Symmetry Groupoids in Coupled Cell Networks

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Abstract
In this paper, we review the application of symmetry groupoids in coupled cell networks, which intuitively speaking allows for the modeling of local symmetry and synchrony in directed graphs. We draw directly and heavily from the works of Golubitsky and Stewart (see [5], [11]) while presenting the groupoid-theoretic approach, in addition to providing a few original examples to motivate the theory. In the future, the author hopes to pursue an original application of the so-called “groupoid formalism” in computational biology.

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1 Introduction
Group theory has well been used to model the symmetry apparent in many biological and chemical phenomena; for an authoritative work consult [1] and [6]. Groupoid theory, however, has only been recently introduced in [2], [5], [8], and [11] to examine the symmetry in coupled cell networks, which loosely speaking can be thought of as (locally finite) directed graphs with cells (nodes or objects) representing states and directed edges (morphisms)
representing interactions between cells. These cells and edges may have some sort of
dynamic, possibilities of which include Markov chains to model transition probabilities,
cellular automata to model discrete conditions, or coupled ODEs to represent continuous
states. Golubitsky and Stewart’s “groupoid formalism” differs from the group-theoretic
approach in that the symmetry groupoid allows for the consideration of the input set of
any given cell, which consists of all cells connected to that cell. In particular, the condition
of global symmetry required to implement group-theoretic techniques is seldom met in
real phenomena, and it would be more sensible and freeing to use groupoid techniques
instead, with which we can talk about local symmetry and partial multiplication in terms
of dynamics. This also allows us to define the important notion of vector fields equivariant
under the symmetry groupoid and to prove properties of synchrony and quotient networks.

The paper is structured as follows: Sections 2 tightly follows Golubitsky and Stew-
art’s exposition, introducing the core concepts of symmetry, synchrony, and a coupled cell
network, and thereafter reviewing the groupoid formalism rigorously introduced in [11].
Section 3 introduces vector fields, which return some physical significance to a coupled
cell network, and quotient networks, which like quotient groups represents the coupled cell
network modded out by some equivalence relation (we again tightly follow [11] in this ex-
position). We conclude the paper with a brief survey of recent literature, and emphasize
the many possibilities for future research in the symmetry groupoid approach to coupled
cell networks.

2 Elements of the Groupoid Formalism

For this section and the next, we will trace linearly through [5] and [11], sometimes pro-
viding original examples but most of the time drawing heavily from [11].

2.1 Motivating Examples

Loosely speaking, a cell can be represented by a system of ODEs, and a coupled cell
system is a dynamical system with cell variables coupled together; e.g. the output of
some cells affects the state of others. We can label a coupled cell system as a system of
ODEs with individual cells producing observable outputs, and considering the set of such
outputs detect synchrony, the state of identical time-evolution in certain cells (this is not
unlike the probability-theoretic notion of coupling random variables or Brownian motion
to synchronize their time-evolution).

We will use the term architecture to broadly refer to the connection between cells and
their equivalence under coupling. Like [5] and [11], as a premise we assume that the ar-
chitecture of a directed graph is fixed, and we do not discuss cases where the architecture
evolves: the local nature of groupoid symmetries implies that small changes in architecture
may preserve the symmetry groupoid. The theory that we establish—namely, seeing how
the network architecture, given relatively weak assumptions, implies synchrony—is likely
useful in cases where the network architecture is fairly repetitive and symmetries abound on the local, but not global, level. We’ll present simple examples as we develop the theory.

**Example 2.1.1.** In fact, here’s a starting example. As motivation for the cell system concept, consider the network created by two cells with identical coupling. The symmetry of this network is between the two nodes and two edges, and can therefore be identified with $\mathbb{Z}_2$, the cyclic group with two elements.

![Figure 2.1: A two-cell network with $\mathbb{Z}_2$ symmetry (by the author).](image)

Without assuming the system dynamics, we can assign to each node some “admissible” differential equations, which are of the form

\[
\begin{align*}
\dot{x}_1 &= g(x_1, x_2), \\
\dot{x}_2 &= g(x_2, x_1),
\end{align*}
\]

(1)

with $x_1, x_2 \in \mathbb{R}^k$ being observable state variables. Note that we can fix the function $g$ to be the same in both variables by the symmetry of the network. Some questions immediately arise: what is the directed graph associated to the system (Figure 2.1), what can we say about the solutions as a result of the symmetry in the network, and is there synchrony between cells?

There are indeed a few consequences of $\mathbb{Z}_2$-symmetry in this example. One is the existence of solutions for which $x_1(t) = x_2(t)$ for all $t$ (this is what we mean by synchrony); e.g. in terms of dynamical systems, the diagonal subspace (which is also the fixed-point subspace) $\Delta = \{(x_1, x_2) : x_1 = x_2\}$ is flow-invariant under the system (1) above. This is because within $\Delta$, the system becomes

\[
\begin{align*}
\dot{x}_1 &= g(x_1, x_1), \\
\dot{x}_1 &= g(x_1, x_1),
\end{align*}
\]

(2)

and $x_1(0) = x_2(0) \Rightarrow x_1(t) = x_2(t)$ for all $t$. That is, synchrony of initial conditions for solutions imply synchrony of the solutions.

Another consequence of $\mathbb{Z}_2$-symmetry is that there exists an open set of functions $g$ for which there is a periodic solution, with period $T$, such that $x_2(t) = x_1(t + T/2)$ for all $t$; that is, the cells have the same period dynamics except for a phase shift. This remark follows from the $H/K$ theorem from bifurcation theory, which we will not cover (see [5], [6]). □
The concepts of synchrony and phase relations occur naturally in contexts where there is local symmetry, and not necessarily a global, group-theoretic symmetry. Here is an example that shows the existence of robust synchrony (robust in that the synchrony is preserved under perturbation) in a three-cell system that does not exhibit global symmetry.

**Example 2.1.2.** Consider the system induced by the ODEs

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_3), \\
\dot{x}_2 &= f(x_2, x_1, x_3), \\
\dot{x}_3 &= g(x_3, x_1),
\end{align*}
\]

(3)

where \(x_1, x_2 \in \mathbb{R}^k\) and \(x_3 \in \mathbb{R}^l\), which we can visually depict as follows (note that squares and circles indicate different types of cells, and different types of arrows indicate different couplings):

![Three-cell network](image)

*Figure 2.1.3: A three-cell network with synchrony (by the author).*

This system obviously has no global symmetry, but the subspace \(\Delta := \{(x_1, x_2) : x_1 = x_2\}\) is flow-invariant because the restriction of the first two equations in (3) to \(\Delta\) gives

\[
\begin{align*}
\dot{x}_1 &= f(x_1, x_2, x_3), \\
\dot{x}_2 &= f(x_1, x_2, x_3).
\end{align*}
\]

(4)

In some ways, cells 1 and 2 can be treated as equivalent in this network under certain subspaces. To make this more precise, we introduce the notion of an input set:

![Input sets](image)

*Figure 2.1.4: The input sets of each cell in the three-cell system (by the author).*
Definition 2.1.5. Informally, the input set $I(j)$ of a cell $j$ is the set consisting of the cell $j$ and all cells $i$ that connect to cell $j$, along with the corresponding morphisms.

It follows that $\Delta$ is flow-invariant as a result of the induced isomorphism between the input sets $I(1)$ and $I(2)$ by the permutation $\sigma : (1 \ 2 \ 3) \to (2 \ 1 \ 3)$. If (3) were equivariant in $\sigma$, then the fixed-point space (which is $\Delta$ here) of $\sigma$ would be flow-invariant (see [11]). But (3) is not equivariant in $\sigma$: applying $\sigma$, we see that the system can be expressed by the equations

$$
\begin{align*}
\dot{x}_2 &= f(x_2, x_1, x_3) \\
\dot{x}_1 &= f(x_1, x_2, x_3) \\
\dot{x}_3 &= g(x_3, x_2),
\end{align*}
$$

which differs from the original system in the third equation. Since the third equation is the same when restricted to $\Delta$, the restriction of the system to $\Delta$ is $\sigma$-equivariant, which makes $\Delta$ flow-invariant. \hfill \square

With the goal of explaining these examples in a general theory, we proceed to formally defining the notions of a coupled cell network, input set, and symmetry groupoid.

