Optimization Theory in Financial Engineering

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Abstract

In this paper, we survey the applications of optimization theory in finance by interpreting some tools of functional analysis. We first prove the hyperplane separation theorem in an economics context, then proceed to develop its applications in asset pricing and arbitrage. We also introduce calculus of variations as a powerful tool to solve related problems in economics, and to provide an extension of portfolio theory. Lastly, we apply some aspects of our theoretical results to historical market data.

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1 Introduction

Theoretical finance is a quantitative field guided by mathematical logic that seeks to solve finance-related optimization problems. Optimization theory in finance seeks to model factors such as portfolio allocation, market completeness, and stock trading strategies. Many results of classical optimization theory can be readily extended to similar fields such as economics.

In general, one may think of the standard optimization problem as maximizing a function denoted by $F(x)$ subject to some constraints $G(x) \leq c$ over a given unit, such as time. It will be crucial to define the notion of convexity to prove many results in optimization theory. Through such results, we can develop theories on arbitrage pricing, asset pricing, and portfolio theory.
This paper is organized as follows. Section 2 develops the tools of functional analysis, first in an economics perspective to motivate the reader, and then in a financial perspective. Section 3 applies functional analysis to theoretical finance. Section 4 develops the tools of calculus of variations and modern portfolio theory through a few examples. Section 5 is a brief, applied sampling of our theory using statistical methods.

2 Separating Hyperplanes

Let us remind ourselves of convexity and some associated definitions, then proceed to prove and interpret the hyperplane separation theorem in economics.

Definition 2.1. Given a vector space $V$ and a field $F$, a functional is a map $f : V \rightarrow F$.

Definition 2.2. A set $S$ in a vector space $V$ is convex if and only if given $x_1, x_2 \in S$, all points of the form $\alpha x_1 + (1 - \alpha) x_2$ for $\alpha \in [0, 1]$ are in $S$.

Definition 2.3. The translation of a subspace is a linear variety.

Definition 2.4. If $S$ is a nonempty subset of a vector space $V$, then the set $v(S)$, the intersection of all linear varieties in $V$ that contain $S$, is called the linear variety generated by $S$.

Definition 2.5. A hyperplane $F$ in a linear vector space $X$ is a maximal proper linear variety; that is, a linear variety $H$ such that $H \neq X$, and if $V$ is any linear variety containing $H$, then either $V = X$ or $V = H$.

Due to [10], we can provide an economics-based proof of the Eidelheit separation theorem. The strategy is as follows. We desire an output, $y$, from an $n$-dimensional input vector $x$. Define our "production set" as $S$; $S$ is closed and $(n + 1)$-dimensional. So $(y, x) \in S$. Let $(y^*, x^*)$ be an "efficient point," or boundary point of $S$. Suppose the induction assumption that there exists an $n$-dimensional price vector $p_\alpha$ that causes the input vector $\hat{x}$ to minimize the production cost. We desire to prove that there exists an $(n + 1)$-dimensional price vector $P = (P_y, P_x)$ that minimizes the production cost of the input-output combination $(\hat{y}, \hat{x})$, and from an economic standpoint this seems feasible: if the choice of inputs could be decentralized through cost minimization by naming the costs of outputs, then there should exist an output price that maximizes the profits from a choice of outputs and inputs. This price can be constructed thus:

Denote the marginal cost of an input price $p_x$ and an output $\hat{y}$ as $MC(\hat{y}|p_x)$. Then $p_y = MC(\hat{y}|p_x)$, and $p_y$ is the marginal trade-off. This price should support the output $\hat{y}$. The $(n + 1)$-dimensional vector that supports $(\hat{y}, x)$ is $P := (p_y, -p_x)$; this is equivalent to asserting that $(\hat{y}, \hat{x})$ maximizes $p_y y - p_x x$ for $(y, x) \in S$.

Using this reasoning as a guide and induction of the number of input dimensions, we will prove a version of the Eidelheit separation theorem:

Lemma 2.1. If $K \subseteq \mathbb{R}^1$ is closed and convex, then $K$ has one of the following three forms: $(-\infty, p_1], [p_2, \infty)$, or $[p_3, p_4]$, where $p_1, p_2, p_3, p_4$ are finite boundary points of $K$ and $p_3 \leq p_4$.

Lemma 2.2. (Separating Hyperplane Theorem in one dimension). If $K \subseteq \mathbb{R}^1$ is closed and convex with boundary $b$, then there exists a scalar $\alpha \neq 0$ such that $\alpha x < ab$ for $x \in K$.

Lemma 2.3. Let $f(y)$ be a convex function defined over the closed one-dimensional convex set $K$. Then, for any $a \in K$, there exists a number $f'(a)$ such that, for all $y \in K$, $f(y) - f(a) \geq f'(a)(y - a)$. 

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The proofs of these lemmas are readily apparent and will be omitted.

**Theorem 2.4.** (Separating Hyperplane Theorem, general version) Let $K \subset \mathbb{R}^m$ be closed and convex. Let $\hat{z}$ be any boundary point of $K$. Then there exists an $m$-dimensional vector $P \neq 0$ such that, for any $z \in K$, $\langle P, z \rangle \leq \langle P, \hat{z} \rangle$.

**Proof.** This is a proof by induction. The base case $m = 1$ is true by Lemma 2.2. Now suppose the theorem holds for $n$; we must show that the theorem holds for $m = n + 1$.

Let $z$ be an $(n + 1)$-dimensional vector, $x$ be an $n$-dimensional vector, and $y$ be a scalar. Let $z = (y, x)$. Define the $n$-dimensional set $X(y)$ as $X(y) := \{ x : (y, x) \in K \}$. $X(y)$ is called an input set. Since $K$ is convex and closed, $X(y)$ is also convex and closed. Since $(y, \hat{x})$ is a boundary point of $K$, $\hat{x}$ is a boundary point of $X(\hat{y})$. By our assumption for dimension $n$, there exists an input price vector $p_x \neq 0$ such that, for $x \in X(\hat{y})$, $\langle p_x, x \rangle \leq \langle p_x, \hat{x} \rangle$.

