenation, we applied the sequential imputation procedure with different restarting thresholds for the first 200 simulated observations. Table 1 summarizes the result of using different threshold values (i.e., ESS) for rejuvenation found in the first row. The second row contains the number of misclassified signals for real time restoration with delay time 3. The numbers in the third row are the misclassified signals using final weight. The fourth row records the number of times that the algorithm rejuvenates for each threshold value.

Here a threshold of \(ESS = 1\) implies that we never rejuvenate. From Table 1, it is obvious that rejuvenation is important. But the effect of the threshold value (other than 1) is very small regarding the final results. Because rejuvenation requires computing time, a small ESS threshold is preferred.

To illustrate the effect of time delay in the real time restoration, we simulated 200 signal sequences, each with 200 observations, from the system (19). Figure 2 is the boxplot of the number of misclassifications for real time restorations with delay time 0 to 4 and the restoration using the final weights, where \(N = 1,000\) for all the simulations. We can see that a certain delay time is needed to get acceptable results. For the estimates using the final weights, the sequential imputation procedure misclassified more than 10 input signals 22 times in the 200 simulated data sets.

6.2 Combining Gibbs Sampling and Sequential Imputations

In this section we study a large system,

$$y_i = 0.9162x_i + 1.8333x_{i-1} + 0.4812x_{i-2} - 0.1987x_{i-3} + e_i,$$

where the \(x_i\) were randomly drawn from 16 possible levels \{-15, -13, \ldots, -1, 1, \ldots, 13, 15\} with equal probabilities. The noise \(e_i\) were iid from a normal distribution with mean zero and standard deviation 0.3144, with a signal-to-noise ratio of 30 dB. A similar example has also been used by Hatziakos and Nikias (1991) and Chen and Li (1993). Figure 3(a) shows the first 200 output signals \(y_i\) (\(i = 1, \ldots, 198\)). Because of the large number of possible levels that \(x\) can take, the information at early stages is poor. Plain sequential imputation is not practical in this situation for the reasons discussed in Section 5.2. Note that in assigning the presignal values \(x_{-3}, x_{-1}, x_0\), there are \(16^3 = 4,096\) possible combinations. Experience shows that 5,000 imputation se-

![Figure 2. The Boxplot of the Number of Signals Misspecified by Real Time Restoration With Delay Time 0 to 4 and Using the Final Weight in 200 Simulated Sequences.](image-url)
quences are not enough for such a system. The procedure proposed in Section 5.2 is more appropriate. For this example, 10 independent Gibbs sampling chains were applied to the first 55 observations, each with 900 iterations. The first 500 iterations for each chain were discarded, and 20 imputations were saved from the rest of the 400 iterations by choosing 1 in every 20 iterations. The imputations of the last 5 signals were discarded, because the Gibbs sampler is usually not very accurate in estimating the last several signals (Chen and Li 1993). Thus at the end of the Gibbs sampling procedure, 200 imputed signal sequences, each 50 signals long, were constructed. The posterior mean vector and covariance matrix of $H$ were then computed and saved for each of the 200 imputed sequences. Sequential imputations (i.e., Steps 1 and 2) were performed to process the rest of the data.

Figure 3 shows some of the results. The dots are the true signals, and the solid line in Figure 3b indicates the signals recovered using the Gibbs sampler imputations with the final sequential imputation weights. The dashed lines are the realistic signal restoration when performed with sequential imputations. The total number of misclassified signals was 14. Figure 3c plots the estimated input signals using the final weight. It misclassified 6 signals for the entire sequence of 204 signals.

7. SUMMARY

In this article we proposed a sequential imputation algorithm for blind deconvolution. The basic idea of the method is to treat the unobserved signals as missing data and to impute them sequentially. Rubin (1987b) suggested multiple imputations to incorporate the extra variations caused by missing data. In our case, however, multiple imputed sequences are needed to enhance a competition among them in the predictibility of the future observables, so as to direct the signal recovery. A particular strategy termed rejuvenation is introduced to reduce the Monte Carlo variations when the coefficient of variation in the sequential imputation weight is large. We expect the method to be useful in many other situations.

The advantage of the algorithm is its ability to recover system coefficients sequentially without a training sequence.
It also requires less of the system than methods of other investigators. Our simulations show that the procedure works well in complicated systems with small sample sizes. Furthermore, while the Gibbs sampler can become trapped in a local mode (Chen and Li 1993), sequential imputation does not. This is partly because $N$ independent sequences are simulated and competing. Those sequences trapped in a local mode will lose their predictabilities gradually, and the ones with better predictabilities will eventually dominate.

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REFERENCES