2.2 Coupled Cell Networks

The “coupled cell networks” we have presented above are basically directed graphs with ODEs at each cell. Let’s suppress the ODEs (we’ll reintroduce them in the vector fields of Section 3), and abstract the directed graph notion more precisely:

Definition 2.2.1. A coupled cell network $G$ consists of the following:

(a) **Cells**: A finite set $\mathcal{C} = \{1, \ldots, C\}$ of cells, or nodes.

(b) **Edges**: A finite set of ordered pairs $\mathcal{E} \subseteq \mathcal{C} \times \mathcal{C}$ of arrows, or directed edges. Each edge $(c, d) \in \mathcal{E}$ has a tail $c$ and a head $d$.

(c) **Equivalence on cells**: An equivalence relation $\sim_{\mathcal{C}}$ on cells in $\mathcal{C}$. The *type* of a cell $c \in \mathcal{N}$ is the $\sim_{\mathcal{C}}$-equivalence class $[c]_{\mathcal{C}}$ of $c$.

(d) **Equivalence on edges**: An equivalence relation $\sim_{\mathcal{E}}$ on edges in $\mathcal{E}$. The *type* is, similarly, the $\sim_{\mathcal{E}}$-equivalence class $[e]_{\mathcal{E}}$ of $e$. An edge $(c, c)$ is called an internal edge, and a cell is called *active* if it has an internal edge. Hereafter we assume that every cell is active.

(e) **Compatibility condition 1**: equivalent edges have equivalent tails and heads; e.g. if $(i, c) \sim_{\mathcal{E}} (j, d)$, then $i \sim_{\mathcal{C}} j$ and $c \sim_{\mathcal{C}} d$. 

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(f) Compatibility condition 2: internal edges are equivalent if their tails are equivalent; furthermore, internal edges and noninternal edges are never equivalent. That is, for all \( c, d, d' \in C \), 
\[(c, c) \sim_E (d, d') \iff d = d' \text{ and } d \sim_C c.\]

Denote \( G \) by \( G = (C, E, \sim_C, \sim_E) \). To visualize \( G \), we assign a distinct cell symbol to each \( \sim_C \)-equivalence class of cells and a distinct arrow symbol to each \( \sim_E \)-equivalence class of non-internal edges. Note that our definition is of coupled cell network is basically the definition of a directed graph with added labeling.

**Example 2.2.2.** The coupled cell network \( G = (C, E, \sim_C, \sim_E) \) with 
\[C = \{1, 2, 3, 4, 5\}, \quad E = \{(1, 3), (1, 5), (2, 4), (2, 5), (3, 1), (3, 2), (4, 5)\}, \quad \sim_C \text{ having equivalence classes } \{1\}, \{2\}, \{3, 4\}, \{5\}, \quad \text{and } \sim_E \text{ having equivalence classes } \{(1, 3), (2, 4), (2, 5)\}, \{(3, 1)\}, \{(1, 5), (3, 2), (4, 5)\} \]
can be represented by the following diagram. □

![Diagram of the coupled cell network in Example 2.2.2](image)

*Figure 2.2.3: Diagram of the coupled cell network in Example 2.2.2 (by the author).*

We’ll give a few more definitions leading to the symmetry groupoid.

**Definition 2.2.4.** The input set \( I(c) \) of a cell \( c \) is defined as \( I(c) = \{i \in C : (i, c) \in E\} \). By assumption, \( c \in I(c) \), and \( c \) is called the base cell of \( I(c) \). Two cells \( c \) and \( d \) with \( I(c) \cong I(d) \) are called input equivalent.

**Definition 2.2.5.** The equivalence relation \( \sim_I \) between inputs on \( C \) is given by \( c \sim_I d \iff \exists \beta : I(c) \to I(d) \) such that \( \beta(c) = d \) and \( (i, c) \sim_E (\beta(i), d) \) for all \( i \in I(c) \), where \( \beta \) is a base-cell perserving bijection. \( \beta \) is called an input isomorphism from \( c \) to \( d \), and we denote by \( B(c,d) \) the set of all such \( \beta \)'s.
Note that two cells have the same type if they are input equivalent, but the converse is false. Also note that $B(c, d) = \emptyset$ if $c \not\sim_I d$.

**Definition 2.2.6.** A **homogeneous network** is a coupled cell network with $B(c, d) \neq \emptyset$ for all $c, d$.

![Figure 2.2.7: Example of a homogeneous four-cell network (by the author).](image)

**Example 2.2.8.** Taking the same coupled cell network $G$ as in Example 2.2.2, we see that there are four $\sim_I$-equivalence classes: $\{1\}$, $\{2\}$, $\{3, 4\}$, and $\{5\}$. The isomorphism $\beta$ between $I(3)$ and $I(4)$ is given by $\beta : \{3, 1\} \rightarrow \{4, 2\}$, $\beta(3) = 4$ and $\beta(1) = 2$. □

**The Symmetry Groupoid.** We'll now introduce the main notion of the *symmetry groupoid*. The symmetry groupoid of a coupled cell network generalizes the notion of the symmetry group of a symmetric network, but includes the local symmetry of subgraphs (most importantly the symmetry of input sets) instead of only global symmetry.

**Definition 2.2.9.** The *symmetry groupoid* of a coupled cell network $G$ is the disjoint union

$$ \mathcal{B}_G = \bigsqcup_{c,d \in \mathcal{C}} B(c, d), $$

where $B(c, d)$ is the collection of all input morphisms from $c$ to $d$. Note the following properties of this groupoid:

(a) Morphism composition is defined as follows: the product of $\beta_1 \in B(c, d)$ and $\beta_2 \in B(c', d')$ is defined if and only if $c' = d$, and $\beta_2 \beta_1 = \beta_2 \circ \beta_1 \in B(c, d')$ where $\circ$ is a composition of maps.

(b) The identity elements $\text{id}_{I(c)}$ for $c \in \mathcal{C}$ are the identity elements of the groupoid.

(c) Each morphism $\beta \in B(c, d)$, when defined, has an inverse $\beta^{-1} \in B(d, c)$.

Observe that, for the particular instance where $c = d$, the collection $B(c, c)$ is a group.
called the \textit{vertex group} of $c$ [2].

\textbf{Example 2.2.10.} Consider the coupled cell network whose graph is as follows. The nonempty sets $B(c,d)$ are:

- $B(1, 1)$: The identity map on $\{1, 3, 4\}$.
- $B(2, 2)$: The identity map on $\{2, 1\}$.
- $B(3, 3)$: The identity map on $\{3, 1\}$.
- $B(4, 4)$: The identity map on $\{4, 2, 3\}$ and the permutation $\sigma$ on $\{4, 2, 3\}$ defined by $\sigma(234) = (324)$.
- $B(2, 3)$: The map $\beta : \{2, 1\} \to \{3, 1\}$ with $\beta(1) = 1, \beta(2) = 3$.
- $B(3, 2)$: $\beta^{-1}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{four-cell-network.png}
\caption{A four-cell coupled network. Source: [11]}
\end{figure}

\textbf{Example 2.2.12.} Considering the individual input sets for Example 2.2.2, we can list out all the $B(c,d)$’s similarly. This is left as an exercise to the reader.