Now, consider $A := \{ y : X(y) \neq \emptyset \}$. $A$ is closed and convex, so it takes one of three forms asserted by Lemma 2. Define the cost function induced by input prices $p_x$ as

$$C_{p_x}(y) := \min_{x \in X(y)} \langle p_x, x \rangle.$$

Since $\langle p_x, x \rangle \leq \langle p_x, \hat{x} \rangle$, $C_{p_x}(\hat{y}) = \langle p_x, \hat{x} \rangle$, $C_{p_x}(y)$ is also convex; for the minimum $C_{p_x}(y)$ incurred by $\hat{x}$ and $\hat{x}'$ that corresponds to $y$ and $y'$, $(\lambda y + (1 - \lambda)y', \lambda \hat{x} + (1 - \lambda)\hat{x}') \in K$ for $\lambda \in [0,1]$. Because $C_{p_x}(y)$ is a minimum function,

$$C_{p_x}(\lambda y + (1-\lambda)y') \leq \langle p_x, (\lambda \hat{x} + (1-\lambda)\hat{x}') \rangle = \lambda \langle p_x, \hat{x} \rangle + (1-\lambda)\langle p_x, \hat{x}' \rangle = \lambda C_{p_x}(y) + (1-\lambda)C_{p_x}(y').$$

By Lemma 2.2 and our definition of $C_{p_x}(y)$, we can define the "marginal cost of producing $\hat{y}$" as the $p_y$ that satisfies, for all $y \in K$, the following condition: $C_{p_x}(y) - C_{p_x}(\hat{y}) \geq p_y(y - \hat{y})$. Also, for $(y, x) \in K$, $C_{p_x}(y) \geq \langle p_x, x \rangle$. So

$$\langle p_x, x \rangle \geq \langle p_x, \hat{x} \rangle \geq p_y(y - \hat{y}).$$

Define the $(n + 1)$-dimensional price vector $P := (p_y, -p_x)$. $p_x \neq 0$, so $P \neq 0$. The latter two equations, along with $z = (y, x)$, result in

$$\langle P, z \rangle \leq \langle P, \hat{z} \rangle,$$

for $z \in K$. So our inductive hypothesis holds for $m = n + 1$. 

**Remark 1.** Our proof of the hyperplane separation theorem leads us to the concept of **shadow prices**, prices that support optimal production outcomes by decentralized profit maximization. Lagrange multipliers are often used to compute optimal outcomes in market situations (see section 4).

**Remark 2.** The hyperplane separation theorem is a specific case of the more general Hahn-Banach Theorem, which will be omitted from this paper.

### 3 Applications of Separating Hyperplanes

The separating hyperplanes theorem is a powerful tool in financial modeling, and can be used to prove theorems in asset pricing. We will proceed to develop some on these results below.

#### 3.1 Arbitrage Pricing

In the spirit of [4], we will start to develop the financial aspects of this paper, starting with arbitrage pricing. Assume a single period model; mainly, that the market is only observable.
at time zero and a future, fixed time $T$. If there are $n$ tradable assets in the market, their prices at time $t = 0$ can be defined as

$$S_0 = (S_0^1, S_0^2, ..., S_0^n)^T,$$

where $T$ denotes taking the transpose of the matrix. To account for market fluctuations, we let the time variable $t$ take on values from 1 to $m$; each of these time values represents different states of the market. We can encompass both the time variable and the price variable in an $n \times m$ matrix $D = (D_{ij})$, where $D_{ij}$ denotes the value of the $i$th asset at market state $j$. A portfolio of assets can be represented by a vector $\theta = (\theta_1, ..., \theta_n) \in \mathbb{R}$, with asset weights $\theta_i$, whose market value at time $t$ is $\langle S_t, \theta \rangle$.

In general, the portfolio value $\Omega$ can be expressed as the matrix

$$\Omega = D^T \theta = \left( \sum_{i=1}^{n} D_{i1} \theta_i \right) \begin{array}{l} \vdots \\ \sum_{i=1}^{n} D_{im} \theta_i \end{array}.$$

Now we are ready to develop the concept of arbitrage and state price vectors.

**Definition 3.1.1.** An arbitrage is a portfolio $\theta \in \mathbb{R}^n$ with either $\langle S_0, \theta \rangle \leq 0$ and $D^T \theta > 0$ or $\langle S_0, \theta \rangle < 0$ and $D^T \theta \geq 0$.

**Definition 3.1.2.** A state price vector is a vector $\psi \in \mathbb{R}^n_{++)}$ such that $S_0 = D\psi$.

The existence of state prices implies the existence of a positive linear pricing rule. We see that this definition is well-motivated by expanding:

$$\begin{pmatrix} S_0^1 \\ \vdots \\ S_0^n \end{pmatrix} = \psi_1 \begin{pmatrix} D_{11} \\ \vdots \\ D_{n1} \end{pmatrix} + ... + \psi_n \begin{pmatrix} D_{1m} \\ \vdots \\ D_{nm} \end{pmatrix}.$$

The vector $D^{(i)}$ represents the security price vector when the market is in state $i$. If the market is at state $i$ at the end of the time period, $\Omega$ increases by one unit for each $\psi_i$ of investment at time $t = 0$; we see this by considering a portfolio $\{\theta^{(i)}\}_{1 \leq i \leq n}$ such that

$$\theta^{(i)} \cdot D^{(j)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

In other words, we can define a delta function such that we receive additional profits if the end time is indeed $i = j$, and we receive no additional profits otherwise. Then, according to the equation above, the cost of purchasing $\theta^{(i)}$ at time $t = 0$ is

$$S_0 \cdot \theta^{(i)} = (\sum_{j=1}^{n} \psi_j D^{(j)}) \cdot \theta^{(i)} = \psi_i.$$

The portfolios represented by the set $\{\theta^{(i)}\}_{1 \leq i \leq n}$ are called Arrow–Debreu securities. We are now ready to prove a simple form of the Fundamental Theorem of Asset Pricing.

**Theorem 3.1.1.** (Riesz Representation Theorem) Any bounded linear functional on $\mathbb{R}^n$ can be written as $F(x) = \langle v_0, x \rangle$, for $v_0 \in \mathbb{R}^n$.

**Theorem 3.1.2.** (Separating Hyperplane Theorem, for linear separation of cones) Suppose $M$ and $K$ are closed convex cones in $\mathbb{R}^n$ that intersect precisely at 0. If $K$ is not a linear subspace, then there is a nonzero linear function $f$ such that $f(x) < f(y)$ for each $x \in M$ and nonzero $y \in K$. 

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The proofs of Theorems 3.1.1 and 3.1.2 are omitted.

