Alternate definitions of the uncovered set and their implications

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1 Abstract

Different definitions of the uncovered set are commonly, and often interchangeably, used in the literature. If we assume individual preferences are strict over all alternatives, these definitions are equivalent. However, if one or more voters is indifferent between alternatives these definitions may not yield the same uncovered set. This note examines how these definitions differ in a distributive setting, where each voter can be indifferent between any number of alternatives. I show that, defined one way, the uncovered set is equal to the set of Pareto allocations that give over half the voters a strictly positive payoff, while alternate definitions yield an uncovered set that is equal to the entire Pareto set. These results highlight a small error in Epstein (1998) in which the author characterizes the uncovered set for a different definition of covering than claimed.

2 Introduction

Informally, an uncovered policy is any alternative that can majority-defeat any other policy in two or fewer steps. Miller (1980) was the first to define the uncovered set as a general solution concept, useful particularly in the absence of a core. He showed that, under the assumption of an

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odd number of players with strict preference orderings and a finite choice set, the uncovered set is a subset of the Pareto set and subsumes the set of possible sophisticated voting outcomes (the Banks set). The uncovered set has been of particular interest to social scientists because it has been observed that generally, and under many different institutional arrangements, strategic behavior by voters leads to outcomes in the uncovered set.[1]

Shepsle and Weingast (1984) examine finite agendas in a spatial setting and show that no outcome can be a sophisticated voting outcome if an alternative that covers it is also included in the agenda. Banks (1985) returns to Miller's framework of a finite alternative space and voters with strict preferences, and shows that while all sophisticated voting outcomes are uncovered, all uncovered points are not sophisticated voting outcomes. McKelvey (1986) extends some of Miller's results to the more general setting of a convex alternative space $X \subseteq \Re^N$, where voters have weak preferences over alternatives.

In more recent work, Duggan (2005) and Duggan and Jackson (2004) explore some properties of the "deep uncovered set," a weaker version of the uncovered set with nice continuity properties. Dutta, Jackson, and Le Breton (2004) provide some characterizations of the uncovered set, the Banks set and the top cycle set within the broad class of budget allocation problems. Epstein (1998) calculates the uncovered set for the class of distributive (Pareto constant) games and shows that it is equal to the set of all Pareto allocations that give over half the voters more than their minimum utility.[2] Laslier and Picard (2002) prove that the uncovered set equals the Pareto set in Pareto constant games where the covering relation is defined in terms of pluralities instead of majorities.[3] Penn (2004) shows that within three-player distributive games the uncovered set is equal to the Banks set, or set of sophisticated voting outcomes.

It has proven to be quite challenging to characterize the uncovered set analytically in settings more complex than a divide-the dollar game. However, there has recently been a resurgence in

[1]Banks, Duggan and Le Breton (2002) and McKelvey (1986), among others.

[2]Epstein proves his result using a different definition of covering than claimed.

[3]While the covering definition Laslier and Picard use is not defined in terms of majority rule, it is comparable to Definition (2) that follows.

interest in the uncovered set due to the progress made in computational processing speed. Bianco, Jeliazkov and Sened (2004) use computer simulation and NOMINATE scores to estimate the size, shape, and location of the uncovered set in six sessions of the U.S. House of Representatives, and they show that the concept has considerable explanatory power.

Interestingly, these authors do not use a consistent definition of C, the covering relation. Letting $\succ$ denote the strict majority preference relation, Miller, Banks, Penn, and Dutta, Jackson and Le Breton use the definition

$$yCx \text{ iff } y \succ x \text{ and } [x \succ z \Rightarrow y \succ z] \tag{1}$$

while Shepsle and Weingast, Cox, Bianco, Jeliazkov and Sened and Epstein use the definition

$$yCx \text{ iff } y \succ x \text{ and } [z \succ y \Rightarrow z \succ x] \tag{2}$$

and McKelvey and Jackson and Duggan use the definition

$$yCx \text{ iff } y \succ x \text{ and } [z \succ y \Rightarrow z \succ x] \text{ and } [x \succ z \Rightarrow y \succ z]. \tag{3}$$

Furthermore, the particular definition used in each instance does not appear to arise entirely from the assumed structure of individual preferences or the policy space. McKelvey and Shepsle and Weingast assume individuals have a weak preference ordering over a convex subset of $\Re^N$, yet use different definitions of the covering relation, although when preferences are strictly quasi-concave, Definitions (2) and (3) agree. Similarly, Epstein uses a different definition than Dutta, Jackson and Le Breton and Penn, while all examine Pareto constant games with linear preferences.