With the notion of symmetry groupoid now defined, we proceed to talk about how it allows us to combine features of groups with features of graphs. For this we’ll need the group-theoretic notion of a subgroupoid, which corresponds to that of a connected graph component.

\textbf{Definition 2.2.13.} A subset $S \subset B_G$ is called a \textit{subgroupoid} if $S$ is closed under taking products (when defined) and inverses. Also, if $I$ is a $\sim_I$-equivalence class, then the subgroupoid

$$S(I) = \bigsqcup_{c,d \in I} B(c,d) \quad (7)$$

is called a \textit{connected component} of $B_G$.

\textbf{Lemma 2.2.14.} $B_G$ is the disjoint union of its connected components, e.g.

$$B_G = \bigsqcup_I S(I) \quad (8)$$
where \( I \) runs through the \( \sim I \) equivalence classes in \( C \). If \( I, I' \) are two such distinct classes and \( \beta \in S(I), \beta' \in S(I) \), then the product \( \beta \beta' \) is not defined.

**Proof.** This is a standard fact of groupoid, so we’ll omit it in the interest of space (see [2]). \( \blacksquare \)

Two cells \( c, d \in C \) are said to be in the *same connected component* of \( B_G \) if and only if \( c \sim_I d \). If \( c, d \) belong to the same connected component, then it is clear that the vertex groups \( B(c, c) \) and \( B(d, d) \) are *conjugate*, e.g. there exists \( g \in B(c, d) \) such that

\[
B(c, c) = g^{-1} B(d, d) g,
\]

and \( B(c, c) \) and \( B(d, d) \) is isomorphic in the group-theoretic sense.

**The General Structure of*** \( B(c, d) \).** Note that the follow properties hold for the sets \( B(c, d) \):

(a) \( c \neq_I d \Rightarrow B(c, d) = \emptyset \).

(b) If \( c = d \), then define an equivalence relation \( \equiv_c \) on the input set \( I(c) \) as follows:

\[
j_1 \equiv_c j_2 \iff (j_1, c) \sim_E (j_2, c) \quad \text{for} \quad j_1, j_2 \in I(c).
\]

Denote the \( \equiv_c \)-equivalence classes of \( I(c) \) as \( E_0, ..., E_k \) for \( k = k(c) \) so that

\[
I(c) = \bigsqcup_k E_j.
\]  

Let \( E_0 = \{c\} \) and \( e_r = |E_r| \) for \( 0 \leq r \leq k \). Then \( B(c, c) = S_{e_1} \times ... \times S_{e_k} \), where each \( S_{e_r} \) consists of all permutations of \( E_r \) extended by the identity on \( I(c) \setminus E_r \).

(c) If \( c \sim_I d \) and \( c \neq d \), we can define \( \equiv_d \) on \( I(d) \) similarly to see that if \( \beta \in B(c, d) \) and \( F_r = \beta(E_r) \), then \( \beta \) and \( \beta^{-1} \) preserve \( \sim_E \) and the \( \equiv_d \)-equivalence classes are the \( F_r \)'s. For any \( \beta_0 \in B(c, d) \) with this property, \( B(c, d) = B(d, d) \beta_0 = \beta_0 B(c, c) \), and conversely any \( \beta_0 : I(c) \to I(d) \) with \( F_r = \beta_0(E_r) \) is contained in \( B(c, d) \).

### 3 Deducing Pattern from Structure

Having established the algebraic foundations of coupled cell networks, we proceed (with a flavor of differential topology) to give the cells added meaning through vector fields. This will allow us to talk in generality about when synchrony should exist, and define the concept of a *quotient network* to good use.

#### 3.1 Vector Fields on a Coupled Cell Network

Consider the class \( \mathcal{F}_G \) of vector fields corresponding to a given coupled cell network \( G \), which contains all vector fields that are “symmetric” under the groupoid \( B_G \). The class
depends on the choice of the fixed “total phase space” $P$ (a terminology that seems to be used mainly by [11]). In the two-cell example, for instance, we had $P = \mathbb{R}^k \times \mathbb{R}^k$. We define $P$ as follows.

**Definition 3.1.1.** For each $c \in \mathcal{C}$ we can define a cell phase space $P_c$ to be a smooth Euclidean manifold of dim $\geq 1$, so that $c \sim_C d \Rightarrow P_c = P_d$ and the coordinates on $P_c$ and $P_d$ are the same. Note that this identification is only one-way: even if $c$ and $d$ have the same phase space, they may not necessarily have conjugate dynamics.

**Definition 3.1.2.** The total phase space is defined to be the product

$$P = \prod_{c \in \mathcal{C}} P_c,$$

with the coordinate system $x = (x_c)_{c \in \mathcal{C}}$. The cell projection of $c$ is the canonical projection $\pi_c : P \to P_c$. Moreover, we can define the product

$$P_D = \prod_{c \in D} P_c$$

for $D \subseteq \mathcal{C}$ and $\pi_D : P \to P_D$ being a projection, with coordinates $x_D = \pi_D(x)$.

**Definition 3.1.3.** If $D_1, D_2 \subseteq \mathcal{C}$ and $\exists$ a bijection $\beta : D_1 \to D_2$ with $\beta(d) \sim_C d$ for all $d \in D_1$, we can define the pullback $\beta^* : P_{D_2} \to P_{D_1}$ of $\beta$ by

$$(\beta^*(z))_j = z_{\beta(j)}$$

for all $j \in D_1, z \in P_{D_2}$. The pullback allows us to figuratively pull back components of the vector field associated with a coupled cell network into each other, and has the following properties which can be easily checked: $(\beta \gamma)^* = \gamma^* \beta^*$, $\text{id}^* = \text{id}$, and $(\beta^{-1})^* = (\beta^*)^{-1}$.

What vector fields can be pulled back, or encoded by a coupled cell network? The following definition gives us a notion.

**Definition 3.1.4.** A vector field $f : P \to P$ is said to be $\mathcal{B}_G$-equivariant or $G$-admissible if it satisfies the following two conditions:

(a) **Domain condition:** For all $c \in \mathcal{C}$, the component $f_c(x)$ depends only on $x_{I(c)}$; that is, there exists $\hat{f}_c : P_{I(c)} \to P_c$, with

$$f_c(x) = \hat{f}_c(x_{I(c)}).$$

(b) **Equivariance condition:** For all $c, d \in \mathcal{C}$ and $\beta \in B(c, d)$, we have

$$\hat{f}_d(x_{I(d)}) = \hat{f}_c(\beta^*(x_{I(d)}))$$
for all \( x \in P \). We can also write this as
\[
f_{\beta(c)}(x) = f_c(\beta^*(x))
\]
for all \( x \in P \). Note that here \( f_d(x) \) must depend only on \( x_{I(d)} \), else \( \beta^*(x) \) is not defined.

If \( \beta \in B(c,c) \), then by (16) we have \( f_c(\beta^*(x)) = f_c(x) \) for all \( x \in P \), which means that \( f_c \) is \( B(c,c) \)-invariant. This property is the same as the invariance under a group with \( B(c,c) \) acting on \( P_{I(c)} \).

**Definition 3.1.5.** For given \( P_c \), the class \( F^P_G \) is the set of all \( G \)-admissible vector fields on \( P \).

**Example 3.1.6.** We’ll describe \( F^P_G \) for the four-cell network example in the previous section (Example 2.2.10). Let the cell types \( \bigcirc, \square, \triangle \) have corresponding phase spaces \( U \), \( V \), and \( W \). The state variable is therefore \( x = (x_1,x_2,x_3,x_4) \), with \( x_1 \in U \), \( x_2 \in V \), \( x_3 \in V \), and \( x_4 \in W \). There are four arrow types, and we claim that the \( G \)-admissible vector fields \( f \) take the form
\[
\begin{align*}
  f_1(x) &= A(x_1,x_3,x_4), & \text{where } A : U \times V \times W \to U, \\
  f_2(x) &= B(x_2,x_1), & \text{where } B : V \times U \to V, \\
  f_3(x) &= B(x_3,x_1), \\
  f_4(x) &= C(x_4,x_2,x_3), & \text{where } C : W \times V \times V \to W,
\end{align*}
\]
and \( C \) is symmetric in \( x_2,x_3 \). We now prove that we have all the \( G \)-admissible vector fields \( f \) here.