**Theorem 3.1.3.** (Fundamental Theorem of Asset Pricing) The following are equivalent:

(i) the absence of arbitrage

(ii) the existence of a state price vector

(iii) the existence of an optimal portfolio

**Proof.** *(i) ⇔ (ii)* [3] and [4] provides us with a clean proof. Let \( L = \mathbb{R} \times \mathbb{R}_n \) and set \( M = \{ ( - S_0, \theta ), D^t \theta : \theta \in \mathbb{R}^N \} \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{1+n}, K = \mathbb{R}_+ \times \mathbb{R}_n^*. \)

Note that \( K \) is a cone and not a linear space because it does not contain any negative elements, while \( M \) is a linear space. There is no arbitrage if and only if \( K \) and \( M \) intersect precisely at the origin (show this by negating the definition of arbitrage). We must prove that \( K \cap M = \{ 0 \} \) if and only if there is a state price vector.

For one direction, assume that \( K \cap M = \{ 0 \} \). From the hyperplane separation theorem, there is a linear function \( F : L \rightarrow \mathbb{R} \) such that \( F(y) < F(x) \) for all \( y \in M \) and non-zero \( x \in K \).

We first show that \( F(y) = 0 \) for all \( y \in M \). \( M \) is a linear space, so by definition \( F(0) = 0 \) for \( x \in K \) and \( F(x) > 0 \) for \( x \in K/\{ 0 \} \). Let \( x_0 \) be nonzero, with \( x_0 \in K \). Then, by the hyperplane separation theorem, \( F(y) < F(x_0) \) for any \( y \in M \), but since \( M \) is a linear space, by linearity \( cF(y) = F(cy) < F(x_0) \) for all \( c \in \mathbb{R} \). This can only hold if \( F(y) = 0 \).

We now construct a state price vector from \( F \). By the Riesz Representation Theorem, \( F \) can be expressed as \( F(x) = \langle v_0, x \rangle \) for some \( v_0 \in L \). Write \( v_0 = (\alpha, \phi) \) where \( \alpha \in \mathbb{R} \) and \( \phi \in \mathbb{R}_n \), so that \( F(a, b) = \alpha \alpha + \langle \phi, b \rangle \) for any \( (a, b) \in L \).

Since \( F(x) > 0 \) for all nonzero \( x \in K \), \( \alpha > 0 \) and \( \phi > 0 \). But \( F(y) = 0 \) for all \( y \in M \), so

\[
-\alpha \langle S_0, \theta \rangle + \langle \phi, D^t \theta \rangle = 0, \quad \forall \theta \in \mathbb{R}^N.
\]

\[
\langle \phi, D^t \theta \rangle = \langle D\phi, \theta \rangle, \quad -\alpha \langle S_0, \theta \rangle + \langle D\phi, \theta \rangle = 0, \quad \forall \theta \in \mathbb{R}^N.
\]

This implies that \( -\alpha S_0 + D\phi = 0 \). So \( S_0 = D(\phi/\alpha) \), and \( \psi = \phi/\alpha \) is a state price vector.

To prove the other direction, suppose that there is a state price vector \( \psi \); we must show that \( K \cap M = \{ 0 \} \). \( S_0 = D\psi \) by definition, so for any portfolio \( \theta \),

\[
\langle S_0, \theta \rangle = \langle D\psi, \theta \rangle = \langle \psi, D^t \theta \rangle.
\]

Suppose that \( ( - S_0, \theta ), D^t \theta \in K \) for some \( \theta \). Then \( D^t \theta \in \mathbb{R}_n^* \), and \( -\langle S_0, \theta \rangle \geq 0 \). But since the state price vector exists, \( \psi > 0 \). If \( D^t \theta \in \mathbb{R}_n^+ \), then \( \langle \phi, D^t \theta \rangle \leq 0 \). By the above equation, \( \langle S_0, \theta \rangle \geq 0 \). Thus \( \langle S_0, \theta \rangle = 0 \) and \( D^t \theta = 0 \), showing that \( K \cap M = \{ 0 \} \), as desired.

We now wish to show that there are no arbitrage opportunities if and only if there is no optimal portfolio. If there existed arbitrage opportunities, the expected utility function would be unbounded because agents will prefer to take unbounded positions to profit from arbitrage. To this extent we consider an increasing utility function \( U : C \rightarrow \mathbb{R} \), with \( C \) defined to be the set of all consumption vectors \( c \) in \( S \) possible states. If the agent begins with an endowment \( e \), we can state the optimization problem as finding max \( U(c) \) subject to the conditions \( (c, e) + D^t \theta = (S_0, \theta) = 0 \) with \( c \geq 0 \) in all states. We must prove that there is a solution to the optimization problem if and only if there is no arbitrage and \( U \) is continuous.

We prove \((iii)\) in two steps, assuming for simplicity that any consumption vector \( c \) is one-dimensional.

\((i) \Rightarrow (iii)\) If there existed no arbitrage, any consumption \( c \) would be bounded by an upper limit \( M \). Since \( U \) is increasing, \( U(M) \) denotes a maximum value for the utility function.
(iii) ⇒ (i) Let $\max U(c)$ denote the solution to the optimization problem above. If an arbitrage existed, all agents who prefer more to less are made better off, and in general a change in consumption $\Delta c$ is positive. But since $U$ is increasing, $\Delta c > 0$ implies that, for $c' > c$, $\max U(c') > \max U(c)$. So we have a contradiction. ■

If we introduce the notion of martingales, the Fundamental Theorem can be restated.

**Definition 3.1.3.** A **martingale** with respect to a probability $P^*$ on a set of scenarios $\Omega$ is a sequence of random variables $X(0), X(1), X(2), ...$ such that, for each $n = 0, 1, 2, ..., E^*(X(n + 1)|S(n)) = X(n)$. 

**Theorem 3.1.4.** (Fundamental Theorem of Asset Pricing, with martingales) [2] The no-arbitrage condition is equivalent to the existence of a probability $P^*$ on the set of scenarios $\Omega$ such that $P^*(\omega) > 0$ for each scenario $\omega \in \Omega$ and the discounted stock prices $\tilde{S}_i(n) = \frac{S_i(n)}{A(n)}$ satisfy 

$$E^*(\tilde{S}_j(n + 1)|S(n)) = \tilde{S}_j(n),$$

for any $j = 1, ..., m$ and $n = 0, 1, ..., $ where $E^*(\cdot|S(n))$ denotes the conditional expectation with respect to probability $P^*$ computed once the stock price $S(n)$ becomes known at time $n$.

We omit the proof of this version.

**Remark 1.** The condition above is identical to having the discounted stock prices $\tilde{S}_j(0)$, $\tilde{S}_j(1)$, $\tilde{S}_j(2)$, ... form a martingale with respect to $P^*$, which is called a **risk-neutral or martingale probability** on the set of scenarios $\Omega$. $E^*$ is called a **risk-neutral or martingale expectation**.

### 3.2 Examples

The Fundamental Theorem of Asset Pricing allows us to analyze the possibility of arbitrage opportunities. Consider the following binomial model example:

**Example 3.1.1.** Let the money market prices $A$ be $A(0) = 50$, $A(1) = 60$, $A(2) = 80$, and suppose that stock prices can follow four scenarios:

- $\omega_1 : S(0) = 40, S(1) = 60, S(2) = 85$,
- $\omega_2 : S(0) = 40, S(1) = 60, S(2) = 40$,
- $\omega_3 : S(0) = 40, S(1) = 30, S(2) = 45$,
- $\omega_4 : S(0) = 40, S(1) = 30, S(2) = 25$.

Is there an arbitrage opportunity?

**Solution.** The probability tree can be represented as follows:
where the branching probabilities at each node are represented by \( p^*, q^*, \) and \( r^* \). We then apply Theorem 3.1.4 to see that, for \( \bar{S} = S(n)/A(n) \),

\[
\frac{60}{60} p^* + \frac{30}{60} (1 - p^*) = \frac{40}{50},
\]
\[
\frac{85}{80} q^* + \frac{40}{80} (1 - q^*) = \frac{60}{60},
\]
\[
\frac{45}{80} r^* + \frac{25}{80} (1 - r^*) = \frac{30}{60}.
\]

The solution to this system is \( (p^*, q^*, r^*) = (3/5, 8/9, 3/4) \). The risk-neutral probability for each path through the tree is therefore

\[
P^*(\omega_1) = p^* q^* = (3/5)(8/9) = 8/15,
\]
\[
P^*(\omega_2) = p^*(1 - q^*) = (3/5)(1 - 8/9) = 1/15,
\]
\[
P^*(\omega_3) = (1 - p^*) r^* = (1 - 3/5)(3/4) = 3/10,
\]
\[
P^*(\omega_4) = (1 - p^*)(1 - r^*) = (1 - 3/5)(1 - 3/4) = 1/10,
\]

Because such a risk-neutral probability exists, there is no arbitrage.

**Example 3.1.2.** We will consider another example that relates more to the current world. Consider the period from January 3, 2011, to February 28, 2011, in which equity traders were considering the stock NFLX. Assume an investment of $17500. Then, the money market prices are

\[
\begin{align*}
A(0) &= 17500, \\
A(1) &= 17614, \\
A(2) &= 17725, \\
A(3) &= 17832, \\
A(4) &= 17947.
\end{align*}
\]

Investors were speculating on the movement of NFLX stock, and suggested that stock prices could follow the scenarios below:

\[
\begin{align*}
\omega_1 : S(0) &= 175, S(1) = 185, S(2) = 215, S(3) = 230, S(4) = 240 \\
\omega_2 : S(0) &= 175, S(1) = 185, S(2) = 215, S(3) = 230, S(4) = 205 \\
\omega_3 : S(0) &= 175, S(1) = 185, S(2) = 215, S(3) = 200, S(4) = 210 \\
\omega_4 : S(0) &= 175, S(1) = 185, S(2) = 215, S(3) = 200, S(4) = 185 \\
\omega_5 : S(0) &= 175, S(1) = 185, S(2) = 170, S(3) = 180, S(4) = 200 \\
\omega_6 : S(0) &= 175, S(1) = 185, S(2) = 170, S(3) = 180, S(4) = 175 \\
\omega_7 : S(0) &= 175, S(1) = 185, S(2) = 170, S(3) = 160, S(4) = 180 \\
\omega_8 : S(0) &= 175, S(1) = 185, S(2) = 170, S(3) = 160, S(4) = 150 \\
\omega_9 : S(0) &= 175, S(1) = 165, S(2) = 180, S(3) = 190, S(4) = 200 \\
\omega_{10} : S(0) &= 175, S(1) = 165, S(2) = 180, S(3) = 190, S(4) = 170 \\
\omega_{11} : S(0) &= 175, S(1) = 165, S(2) = 180, S(3) = 160, S(4) = 170 \\
\omega_{12} : S(0) &= 175, S(1) = 165, S(2) = 180, S(3) = 160, S(4) = 150 \\
\omega_{13} : S(0) &= 175, S(1) = 165, S(2) = 140, S(3) = 150, S(4) = 165 \\
\omega_{14} : S(0) &= 175, S(1) = 165, S(2) = 140, S(3) = 150, S(4) = 140 \\
\omega_{15} : S(0) &= 175, S(1) = 165, S(2) = 140, S(3) = 135, S(4) = 140 \\
\omega_{16} : S(0) &= 175, S(1) = 165, S(2) = 140, S(3) = 135, S(4) = 120
\end{align*}
\]
Is there an arbitrage opportunity?

Solution. Assume that we invest $17,500 in NFLX. We apply Theorem 3.1.4 to see that, for $\bar{S} = S(n)/A(n)$,

\[
\begin{align*}
(18500/17614)a + (16500/17614)(1 - a) &= (17500/17500), \\
(21500/17725)b + (17000/17725)(1 - b) &= (18500/17614), \\
(18000/17725)c + (14000/17725)(1 - c) &= (16500/17614), \\
(23000/17832)d + (20000/17832)(1 - d) &= (21500/17725), \\
(18000/17832)e + (16000/17832)(1 - e) &= (17000/17725), \\
(19000/17832)f + (16000/17832)(1 - f) &= (18000/17725), \\
(15000/17832)g + (13500/17832)(1 - g) &= (14000/17725), \\
(24000/17947)h + (20500/17947)(1 - h) &= (23000/17832), \\
(21000/17947)i + (18500/17947)(1 - i) &= (20000/17832), \\
(20000/17947)j + (17500/17947)(1 - j) &= (18000/17832), \\
(18000/17947)k + (15000/17947)(1 - k) &= (16000/17832), \\
(20000/17947)l + (17000/17947)(1 - l) &= (19000/17832), \\
(17000/17947)m + (15000/17947)(1 - m) &= (16000/17832), \\
(16500/17947)n + (14000/17947)(1 - n) &= (15000/17832), \\
(14000/17947)o + (12000/17947)(1 - o) &= (13500/17832),
\end{align*}
\]

The solution to this system is $(a, b, c, d, e, f, g, h, i, j, k, l, m, n, o) = (.557, .359, .651, .543, .551, .703, .390, .757, .652, .246, .368, .708, .552, .439, .794)$. The risk-neutral probability for each path through the tree is therefore greater than zero, and there are no arbitrage opportunities.