**Proof.** Consider the equivariance condition (16) for all bijections \( \beta \in B(c,d) \) listed before. The nontrivial bijections are the collection \( B(2,3) \) and \( B(4,4) \). For \( B(2,3) \), consider the bijection \( \beta : I(2) \to I(3) \) sending \((2,1) \mapsto (3,1)\). Define the function \( B : P_{I(2)} \to P_2 \) by \( B(x_2,x_1) = f_2(x) \), so that \( B = \hat{f}_2 \) and \( f_3 = \hat{f}(x_3,x_1) \). The reader should verify that the pullback of \( \beta \) is given by \( \beta^* : (x_3,x_1) \mapsto (\hat{x}_3,x_1) \) (note that this is not an identity—why?), from which it follows from (16) that \( \hat{f}_3(x) = B(x_3,x_1) \). Similarly, for \( \gamma \in B(4,4) \), we can verify that the pullback \( \gamma^* : P_{I(4)} \to P_{I(4)} \) is given by \( \gamma^* : (x_4,x_2,x_3) \mapsto (\hat{x}_4,x_3,x_2) \), from which it follows from (16) that \( \hat{f}_4(x) = C(x_4,x_3,x_2) = C(x_4,x_2,x_3) \).

For notation, we’ll bar over the variables in the argument that the component \( \hat{f}_c \) is symmetric in. For example, we would write \( \hat{f}_4(x) = C(x_4,\overline{x_2},\overline{x_3}) \) above. \( \Box \)

**Admissible Vector Fields.** There is a certain kind of special \( G \)-admissible vector fields for which the theorems presented later in this paper hold, so we’ll start describing these
now. In the previous example, the general form of the $G$-admissible vector fields depended on the independent functions $A, B, C$, one for each $\sim_I$ equivalence class of cells (each connected component of $\mathcal{B}_G$).

Note that if $c \in C$, then the vector field $f_c$ is $B(c, c)$-invariant. If $d \sim_I c$, then $f_d$ can be uniquely defined by $f_c$ because of $\mathcal{B}_G$-equivariance. In particular, because a groupoid is the disjoint union of its connected components (Lemma 2.2.14), we can relate the components $f_c$ and $f_d$ of $f$ when $c, d$ are in the same connected component of $\mathcal{B}_G$. It follows that we can construct certain $G$-admissible vector fields $g$ on $P$ with components $g_c = 0$ for all $c$ outside a $\sim_I$-equivalence class, and furthermore have these vector fields span $\mathcal{F}_G^P$.

**Definition 3.1.7.** Now let $Q \subseteq C$ be a $\sim_I$-equivalence class, and define

$$\mathcal{F}_G^P(Q) = \{ f \in \mathcal{F}_G^P : f_s(x) = 0 \ \forall s \notin Q \}.$$ 

Vector fields in $\mathcal{F}_G^P(Q)$ are said to be supported on $Q$, and the subset $\mathcal{F}_G^P(Q) \subseteq \mathcal{F}_G^P$ is a linear subspace of $\mathcal{F}_G^P$.

The main condition for a vector field to be in $\mathcal{F}_G^P(Q)$ is $B(q, q)$-equivariance for arbitrary $q \in Q$. This notion is made more precise in the following lemma.

**Lemma 3.1.8.** Let $Q \subseteq C$ be a $\sim_I$-equivalence class, $q \in Q$, and $g_q : P_I(q) \to P_q$ be a $B(q, q)$-invariant mapping. Then $g_q$ extends uniquely to a vector field in $\mathcal{F}_G^P(Q)$.

**Proof.** Suppose $r \in Q$ and choose $\beta_0 \in B(q, r) : \beta_0(q) = r$. By equivariance, we have $g_r(y) = g_q(\beta_0(y))$ for all $y \in P_I(r)$, so the extension to $r \in Q$ is unique if it exists. It’s easy to see that $g_r$ does not depend on $\beta_0$, and if $r \notin Q$, we define $g_r(x) = 0$.

Now we claim that $g \in \mathcal{F}_G^P(Q)$. Since $g_r(x) = 0$ if $r \notin Q$, it suffices to show that if $r \in Q$, $\gamma \in B(r, s)$, and $z \in P_I(s)$, then $g_s(z) = g_r(\gamma^*(z))$. The component $g_s$ can be defined by $g_s(y) = g_q(\beta^*_1(y))$ for all $y \in P_I(s)$ and a choice of $\beta^*_1 \in B(q, s)$. Now let $\alpha = \beta^*_1 \gamma \beta_0 \in B(q, q)$, so that $\gamma = \beta_1 \alpha \beta_0^{-1}$ and from the properties of the pullback we see that $g_r(\gamma^*(z)) = g_q(\alpha^*(\beta^*_1(z))) = g_q(\beta^*_1(z)) = g_s(z)$, where the middle equality follows from the fact that $g_q$ is $B(q, q)$-invariant and $\beta^*_1 \in I(q)$.

We can now prove that such vector fields span $\mathcal{F}_G^P$.

**Lemma 3.1.9.** Let $Q$ run over the $\sim_I$-equivalence classes of $G$. Then

$$\mathcal{F}_G^P = \bigoplus_Q \mathcal{F}_G^P(Q).$$

**Proof.** Suppose that $f \in \mathcal{F}_G^P$, so $f$ is $\mathcal{B}_G$-equivariant. Let $Q$ denote a $\sim_I$-equivalence class. For $q \in Q$, define $g \in \mathcal{F}_G^P(Q)$ as $g_q(x) = f_q(x)$ for all $x \in P$, which is $B(q, q)$-invariant because $f$ is $\mathcal{B}_G$-equivariant. As a result, we have $g_r(x) = f_r(x)$ for all $r \in Q, x \in P$, where
\( g_r \) is defined as in the lemma above. We know that \( g_s(x) = 0 \) for \( s \notin Q \), so repeating this argument for all \( Q \) we have that

\[
\mathcal{F}_G^P = \sum \mathcal{F}_G^P(Q) \implies \mathcal{F}_G^P(Q) \cap \sum\limits_{R \neq Q} \mathcal{F}_G^P(R) = \{0\},
\]

so the sum is direct and we’re finished with the proof.

\[\blacksquare\]

3.2 Theorems on Synchrony

What are the conditions under which certain cells in a coupled cell network can synchronize purely as a result of the network architecture? We will answer this question with the following theorem:

**Theorem 3.2.1.** If \( \bowtie \) is an equivalence relation on a coupled cell network, then \( \bowtie \) is robustly polysynchronous if and only if \( \bowtie \) is balanced.

But before we decipher it, let’s work through an example of synchrony for motivation.

For surveys of synchronous cells in various architectures, see [1] and [3].

**Example 3.2.2.** Consider the ten-cell network \( G_1 \) with two types of cells, three arrow types, and three classes of \( \mathcal{C} \), as divided by the shape and shading: \( \{0, 1\} \), \( \{2, 3, 6, 8\} \), \( \{4, 5, 7, 9\} \). The three distinct arrow types are also illustrated below.

![Figure 3.2.3: The ten-cell network \( G_1 \), with synchronous cells shaded. Source: [11]](image-url)
With the appropriate choice of phase spaces $U$, $V$, and $W$, a vector field $f \in \mathcal{F}_{G_1}^P$ takes the form
\begin{align*}
  f_0 &= A(x_0, x_2, x_3, x_4, x_5), & f_5 &= C(x_5, x_0, x_2), \\
  f_1 &= A(x_1, x_6, x_7, x_8, x_9), & f_6 &= B(x_6), \\
  f_2 &= B(x_2), & f_7 &= C(x_7, x_0, x_8), \\
  f_3 &= B(x_3), & f_8 &= B(x_8), \\
  f_4 &= C(x_4, x_1, x_3), & f_9 &= C(x_9, x_1, x_8).
\end{align*}

Now consider the space $Y = \{(u, u, v, v, w, w, v, v, w, w) : u \in U, v \in V, w \in W\}$ determined by making the entries constant on the three classes $\{0, 1\}, \{2, 3, 6, 8\}, \{4, 5, 7, 9\}$. $f|_Y$ is then given by
\begin{align*}
  g_0 &= A(u, u, v, v, w, w), & g_5 &= C(w, u, v), \\
  g_1 &= A(u, u, w, v, w), & g_6 &= B(v), \\
  g_2 &= B(v), & g_7 &= C(w, u, v), \\
  g_3 &= B(v), & g_8 &= B(v), \\
  g_4 &= C(w, u, v), & g_9 &= C(w, u, v),
\end{align*}
and $g_0$ and $g_1$ are identical under symmetry. So $Y$ is flow-invariant for $f$. If we identify elements of $Y$ with the corresponding 3-tuple $(u, v, w)$, we get an induced vector field $\overline{f}$ of the form
\begin{align*}
  \overline{f}_0 &= A'(u, v, w), \\
  \overline{f}_1 &= B(v), \\
  \overline{f}_3 &= C(w, u, v),
\end{align*}
where $A'(u, v, w) = A(u, v, v, w, w)$. This is also the class of admissible vector fields for the quotient coupled cell network $G_2$ shown below. Here cells $v, w$ have the same type, but cell $v$ is shaded to show which equivalence class it corresponds to.
What structure in $G_1$ makes $Y$ flow-invariant for all $f \in \mathcal{F}_{G_1}^P$, and allows the reduction to $G_2$ on $Y$? We will start answering this question with a theory of flow-invariant subspaces, and develop the theory behind quotient networks later. □

**Patterns of Synchrony.** The three equivalence classes $\{0, 1\}, \{2, 3, 6, 8\}, \{4, 5, 7, 9\}$ from the example above can be seen as the components of an equivalence relation, relative to the symmetry groupoid of the network, that controls the existence of the flow-invariant subspace $Y$ and the quotient network $G_2$.

In particular, let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network. Fixing a total phase space $P$, we let $\bowtie$ be an equivalence relation on $\mathcal{C}$ that gives a set of equivalence classes. $\bowtie$ can also be assumed to be a refinement of $\sim_C$; that is, if $c \bowtie d$, then $c$ and $d$ are of the same type. Moreover, the polydiagonal subspace, a linear subspace of $P$ given by

$$\Delta_{\bowtie} = \{ x \in P : x_c = x_d \text{ whenever } c \bowtie d \forall c, d \in \mathcal{C} \},$$

is well-defined because $x_c$ and $x_d$ are both contained in $P_c = P_d$. In Example 3.2.2, if $\bowtie$ has the equivalence classes $\{0, 1\}, \{2, 3, 6, 8\}, \{4, 5, 7, 9\}$, then $\Delta_{\bowtie} = Y$.

We now examine patterns of synchrony by defining what it means to be **polysynchronous**:

**Definition 3.2.6.** A trajectory $x(t)$ of $f \in \mathcal{F}_{G_1}^P$ is said to be $\bowtie$-polysynchronous if its components are constant on $\bowtie$-equivalence classes; e.g.

$$c \bowtie d \implies x_c(t) = x_d(t)$$

for all $t \in \mathbb{R}$, or $x(t) \in \Delta_{\bowtie}$ for all $t \in \mathbb{R}$. The trivial polysynchrony occurs when the equivalence relation is equality and $\mathcal{C}$ is partitioned into its individual cells.

**Definition 3.2.7.** Let $\bowtie$ be an equivalence relation on $\mathcal{C}$. If $\Delta_{\bowtie}$ is invariant under every vector field $f \in \mathcal{F}_{G_1}^P$, then $\bowtie$ is robustly polysynchronous. That is, $f(\Delta_{\bowtie}) \subseteq \Delta_{\bowtie}$.
for all \( f \in \mathcal{F}_G^P \). Moreover, this is identical to the condition that if \( x(t) \) is a trajectory of \( f \in \mathcal{F}_G^P \) with initial condition \( x(0) \in \Delta_\infty \), then \( x(t) \in \Delta_\infty \) for all \( t \).

Note that a necessary condition for robust polysynchrony is that cells have isomorphic input sets, because these involve the “same” function in the corresponding components of the admissible vector fields.

**Lemma 3.2.8.** If \( \triangleright \) is robustly polysynchronous, then \( \triangleright \) refines \( \sim_I \). In other words, \( c \triangleright d \implies c \sim_I d \) for all \( c, d \in C \).

*Proof.* From Lemma 2.2.14 we see that it suffices to show that if \( c \triangleright d \), then \( c \) and \( d \) are in the same connected component \( Q \) of \( B_G \). Assume to the contrary that they are not in the same connected component; we’ll show that \( \Delta_\triangleright \) is not flow-invariant. Let \( x(0) \in \Delta_\infty \) be such that \( x_c(0) = x_d(0) \neq 0 \), and pick a \( f \in \mathcal{F}_G^P(Q) \) where \( d \notin Q \). Now because \( f_d \) vanishes for \( f \in \mathcal{F}_G^P(Q) \), \( x_d(0) = x_d(t) \) for all \( t \) and \( x(t) \) being a solution to the differential equation \( f \). Choosing an \( f \) so that \( f_c(x(0)) \neq 0 \) would imply that \( x_c(t) \neq x_c(0) \) for small \( t \), thus giving the result.

To see how we choose such an \( f \), first note that by Lemma 3.1.8 we need only find a \( g_c : P_I(c) \to P_c \) invariant under \( B(c,c) \) such that \( g_c(x(0)) \neq 0 \), as then \( g_c \) would extend to a vector field in \( \mathcal{F}_G^P(Q) \). An example is to take \( g_c(x) = x_c(0) \neq 0 \), which is invariant under \( B(c,c) \).\[\blacksquare\]

We will need one more notion of a balanced equivalence relation, one that ensures that the symmetric components show up the right amount of times under different equivalence relations. For example, if we were to partition the cells from Example 3.2.2 above into the classes \( \{0,1\} \), \( \{2,3,8\} \), \( \{4,5,6,7,9\} \) (making cell 6 gray instead of white), the associated polydiagonal would be \( Y' = \{(u,u,v,v,w,w,w,v,w)\} \), changing the restriction \( f|_Y \) in the second component to \( g_1 = A(u,u,w,w,v,w) \). This makes \( g_1 \) different from \( g_0 \), and we cannot reduce to the quotient three-cell network.

**Definition 3.2.9.** An equivalence relation \( \triangleright \) on \( C \) is said to be balanced if for all \( c, d \in C \) with \( c \triangleright d \) and \( c \neq d \), there exists \( \beta \in B(c,d) \) such that \( i \triangleright \beta(i) \) for all \( i \in I(c) \).

Note that \( B(c,d) \neq \emptyset \iff c \sim_I d \), so balanced equivalence relations refine \( \sim_I \). The reader can check that the equivalence relation for Example 3.2.2 is balanced, whereas the modified example in the discussion above is not. We can, actually, check whether or not an equivalence relation is balanced graphically as follows: color the cells so that two cells share the same color if they are in the same \( \triangleright \)-equivalence class. \( \triangleright \) is balanced if and only if every pair of \( \triangleright \)-equivalent cells is connected by a color preserving morphism.