Remark 1. Observe that, if in $\omega_{16}$ $S(f)$ had been 120000 rather than 120, $o$ would have been greater than 1 and an arbitrage opportunity would have existed.

Remark 2. $\omega_2$ was the real path of the stock.

Example 3.1.3. [5] Assume a zero interest rate, a stock with price 175 that will certainly end at time $\tau$ in the range from 165 to 185, and a call option with strike 175 costing 30. Is there an arbitrage opportunity?

Solution. Denote $\Theta$ as the set of all possible outcomes ($\theta_1, ..., \theta_n$), so that $D^T\Theta$ represents the price of the $n$ assets at outcome $\theta_i$. Theorem 3.1.3 is equivalent to the following statement, presented here without proof: **Arbitrage exists if and only if $S_0$ is not in the smallest closed cone containing the range of $D^T\Theta$.** If $S_t$ is point in the cone closest to $S_0$, then $\psi = S_t - S_0$ is an arbitrage.

Denote the unit vectors in the bond, stock, and call directions as $b, s$, and $c$. So let $S_0 = (1, 175, 30), \Theta = [165, 185]$, and $D^T\Theta = (1, \theta, \max\{\theta - 175, 0\})$. We will show that there is indeed an arbitrage opportunity in this model. The smallest cone containing the range of $D^T(\Theta)$ is spanned by $D^T(165) = (1, 165, 0), D^T(175) = (1, 175, 0)$, and $D^T(185) = (1, 185, 10)$. $S_0$ does not belong to this cone because it lies above the plane determined by the origin, $D^T(165)$, and $D^T(185)$. 

8
We now wish to find the arbitrage \( \psi \), which is described by the vector perpendicular to the plane described by the origin, \( D^T(165) \), and \( D^T(185) \). So \( \psi = (D^T(185) - 0) \times (D^T(165) - 0) = (1, 185, 10) \times (1, 165, 0) = -1650b + 10s - 20c. \)

Because \( \psi = (-165, 1, -2) \), we note an arbitrage opportunity: borrow 165 using the bond, buy one share of the stock, and sell two calls options. We would therefore have made
\[
-S_0 \cdot \psi = (-1, -1, -15, -30) \cdot (-165, 1, -2) = 165 - 175 + 60 = 50.
\]
At expiration the position would be liquidated to pay \( D^T(\Theta) \cdot \psi = -165 + \theta - 2 \max\{\theta - 175, 0\} = 10 - |175 - \theta| \geq 0 \) for \( 165 \leq \theta \leq 185 \).

\section{4 Lagrange Multipliers and Duality}

We now turn from the hyperplane separation and asset pricing theorems to calculus of variations. Calculus of variations is a powerful tool that will allow us to work with infinite-dimensional normed vector spaces, and to extend some results of portfolio theory.

\subsection{4.1 Optimal Investment Problems}

For motivation, consider the following short economics example from [8]. A firm makes profit \( P(x) \) using a capital stock of \( x \) units. Without loss of generality, assume that \( P \) is strictly increasing and twice-differentiable, with \( P(0) = 0 \). Denote the time space of \( P \) as \( [0, \tau] \), where 0 represents today and \( \tau \) represents the final date of the planning horizon. Capital accumulation is modeled by the mapping \( f : [0, \tau] \to \mathbb{R} \), with \( f \in C^1[0, \tau] \), \( f(x) \geq 0 \quad \forall x \), and \( f(0) = x_0 \), where \( x_0 \geq 0 \) represents the initial capital stock.

By increasing this stock at a given time by \( \epsilon > 0 \), we can also define a cost function \( C(\epsilon) \), where \( C \) is increasing and \( C(0) = 0 \). So at time \( t \), the investment cost of the firm is \( C(f'(t)) \), while the total profit is \( P(f(t)) - C(f'(t)) \). If the firm discounts the future according to a discount function \( D \) such that \( D' < 0 \) and \( D(0) = 1 \), we find that we have to maximize
\[
\int_0^\tau D(t)(P(f(t)) - C(f'(t))) \, dt,
\]
for \( f(0) = x_0 \) and \( f, f' \geq 0 \).

Suppose the firm has a target level of capital stock at time \( \tau \); denote this as \( x_\tau > 0 \). Now we desire to maximize
\[
\int_0^\tau D(t)(P(f(t)) - C(f'(t))) \, dt,
\]
for \( f(0) = x_0 \), \( f(\tau) = x_\tau \), and \( f' \geq 0 \). Assume that \( P \) is bounded and \( P(0) = 0 \), \( P' > 0 \), \( P'' \leq 0 \), and \( P'_\tau(0) = \infty \). Also assume that \( C(0) = C'(0) = 0 \), \( C'' > 0 \), and \( C''' > 0 \).

Euler-Lagrange then yields
\[
P'(f(t)) = -\frac{D'(t)}{D(t)} C'(f'(t)) - C''(f'(t)) f''(t),
\]
with \( t \in [0, \tau] \).

In economic terms, \( P'(f(t)) \) is the marginal benefit of the firm at time \( t \) that corresponds to the capital accumulation plan \( f \). \( -\frac{D'(t)}{D(t)} \) is the relative discount rate at time \( t \), so \( -\frac{D'(t)}{D(t)} C'(f'(t)) \) is the foregone interest on the capital invested. The last term, \( C''(f'(t)) f''(t) \), is the capital gains or losses that have occurred up to time \( t \). Thus, the result above asserts that a firm should balance marginal benefit and cost at any given point in time.

If the requirement that \( f'(\tau) = 0 \) is removed and an optimum is achieved with \( C'(f'(\tau)) = 0 \), then \( f'(\tau) = 0 \). This means that the firm should not make any additional investments at the final time \( \tau \), which is reasonable.
4.2 The Efficient Frontier of Multiple Assets in Portfolio Theory

In this section, we will present Markowitz’s mean-variance analysis in \( n \) assets, and consider the question of how to measure risk.

4.2.1 Review of Mean-Variance Analysis

[6] provides us with a concise review. Recall that, in Markowitz’s method of mean-variance analysis, we would like to consider the mean and a measure of risk, typically taken as the variance, of the returns of an asset. The mean portfolio return provides us with a concise review. Recall that, in Markowitz’s method of mean-variance analysis, we would like to consider the mean and a measure of risk, typically taken as the variance, of the returns of an asset. The mean portfolio return is the weighted linear combination of the average mean returns of each asset contained in the portfolio. If \( r \) denotes the average rate of return, we have \( r_P = w \cdot r \), where \( w = (w_1, \ldots, w_n) \) denotes the portfolio allocation and \( r = (r_1, \ldots, r_n) \) denotes the average returns of each \( n \) assets. We also have \( \sigma_P = \sqrt{w \cdot (C(r))} \), where \( \sigma_P \) denotes the portfolio risk and \( C(r) \) denotes the covariance matrix of individual asset returns.