We can now prove the theorem we stated at the beginning of this section:

**Theorem 3.2.10.** If \( \triangleright \) is an equivalence relation on a coupled cell network, then \( \triangleright \)
is robustly polysynchronous if and only if $\bowtie$ is balanced.

**Proof.** This proof is fairly involved, and in the interest of space we will omit it (consult [11]).

### 3.3 Quotient Networks

We noted that in Example 3.2.2, the restriction $f|_Y$ on the coupled cell network $G_1$ where $Y$ was a polysynchronous subspace gave an admissible vector field for a so-called “quotient” coupled cell network $G_2$. We will more formally go through these notions in this section.

**Definition 3.3.1.** Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network and $\bowtie$ denote a balanced equivalence relation. The **quotient network** $G_{\bowtie} = (\mathcal{C}_\bowtie, \mathcal{E}_\bowtie, \sim_{C\bowtie}, \sim_{E\bowtie})$ corresponding to the polysynchronous subspace $\Delta_\bowtie$ has components constructed as follows:

(a) **Cells:** If $[c]$ denotes the $\bowtie$-equivalence class of $c \in \mathcal{C}$, then $\mathcal{C}_\bowtie = \{[c] : c \in \mathcal{C}\} = \mathcal{C}/\bowtie$.

(b) **Edges:** $\sim_{C\bowtie}$ is given as follows: $[c] \sim_{C\bowtie} [d] \iff c \sim_C d$. This is well-defined because $\bowtie$ refines $\sim_C$.

(c) **Equivalence of cells:** $\mathcal{E}_\bowtie = \{([i], [c]) : (i, c) \in \mathcal{E}, i \neq c\} \cup \{([c], [c]) : c \in \mathcal{C}\}$.

(d) **Equivalence of edges:** $\sim_{E\bowtie}$ is given as follows: let $(j, d) \in \mathcal{E}_\bowtie$, and $[c] = d$ for $c \in \mathcal{C}$.

Define

$$\Omega_c(j) = \{i \in I(c) : [i] = j\}.$$ 

If $(j_1, d_1), (j_2, d_2) \in \mathcal{E}_\bowtie$, then $(j_1, d_1) \sim_{E\bowtie} (j_2, d_2)$ if and only if for some $c_1, c_2 \in \mathcal{C}$ with $[c_1] = d_1$ and $[c_2] = d_2$ there exists $\beta \in B(c_1, c_2)$ such that

$$\beta(\Omega_{c_1}(j_1)) = \Omega_{c_2}(j_2).$$

Pictorially, $\sim_{E\bowtie}$ can be thought of as follows. If we shade the cells of $\mathcal{C}$ according to their equivalence classes under $\bowtie$, the set $\Omega_c(j)$ consists of cells in the input set $I(c)$ with color $j$. The arrows $(j_1, d_1)$ and $(j_2, d_2)$ are $\sim_{E\bowtie}$ equivalent if there is an input equivalence of $I(c_1)$ to $I(c_2)$ that maps cells of color $j_1$ to cells of color $j_2$, and in particular the number of cells should match. Note that, however, $\sim_{E\bowtie}$ does not depend on the choice of $c_1$ and $c_2$.

**Lemma 3.3.2.** If $\bowtie$ is balanced, $c_1, c_2, c'_1, c'_2 \in \mathcal{C}$ with $c_1 \bowtie c'_1$ and $c_2 \bowtie c'_2$, $j_1, j_2 \in \mathcal{C}_\bowtie$, and if there exists $\beta \in B(c_1, c_2)$, $\beta(\Omega_{c_1}(j_1)) = \Omega_{c_2}(j_2)$, then there exists $\beta' \in B(c'_1, c'_2) : \beta'(\Omega_{c'_1}(j_1)) = \Omega_{c'_2}(j_2)$.

**Proof.** By definition of balanced, $\exists \gamma_k \in B(c_k, c'_k)$ for $k = 1, 2$ with the property that $\gamma_k(i) \bowtie i$ for all $i \in I(c_k)$, which gives us $\gamma_k(\Omega_{c_k}(j_k)) = \Omega_{c'_k}(j_k)$. $\beta' = \gamma_2 \gamma_1^{-1}$ is an input isomorphism under which $\beta'(\Omega_{c'_1}(j_1)) = \Omega_{c'_2}(j_2)$. ■
It remains to verify that $G_{\bowtie}$ is indeed a coupled cell network, which we accomplish by verifying the two compatibility conditions:

**Compatibility condition 1:** If $(j_1, d_1) \sim_{E_{\bowtie}} (j_2, d_2)$, then $j_1 \sim_{C_{\bowtie}} j_2$ and $d_1 \sim_{C_{\bowtie}} d_2$. Let $c_1, c_2 \in C$ with $[c_1] = d_1$ and $[c_2] = d_2$. By definition of $\sim_{E_{\bowtie}}$, there exists $\gamma_1 \in B(c_1, c_2)$ such that $\gamma_1(\Omega_{c_1}(j_1)) = \Omega_{c_2}(j_2)$, and since $\gamma_1$ is an input isomorphism, it preserves cell type, and in particular $c_1 \sim_C j_2$. By definition of $\sim_{C_{\bowtie}}$, $d_1 \sim_{C_{\bowtie}} d_2$. Now choose any $i \in \Omega_{c_1}(j_1)$. Then $\gamma_1(i) \in \Omega_{c_2}(j_2)$, and $i \sim_C \gamma_1(i) \Rightarrow j_1 \sim_{C_{\bowtie}} j_2$.

**Compatibility condition 2:** $(j_1, j_1) \sim_{E_{\bowtie}} (j_2, d_2) \iff j_2 = d_2$ and $j_2 \sim_{C_{\bowtie}} j_1$. This is left to the reader.

We will prove later that the restriction of each $G$-admissible vector field to $\Delta_{\bowtie}$ is a $G_{\bowtie}$-admissible vector field, which involves the notion of quotient maps.

**The Symmetry Groupoid of the Natural Quotient.** $G/ \bowtie$ can be thought of in terms of $G$ and $\bowtie$ as follows. Let

$$\Sigma^\bowtie(c, d) := \{\sigma \in B(c, d) : \sigma(i) \bowtie i \forall i \in I(c)\},$$

$$T^\bowtie = \{\tau \in B(c, d) : i \bowtie j \iff \tau(i) \bowtie \tau(j) \forall i, j \in I(c)\}.$$  

Define two subgroupoids $\Sigma^\bowtie, T^\bowtie \subseteq B_G$ as follows:

$$\Sigma^\bowtie = \bigcup_{c, d \in C} \Sigma^\bowtie(c, d),$$

$$T^\bowtie = \bigcup_{c, d \in C} T^\bowtie(c, d).$$

Then $T^\bowtie$ consists precisely of the $\bowtie$-compatible elements of $B_G$, and $B_{G/\bowtie}$ consists precisely of the bijections induced on $C/ \bowtie$ by $T^\bowtie \subseteq B_G$. The elements of $\Sigma^\bowtie$ are, furthermore, the identity on $C/ \bowtie$, and $B_{G/\bowtie} = T^\bowtie/\Sigma^\bowtie$, with $T^\bowtie$ being the normalizer of $\Sigma^\bowtie$ in $B_G$ (see [2], [11]).

Note that admissible vector fields under the quotient network $G_{\bowtie}$ may not always lift to admissible vector fields under the original network $G$ (for examples and characterizations, see [2]). Now let’s generalize the notion of quotient networks by looking at them under quotient maps.

**Definition 3.3.3.** Let $G_1$ and $G_2$ be coupled cell networks. The map $\phi : C_1 \to C_2$ is a **quotient map** if $\phi$ satisfies the following:

(a) **Cells lift:** $\phi$ is surjective.