Also recall the efficient frontier with two assets, which seeks to minimize \( \sigma_P \) for any given \( r_P \). Suppose we have two assets, so that \( r = (r_1, r_2) \); we can show that \( r_P \) takes on the form \( w_1r_1 + (1-w_1)r_2 \), and that the variance at \( r_P \) is equal to \( w_1^2\sigma_1^2 + 2w_1(1-w_1)v_{12} + (1-w_1)^2\sigma_2^2 \), for \( v_{xy} \) denoting the variance between \( x \) and \( y \). Differentiating, we see that the optimal allocation is

\[
w_1^* = \frac{v_{22} - v_{12}}{v_{11} + v_{22} - 2v_{12}\sqrt{v_{11}v_{22}}},
\]

4.2.2 Generalizing to Multiple Assets

Motivated by [9], we extend the two-asset frontier using Lagrange multipliers. Denote the returns of \( n \) assets as \( r = (r_1, \ldots, r_n)^T \), the portfolio weights as \( w = (w_1, \ldots, w_n)^T \), and expected returns as \( x = (x_1, \ldots, x_n)^T \). Also denote the \( n \times n \) matrix of return variances as \( V = (\sigma_{ij}) \), for \( i, j \in [1, n] \). We would then like to minimize the variance of \( r_P = \sum_{i=1}^n w_ir_i \), or \( \text{var}(r_n) = w^T V w \), subject to \( \sum_{i=0}^n w_i = 1 \) and \( w^T x = r \).

Let \( l = (1, \ldots, 1)^T \), a column of \( n \) ones. Using Lagrange multipliers, we see that

\[
L = w^T V w = \lambda_1(w^T x - r) - \lambda_2(w^T l^T - 1)
\]

\[
\frac{\partial L}{\partial w} = 0 \Rightarrow V w = (1/2)(\lambda_1 x + \lambda_2 l).
\]

So \( w = \frac{1}{2} V^{-1}(\lambda_1 x + \lambda_2 l) = \frac{1}{2} V^{-1}(x, l)(\lambda_1, \lambda_2)^T \). From this we see that \( (x, l)^T w = \frac{1}{2}(x, l)^T V^{-1}(x, l)(\lambda_1, \lambda_2)^T \), which implies

\[
\frac{1}{2}(\lambda_1, \lambda_2)^T = ((x, l)^T V^{-1}(x, l))^{-1}(x, l)^T w = ((x, l)^T V^{-1}(x, l))^{-1}(r, l)^T,
\]

so \( w = \frac{1}{2} V^{-1}(\lambda_1 x + \lambda_2 l) = V^{-1}(x, l)((x, l)^T V^{-1}(x, l))^{-1}(r, l)^T \) on the effective frontier. Furthermore, because the variance \( \sigma^2 = w^T V w \), \( \sigma^2 = (r, 1)((x, l)^T V^{-1}(x, l))^{-1}(r, 1)^T \). The efficient frontier equation is \( \sigma_F^2 = (x, p)^T ((x, l)^T V^{-1}(x, l))^{-1}(x, l)^T \).

Now, if we let \( A = ((x, l)^T V^{-1}(x, l)) \), we can show that \( A^{-1} \) takes on the form \( M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \), for \( a, b, c \in \mathbb{R} \);

\[
A^{-1} = ((x, l)^T V^{-1}(x, l))^{-1} = \begin{pmatrix} x^T V^{-1}x & x^T V^{-1}l \\ l^T V^{-1}l & l^T V^{-1}l \end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix} x^T V^{-1}x \theta & x^T V^{-1}l \\ l^T V^{-1}l \theta & l^T V^{-1}l \theta \end{pmatrix},
\]

where \( \theta = (x^T V^{-1}x)(l^T V^{-1}l) - (x^T V^{-1}l)^2 \) and \( \theta = (x^T V^{-1}l)(l^T V^{-1}l) - (x^T V^{-1}l)^2 \).
So the efficient frontier curve can also be written as \( \sigma_P^2 = c + 2bx_P + ax_P^2 \). We can rewrite this as
\[
\sigma_P^2 = c + 2bx_P + ax_P^2
\]
\[
\sigma_P^2 - (2bx_P + ax_P^2 + (b/a)^2) = c + (b/a)^2
\]
\[
\frac{\sigma_P^2}{c + (b/a)^2} - \frac{(ax_P + (b/a))^2}{c + (b/a)^2} = 1.
\]
Thus the frontier takes the shape of a hyperbola.

4.2.3 Alternative Risk Measures

What if, however, we decide to use other risk measures such as VaR (value at risk) and GLS (gain-loss spread)? We will generalize the motion of risk measures as a mappings in this section, and examine the risk measure of conditional value at risk in the next.

In general, given a set of returns \( \Phi \), the risk function is a mapping \( R : \Phi \to \mathbb{R} \) that quantifies a property of the return-space. Observe that \( R \) is a functional, and as a result it is contained in the space \( \mathfrak{R} \) dual to \( \Phi \). It may be interesting to note that \( \mathfrak{R} \) is isomorphic to \( \Phi \) as a result.

Assume that \( R \) is unbounded and convex, and that \( \Omega \subset \Phi \) is a convex subspace. In general, higher values of \( R(\phi) \) for \( \phi \in \Omega \) are less desired. This becomes an optimization problem; we must find \( \inf R(\phi) \) subject to a constraint \( G(\phi) \leq \theta \) and \( \phi \in \Omega \). The mapping \( G : \Phi \to \mathbb{R} \) in this case will be a convex returns function. We can then invoke Lagrange’s duality theorem to redefine our problem, presented here without proof:

**Theorem 4.2.1.** (Lagrange Duality) [7] Let \( R \) be a real-valued convex functional defined on a convex subset \( \Omega \) of a vector space \( \Phi \), and let \( G \) be a convex mapping of \( \Phi \) into \( \mathbb{R} \) endowed with the standard inner product. Suppose that there exists an \( \phi_1 \) such that \( G(\phi_1) < 0 \) and that \( \mu_0 = \inf \{ R(\phi) : G(\phi) \leq \theta, \phi \in \Omega \} \) is finite. Then
\[
\inf_{G(\phi) \leq \theta, \phi \in \Omega} R(\phi) = \max_{z^* \leq \theta} \psi(z^*),
\]
and the maximum on the right is achieved by some \( z_0^* \leq \theta \). If the infimum on the left is achieved by some \( \phi_0 \in \Omega \), then \( \langle G(\phi_0), z_0^* \rangle = 0 \) and \( \phi_0 \) minimizes \( R(\phi) + \langle G(\phi), z_0^* \rangle \) for \( \phi \in \Omega \). \( \psi(z^*) = \inf_{\phi \in \Omega} \{ R(\phi) + \langle G(\phi), z^* \rangle \} \) is called a dual functional.

Interpretatively, this means that, for a given possibility of portfolios, the problem of finding the portfolio with lowest risk subject to returns at or below a certain mark is equivalent to the problem of maximizing, for some other return below or equal to the desired return (call this the alternate return), the smallest possible value of the risk and the product of desired and alternate returns.

Modern financial theory has more explicitly defined our \( \mathfrak{R} \) space as spaces of convex risk measures and convex deviation measures.

**Definition 4.2.1.** [1] The function \( R : \Phi \to \mathbb{R} \) is a convex risk measure if the following properties are fulfilled:

1. translation invariance: \( R(\phi + b) = R(\phi) - b \) for all \( \phi \in \Phi, b \in \mathbb{R} \)
2. monotone: if \( X \leq Y \), then \( R(X) \geq R(Y) \) for all \( X, Y \in \Phi \)
3. convexity: \( R(\lambda X + (1 - \lambda)Y) \leq \lambda R(X) + (1 - \lambda)R(Y) \) for all \( X, Y \in \Phi, \lambda \in [0, 1] \).

**Definition 4.2.2.** [1] The function \( d : \Phi \to \mathbb{R} \) is a convex deviation measure if the following properties are fulfilled:

1. translation invariance: \( d(\phi + b) = d(\phi) - b \) for all \( \phi \in \Phi, b \in \mathbb{R} \)
2. strictness: if \( d(\phi) \geq 0 \) for all \( \phi \in \Phi \).
Recent advances in theoretical finance have used Lagrangian and conjugate duality methods to test optimal portfolio strategies. One recent work, by Haugh, Kogan, and Wang (2006), evaluated non-optimal portfolio strategies by computing upper and lower bounds on the expected utility of the optimal strategy. We will, however, not pursue dual methods for portfolio optimization in this paper.

5 Implementation using Statistical Methods

In this last section, we will apply some aspects of the theory we have developed to test certain examples using \( R \).

5.1 Optimizing Multiple Asset Portfolios

Let the returns of \( n \) assets be denoted, as \( r = (r_1, ..., r_n)^T \), the portfolio weights as \( w = (w_1, ..., w_n)^T \), the expected returns as \( x = (x_1, ..., x_n)^T \), the \( n \times n \) matrix of return variances as \( V = (\sigma_{ij}) \), for \( i,j \in [1,n] \), and \( l = (1, ..., 1)^T \), a column of \( n \) ones From Section 4.2.2, we observed that the efficient frontier curve of \( n \) assets can be expressed in the form

\[
\sigma_P^2 = ax_P^2 + 2bx + c,
\]

for

\[
a = \frac{l^TV^{-1}l}{(l^TV^{-1}l)^2 - (l^TV^{-1}x)^2},
\]

\[
b = \frac{x^TV^{-1}l}{(l^TV^{-1}l)^2 - (l^TV^{-1}x)^2},
\]

\[
c = \frac{x^TV^{-1}x}{(l^TV^{-1}l)^2 - (l^TV^{-1}x)^2}.
\]

Remark 1. Note that we have used variance as our risk metric. It would be a worthwhile exercise to find the efficient frontier curve for another risk metric.

In this section, we will illustrate an example of this model using historical data.

**Example 5.1.1.** Find the minimum risk portfolio of all stocks in the Nasdaq 100, using current market price.

**Solution.** Differentiating the equation of the hyperbola, we see that, for the minimum risk portfolio, the following is true:

\[
x_{min} = \frac{-b}{c} = \frac{x^TV^{-1}l}{(l^TV^{-1}l)^2 - (l^TV^{-1}x)^2}
\]

So

\[
w_{min} = \frac{x_{min} l}{x} = \frac{V^{-1}l}{l^TV^{-1}l}.
\]

We therefore have to compute the matrix \( V^{-1} \) for the Nasdaq stocks, then use the formula above to determine their weights. The variance is then given by \( \sigma_{min}^2 = w^TVw = \frac{1}{l^TV^{-1}l} \).

Our \( R \) code is given by
library(quantmod)
library(PerformanceAnalytics)

# Find the file with the list of stock symbols.
fname=file.choose()

# Read in the stock symbols.
stocknames=read.csv(fname,colClasses="character")

# Verify the data.
dim(stocknames)

# Find V:

n=100
z=1:n
for(i in 1:n) {
  stocks=getSymbols(stocknames[i,2], auto.assign=FALSE)
  z[i]=sd(monthlyReturn(stocks))
}
a_1=z*z[1]
a_2=z*z[2]
a_3=z*z[3]
a_4=z*z[4]
a_5=z*z[5]
a_6=z*z[6]
a_7=z*z[7]
a_8=z*z[8]
a_9=z*z[9]
a_10=z*z[10]
a_11=z*z[11]

# Find V^{-1}
library(MASS)
v=ginv(V)

# So the risk minimizing portfolio weight would be:
l=c(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1
l=matrix(l)
N=v %*% l
D=t(l) %*% v %*% l
w=N/D[1,1]
cbind(stocknames,w)

This code gives an output of

> cbind(stocknames,w)

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<th>Symbol</th>
<th>Weight</th>
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<td>60</td>
<td>Marvell Technology Group, Ltd.</td>
<td>MRVL</td>
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<tr>
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<td>MAT</td>
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<tr>
<td>62</td>
<td>Maxim Integrated Products, Inc.</td>
<td>MXIM</td>
</tr>
<tr>
<td>63</td>
<td>Microchip Technology Incorporated</td>
<td>MCHP</td>
</tr>
</tbody>
</table>
The variance of this portfolio is \( \sigma_{\min}^2 = w^T V w \).

> t(w) %*% V %*% w

[[1,1] 0.01554084

5.2 Calculating Conditional Value at Risk

In recent literature, RiskMetrics has provided the following measures of risk: standard deviation, value at risk, expected shortfall, marginal value at risk, incremental risk, and the more general coherent risk measures.