(b) **Input arrows lift:** $(i, c) \in E_1 \Rightarrow (\phi(i), \phi(c)) \in E_2$. If $(j, d) \in E_2$ and $c \in C_1$ with $\phi(c) = d$, then $\exists i \in C_1 : \phi(i) = j$ and $(i, c) \in E_1$. 

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Lemma 3.3.4. Let \( \phi : C_1 \rightarrow C_2 \) be a quotient map. Then the following are true:

(a) If \( \phi(c) = d \), then \( \phi(I(c)) = I(d) \).

(b) For every \( c, c' \in C_1 \) with the property that \( \phi(c) = \phi(c') \), there is an input isomorphism \( \beta \in B(c, c') \) such that \( \phi(i) = \phi(\beta(i)) \) for all \( i \in I(c) \).

(c) The equivalence relation \( \bowtie \) is balanced.

Proof. (a) and (b) follow from parts (b) and (c), respectively, of the definition of quotient map. Since we know that \( \beta \in B(c, c') \) exists by (b), \( c \sim_{C_1} c' \) and \( \phi(i) = \phi(\beta(i)) \Rightarrow i \bowtie \beta(i) \) by definition of the equivalence relation \( \bowtie \), so \( \bowtie \) is balanced by definition.

Quotient maps do indeed generalize quotient networks:

Theorem 3.3.5. If \( \bowtie \) is a balanced equivalence relation on \( C \), \( C_{\bowtie} \) is the natural network whose cells are the equivalence classes of \( \bowtie \), and if \( [c] \) denotes the \( \bowtie \)-equivalence class of the cell \( c \in C \), then the map \( \phi : C \rightarrow C_{\bowtie} \) given by \( \phi : c \mapsto [c] \) is a quotient map.

Proof. We need only verify properties (a) through (c) in Definition 3.3.3 above. (a) and (b) basically follow by definition. We now show that input isomorphisms lift, e.g. that if \( d, d' \in C_2, \beta_1 \in B(d, d') \), and we choose \( c, c' \in C_1 \) with \( \phi(c) = d \) and \( \phi(c') = d' \), then \( \exists \beta_1 \in B(c, c') : \beta_2(\phi(i)) = \phi(\beta_1(i)) \) for all \( i \in I(c) \). It suffices to find a \( \beta_1 : I(c) \rightarrow I(c') \) satisfying the above.

To do this, we first show that input sets lift, e.g. \( \phi(I(c)) = I(\phi(c)) \) for \( c \in C \). Let \( j \in I(\phi(c)) \). By (b) in the definition, \( \exists i', c' \in C : \phi(i') = j, \phi(c') = \phi(c), \) and \( (i', c') \in E \). \( c' \) and \( c \) are \( \bowtie \)-equivariant under \( C_{\bowtie} \), and since \( \bowtie \) is balanced \( \exists \gamma \in B(c', c) : i = \gamma(i') \bowtie i' \Rightarrow \gamma(i', c') = (i, c) \) and \( \phi(i) = j \). So input sets lift, as desired.

Now we show the existence of the appropriate \( \beta_1 \). Note that from the discussion in the previous session we can choose a finite set \( J \) with \( j \in J : \)

\[
I(c) = \bigsqcup_{j \in J} \Omega_c(j),
\]

and since input sets lift, we have

\[
I(c') = \bigsqcup_{j \in J} \Omega_{c'}(\beta_2(j)).
\]
Corollary 3.3.9. The following are true (see [2], [5], [11]):

\( \beta_1 \) is then the union of all bijections \( \beta_1|_{\Omega_c(j)} : \Omega_c(j) \to \Omega_c'(\beta_2(j)) \) where \( j \) runs through \( J \). Since \( \beta_2 \in B(d,d') \), we have \( )|d \sim_{E_{\infty}} (\beta_2(j),d') \), and by definition of \( \sim_{E_{\infty}} \) we have \( \gamma \in B(c,d') : \gamma(\Omega_c(j)) = \Omega_c'(\beta_2(j)) \implies \beta_2(i) = \phi(\gamma(i)) \) for all \( i \in \Omega_c(j) \), and setting \( \beta_1|_{\Omega_c(j)} = \gamma \) gives the result.

Quotient networks need not be unique (see [11] for an example), but all other quotient networks are quotients of \( G_{\infty} \) of a trivial kind in that distinct cells remain distinct.

**Definition 3.3.6.** Let \( \gamma : J \to K \) be a bijection between \( J,K \subset C \), and let \( \propto \) be an equivalence relation on \( C \). \( \gamma \) is said to be \( \propto \)-compatible if for all \( j_1,j_2 \in J \), \( j_1 \propto j_2 \iff \gamma(j_1) \propto \gamma(j_2) \).

\( \gamma \) can be thought of as a permutation of \( \propto \)-equivalence classes. The stronger condition of “balanced” has \( i \propto \gamma(i) \), in which \( \gamma \) fixed \( \propto \)-equivalence classes.

Such bijections \( \gamma \), moreover, arise naturally. If \( \phi : G \to G' \) is any quotient map and \( \beta \in B_G' \) is an input isomorphism, then by definition of quotient there must exist a lift \( \tilde{\beta} \in B_G \) which implies that \( \tilde{\beta} \) is \( \propto_{\phi} \)-compatible with \( J = I(c) \), \( K = I(d) \), and thus \( \tilde{\beta} \in B(c,d) \) if \( \beta \in B(\phi(c),\phi(d)) \subseteq B_{G'} \).

We can now take on an important theorem.

**Theorem 3.3.7.** If \( \propto \) is a balanced equivalence relation on \( C \), and \( G_{\infty} \) is the associated natural quotient network with quotient map \( \phi \), then the pair \( (G_{\infty},\phi) \) is universal. That is, if \( G' \) is a coupled cell network with quotient map \( \phi' \) with \( \propto_{\phi'} = \propto \), then there is a quotient map \( \zeta : G_{\infty} \to G' \) such that \( \phi'(c) = \zeta(\phi(c)) \) for all \( c \in C \). In this situation we say that \( (G',\phi') \) factors through \( (G_{\infty},\phi) \).

**Proof.** Let \( [c] \in C_{\infty} \), and define \( \zeta([c]) = \phi'(c) \). Then \( \zeta : G_{\infty} \to G' \) is well-defined because \( \propto_{\phi'} = \propto = \propto_{\phi} \), and is a bijection on cells. We can verify that \( \zeta \) is indeed a quotient map routinely.

One more definition allows us to state several corollaries.

**Definition 3.3.8.** Let \( \zeta : G \to G' \) be bijective on cells. Then \( G' \) is called an edge-refinement of \( G \) if \( \zeta(i,c) \sim_{E'} \zeta(j,d) \Rightarrow (i,c) \sim_E (j,d) \).

**Corollary 3.3.9.** The following are true (see [2], [5], [11]):

(a) Every quotient network corresponding to a given balanced equivalence relation \( \propto \) is an edge-refinement of the natural quotient \( G_{\infty} \).

(b) Every edge refinement of \( G_{\infty} \) is a quotient network that corresponds to \( \propto \).

(c) Let \( G',G'' \) be edge refinements of \( G_{\infty} \). Then \( G'' \) is an edge-refinement of \( G' \) if and
only if $B_{G'} \supseteq B_{G''}$.

(d) $B_{G'} \supseteq B_{G''} \iff \mathcal{F}^P_{G'} \subseteq \mathcal{F}^P_{G''}$, for any choice of phase space $P$ on cells, where the cells in $G', G''$ are identified if they correspond to the same $\cong$-class of cells in $G$.

**Induced vector fields are admissible.** We now show that any quotient map $\phi : G_1 \to G_2$ “converts” $G_1$-admissible vector fields into $G_2$-admissible vector fields in a natural way.