Perhaps expected shortfall, or conditional value at risk, is the most efficient measure. CVaR has better mathematical properties than the widely-used VaR (see Artzner et al.), mainly in that it is sub-additive and convex. Because CVaR, like historical calculation methods, do not assume normality, we can use both historical and Monte Carlo methods to
calculate it. We would, then, like to minimize

$$CVAR_\alpha = \frac{1}{\alpha} \int_0^\alpha VaR_\gamma(X) d\gamma,$$

where $X \in \Psi$ is the returns of a portfolio in the future, $\alpha \in [0, 1]$, and $VaR_\gamma$ is the value at risk.

Fortunately, the $R$ package *Performance Analytics* has a CVaR function built in as `ES`. Our job here will simply be to compare the performance of Nasdaq 100 stocks in regards to risk metrics of standard deviation, VaR, and CVaR.

**Example 5.2.1.** Consider a list of the Nasdaq 100 stocks. Compute the standard deviation, value at risk, and conditional value at risk for each stock, and rank them from least to most risk.

*Solution.*** Our $R$ code is as follows:

```r
library(quantmod)
library(PerformanceAnalytics)

# Find the file with the list of stock symbols.
fname=file.choose()

# Read in the stock symbols.
stocknames=read.csv(fname,colClasses="character")

# Verify the data.
dim(stocknames)

# First we rank by standard deviation:
n=100
sdvals=1:n
for(i in 1:n) {
  stockstd=getSymbols(stocknames[i,2], auto.assign=FALSE)
  retsstd=monthlyReturn(stockstd)
  sdvals[i] = sqrt(var(retsstd))
}

A=cbind(stocknames[,2],sdvals)
ord= order(A[,2],decreasing=F)
AA=A[ord,]

# Second we rank by value at risk, using the function given by the PerformanceAnalytics package:
n=100
VaRs=1:n
for(i in 1:n) {
  stockvar=getSymbols(stocknames[i,2], auto.assign=FALSE)
  retsvar=monthlyReturn(stockvar)
  VaRs[i]= VaR(retsvar)
}

B=cbind(stocknames[,2],VaRs)
ord= order(B[,2],decreasing=F)
```

```
# Third we rank by conditional value at risk, using the function given by the PerformanceAnalytics package:

```r
n=100
cVaRs=1:n
for(i in 1:n) {
stockvar=getSymbols(stocknames[i,2], auto.assign=FALSE)
retsvar=monthlyReturn(stockvar)
  CVaRs[i]= ES(retsvar)
}
D=cbind(stocknames[,2],CVaRs)
ord= order(D[,2],decreasing=F)
DD=D[ord,]
```

# Now that we have ranked the stocks according to all four measures, we juxtapose our rankings:
```
cbind(AA[,1],BB[,1],DD[,1])
```

Our final result is as follows, with the stocks ranked by standard deviation, value at risk, and conditional value at risk:

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6 Glossary of Finance Terms

We provide some common and concrete definitions of finance terms used in this paper below:

**arbitrage**: is the practice of making a profit by capitalizing on the imbalance or a price difference between two or more markets. It is the possibility of a risk-free profit at zero cost.

**asset pricing**: attempts to understand the prices of uncertain payments. It maps the general economy at time $t$ into the price of a capital asset at $t$.

**binomial/CRR model**: an options pricing model, named after Cox, Ross, and Rubinstein, that generates a binomial recombinant price tree.

**bond**: a debt security in which the authorized issuer owes a very specific payment to the holder, to be paid on a specific date of maturation (unless it is a bond in perpetuity, where the holder is paid interest indefinitely). In corporations, bondholders are paid before stockholders.

**bond interest rate**: also known as a ”coupon rate,” the bond interest rate is the percentage of the bond’s face value that the issuer pays, at regular intervals, to the bond holder.

**call options with strike**: a call is a financial contract between two parties in which the buyer has the right (but not the obligation) to buy a stock or other financial instrument from the seller at a given date in the future at a given price. The agreed-upon future price is called the ”strike price.”

**conditional value at risk/expected shortfall**: an alternative risk measure to value-at-risk because it is more sensitive to the shape of the loss distribution in the tail, used to evaluate the credit or market risk of a portfolio.
**convex deviation measure**: a mapping from the space of possible portfolios to $\mathbb{R}$ satisfying properties of translation invariance, strictness, and convexity.

**convex risk measure**: a mapping from the space of possible portfolios to $\mathbb{R}$ satisfying properties of translation invariance, monotone, and convexity.

**discounted stock prices**: the difference between an original price of a security and the price that it can fall to in the after-offering market. It is the amount that a security sells below its par value.

**duality methods**: given a primal optimization problem under certain constraints, it is possible to derive a dual problem that is identical to the primal.

**efficient frontier**: the set of all potential portfolios that an investor would consider, keeping in mind that there are many portfolios that can be statistically shown to be poor investments.

**Fundamental Theorem of Asset Pricing**: relates arbitrage opportunities with risk-neutral measures. The following three conditions are equivalent: absence of arbitrage, existence of a positive linear pricing rule (state prices), and the existence of an optimal portfolio for some agent who prefers more to less.

**martingale**: is a stochastic process in which the conditional expected value of an observation at time $t$ is equal to the observation at an earlier time $s$; it is a model of a fair game.

**money market prices**: the prices obtained when an investment is invested into money market instruments, such as certificates of deposit or treasury bills.

**positive linear pricing rule**: assigns positive price to all states with positive probability and zero price to all other states.

**RiskMetrics**: a variance model methodology first used by J.P. Morgan to measure portfolio risk. Several risk factors are taken into account, such as equity prices, foreign exchange rates, commodity prices, interest rates, correlation, and volatility.

**risk metric**: a way of interpreting measurements of volatility in a portfolio. Common risk metrics include beta, delta, gamma, and convexity.

**risk neutral/martingale probability measure**: is a measure that results when the assumption is made that the current value of all financial assets is equal to the expected future payoff of the asset discounted at the risk-free rate. This is used to price derivatives.

**shadow prices**: the marginal utility of relaxing a constraint or the marginal cost of strengthening a constraint. In other words, it is the maximum price that a firm would be willing to pay for an extra unit of a limited resource.

**state price vector**: also known as an Arrow-Debreu security, is a contract that pays one unit of a currency/commodity if a certain state occurs at a certain future time, and pays nothing in all other states. The price of the security is the state price, which can be represented as a vector.
References


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