Let $\Delta_\phi$ denote the polydiagonal subspace corresponding to the equivalence relation $\cong_\phi$; we want to show that the space of $G_1$-admissible vector fields restricted to $\Delta_\phi$ can be identified with a subspace of the space of $G_2$-admissible vector fields. The main consequence of this statement is that certain dynamics in the subspace for the cell system $G_2$ can correspond to the same dynamics in the cell system $G_1$, in which $\cong$-equivalent cells are synchronous.

We’ll motivate this discussion. Let $P_c$ denote the corresponding cell phase space for $c \in C_1$. Then $\phi(c) \in C_2$, and we let the corresponding cell phase space in $C_2$ be $P_{\phi(c)} = P_c$.

(Note that the space $P_{\phi(c)}$ is well-defined because quotient maps preserve $\sim_c$.)

Now choose a set $\mathcal{R}$ with $\mathcal{R} \subseteq C_1$ and for each $d \in C_2$ there exists a unique $c \in \mathcal{R}$ such that $\phi(c) = d$. So the set of all $\phi(c)$ runs through the elements of $C_2$ exactly once when $c$ runs through $\mathcal{R}$. We can then define

$$\overline{P} = \prod_{c \in \mathcal{R}} P_{\phi(c)} = \prod_{c \in \mathcal{R}} P_c.$$  

If $x = (x_c)_{c \in C_1}$ is a coordinate system on $P$, we can consider $y = (y_{\phi(c)})_{\phi(c) \in C_2}$ as a coordinate system on $\overline{P}$. Furthermore, we have uniqueness: $\forall c \in C_1, \exists! r \in \mathcal{R} : \phi(c) = \phi(r)$, and we can then identify $y_{\phi(c)}$ with $y_{\phi(r)}$.

We would hope that a quotient map $\phi : G_1 \to G_2$ induces a natural mapping $\hat{\phi} : \mathcal{F}^P_{G_1} \to \mathcal{F}^P_{G_2}$, where $\overline{P}$ is given by identifying the relevant factors of $P$. This will be our main goal below.

**Quotients preserve admissibility.** We would like to prove that quotient maps induce admissible vector fields. If $\phi : G_1 \to G_2$ is a quotient map, then there exists an injective map $\alpha : P \to P$ defined by $\alpha(y)_c = y_{\phi(c)}$ for all $c \in C_1, y \in P$. $\alpha : P \to \Delta_\phi$ is actually a bijection since the image $\Delta_\phi$ corresponds precisely to the domain $\alpha(\overline{P})$. If we replace $y$ by $\alpha^{-1}x$ for $x \in \Delta_\phi$ and all $c \in C_1$, this becomes

$$(\alpha^{-1}x)_{\phi(c)} = x_c.$$  

**Definition 3.3.10.** Since $\Delta_\phi$ is invariant under $f \in \mathcal{F}^P_{G_1}$, we can define a vector field $\overline{f}$ on $\overline{P}$, which we call the *induced vector field*, by restricting $f$ to $\Delta_\phi$ and projecting the result onto $\overline{P}$ by $\alpha^{-1}$; e.g.

$$\overline{f}(y) = \alpha^{-1}(f(\alpha(y))).$$
for all \( y \in \overline{P} \). We can also write \( \bar{f} = \hat{\phi}(f) \).

This finally brings us to the following theorem:

**Theorem 3.3.11.** For any \( f \in \mathcal{F}_{G_1}^P \), the induced vector field \( \bar{f} \) lies in \( \mathcal{F}_{G_2}^P \).

*Proof.* \( f \) satisfies the domain and equivariance conditions of Definition 3.1.4. In particular, for all \( c \in C_1 \), \( \exists f_c : \)

\[
f_c(x) = \hat{f}_c(x_{I(c)}),
\]

and for all \( c, c' \in C_1 \) and \( \gamma \in B(c, c') \), we have \( \hat{f}_{c'}(x) = \hat{f}_c(\gamma^*(x)) \) for all \( x \in P_{I(c')} \), where

\[
(\gamma^*(x))_i = x_{\gamma(i)}
\]

for all \( i \in I(c) \) and is undefined everywhere else. We must show that \( \bar{f} \) satisfies the domain and equivariance conditions for \( G_2 \). For the former, start by supposing \( d \in C_2 \) and \( \phi(c) = d \). Then \( \bar{f}_d(y) = \bar{f}_{\phi(c)}(y) = (\bar{f}(y))_{\phi(c)} = (\alpha^{-1}(f(\alpha(y))))_{\phi(c)} = f_c(\alpha(y)) \). From (17), we know that \( f_c(\alpha(y)) \) depends on \( (\alpha(y))_{I(c)} \), but since \( \phi(I(c)) = I(d) \), we have \( (\alpha(y))_{I(c)} = y_{\phi(I(c))} = y_d \), as desired.

Before showing the latter condition, we first prove a lemma:

**Lemma 3.3.12.** Let \( d, d' \in C_2 \) and let \( \beta \in B(d, d') \). Choose \( c, c' \in C_1 \) such that \( \phi(c) = d, \phi(c') = d' \). Suppose that \( \beta \) lifts to \( \tilde{\beta} \in B(c, c') \). Then for all \( y \in \overline{P} \) we have

\[
\tilde{\beta}^*(\alpha(y)) = \alpha(\beta^*(y)).
\]

*Proof.* By steps justified previously, \( (\tilde{\beta}^*(\alpha(y)))_i = (\alpha(y))_{\tilde{\beta}(i)} = y_{\phi(\tilde{\beta}(i))} = (\beta^*(y))_{\phi(i)} = (\alpha(\beta^*(y)))_i \). ■

So we must show that for all \( d, d' \in C_2 \) and \( \beta \in B(d, d') \), we have

\[
\bar{f}_{d'}(y) = \bar{f}_d(\beta^*(y)),
\]

for all \( y \in P_{I(d')} \). Note that \( (\beta^*(y))_i = y_{\beta(i)} \) by definition.

Now choose \( c, c' \in C_1 \) with \( \phi(c) = d, \phi(c') = d' \). Lift \( \beta \) to \( \tilde{\beta} \) by \( \phi(\tilde{\beta}(i)) = \phi(\tilde{\beta}(i)) \). Since we know that \( f_{c'}(x) = f_c(\tilde{\beta}^*(x)) \), setting \( x = \alpha(y) \) and applying the lemma above we have that

\[
f_{c'}(\alpha(y)) = f_c(\alpha(\beta^*(y))).
\]

Then we have \( \bar{f}_{d'}(y) = (\alpha^{-1}(f(\alpha(y))))_{\phi(c')} = f_{c'}(\alpha(y)) \), and \( \bar{f}_d(\beta^*(y)) = \bar{f}(\beta^*(y))_d = (\alpha^{-1}(f(\alpha(\beta^*(y)))))_d = (f(\alpha(\beta^*(y))))_c = f_c(\alpha(\beta^*(y))) = f_c(\hat{\beta}^*(\alpha(y))) \), where the latter equality follows from the lemma above. Since \( f_c(\tilde{\beta}^*(\alpha(y))) = f_c(\alpha(y)) \), we’re done. ■

So we have proved that the induced vector field is admissible. This brings us to the end of the paper.
4 Conclusion

As a recap of this paper, we have reviewed the application of symmetry groupoids in coupled cell networks, which loosely speaking allows for the modeling of local symmetry and synchrony in directed graphs. Although the theory as presented in this paper may seem abstract, it can be widely applied to model physical phenomena. Again, for examples of such applications, see [1], [3], [4], [10], and the review of current literature below.

In the future, the author would like to apply the groupoid formalism developed above to directed graphs in biological contexts and illustrate some computational examples in genomics, evolutionary genetics, and molecular cell structure using the statistical programming language R in conjunction with the Bioconductor project.


References


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