This note is not the first to highlight the differences between these definitions. Bordes, Le Breton and Salles (1992) call Definition (1) the Miller's subrelation and Definition (2) the Gillies's subrelation.[4] The authors refer to the uncovered set generated by Definition (1) as the "Miller's set" or "uncovered set." They refer to the uncovered set generated by Definition (2) as the "Gillies's set." They also provide existence theorems for both sets when the policy space is infinite. Bordes (1983) also provides a discussion of the three covering relations defined above, naming the three

[4]Gilles (1959) was the first to define this relation, which he referred to as *majorization*.

relations γ^d, γ^u, and γ, respectively. If the majority-preference relation is strict, then all three definitions agree. However if the majority-preference relation is weak, then the three definitions may or may not yield the same uncovered set.

The remainder of this note calculates the uncovered set under each definition of covering for the class of distributive, divide-the-dollar games. Note that Definition (3) is simply the composition of Definitions (1) and (2), so that if (1) and (2) are equivalent, they are also equivalent to (3). I will show that under Definition (1), the uncovered set is equal to the set of Pareto allocations that give over half the players a strictly positive payoff, while under Definitions (2) and (3), the uncovered set is equal to the entire Pareto set. This result highlights a small error in Epstein (1998), in which the author claimed that in Pareto constant games Definition (2) yields an uncovered set equal to the set of all Pareto allocations that give over half the players a strictly positive payoff.

3 The uncovered set in distributive games

In distributive games we consider the alternative space $\Delta = \{x \in [0,1]^N : \sum_{i=1}^{N} x_i = 1\}$. Let x_i denote the i^{th} component of a vector x, and let x^i be an element of Δ. For each i define $u_i : \Delta \to \Re$ by

$$u_i(x) = x_i \text{ for } x \in \Delta.$$

In words, we are looking at a divide-the-dollar game; preferences are assumed to be linear and the alternative space is the unit simplex.

Proposition 1 *Under Definition (1), the uncovered set is equal to the set of Pareto allocations that give over half the players a strictly positive payoff, while under Definitions (2) and (3), the uncovered set is equal to the entire Pareto set.*

Proof: The proof consists of two parts. Part I calculates the uncovered set under Definition (1), Part II calculates the uncovered set under Definitions (2) and (3).

Part I. (Proof by contradiction.) Assume the first definition of covering. First suppose that there exists a point $x \in U(\Delta)$ such that x gives over half the voters a payoff of zero. Then it must be the case that there exists no $y \in X$ such that $x \succ y$. This is because there exists no other point that gives over half the players a strictly lower payoff than x since x gives over half the players their minimum possible payoff. It follows that any point that is strictly majority-preferred to x covers x by the above definition. Consider the point y such that $y_i = \frac{1}{N}$ for all i. Then $y \succ x$ and if $x \succ z$ then $y \succ z$, trivially. Thus $x \notin U(\Delta)$, a contradiction.

Now suppose that there exists a point $x \notin U(\Delta)$ such that x gives over half the players a strictly positive payoff.[5] Order the players such that all players receiving strictly positive payoffs from policy x come first, so $x_i > 0$ for $i \leq K$, where $K \geq \frac{N+1}{2}$. Let y be a point that covers x. Then over half the players strictly prefer y to x and at least one player strictly prefers x to y, since every point in Δ is Pareto optimal. Let this be the first player. Furthermore, let $A = \sum_{i=2}^{\frac{N+1}{2}} y_i$. We know that A is strictly positive because since y is strictly preferred to x, at least one player i, where $i \in \{2, ..., K\}$, strictly prefers y to x. Now choose a point z such that $z_1 = y_1$, z_2 through $z_{\frac{N+1}{2}} = 0$, and, for $i \in \{\frac{N+3}{2}, ..., N\}$, $z_i = y_i + \frac{A}{\frac{N-1}{2}}$. Because $y_i \leq z_i$ for $i = 1, ..., \frac{N+1}{2}$, it follows that $z \succeq y$. However, $x_i > z_i$ for $i = 1, ..., \frac{N+1}{2}$ implies that $x \succ z$. Thus, y cannot cover x, a contradiction.

Part II. (Proof by contradiction.) Assume the second or third definition of covering. Suppose there exists a point $x \in \Delta$ that is covered. Order the players so that $x_1 \geq x_2 \geq ... \geq x_N$ and let y be a point that covers x. Therefore, over half the players strictly prefer y to x. Pick out $\frac{N-1}{2}$ of the players that strictly prefer y to x, and call this set of players S. Choose one player who strictly prefers x to y (since both points are Pareto optimal, we know such a player exists). Call her Player j. Let $A = \sum_{i \notin S \cup \{j\}} y_i$. We know that A is strictly positive because, since y is strictly preferred to x, there exists some player i, where $i \notin S \cup \{j\}$, such that $y_i > x_i$. Finally, choose some ϵ where $A > \epsilon > 0$ such that $x_j > y_j + \epsilon$.

Construct a point z such that for all $i \in S$, $z_i = y_i + \frac{A-\epsilon}{\frac{N-1}{2}}$, $z_j = y_j + \epsilon$, and for all others, $z_i = 0$.

[5] Adapted from Epstein's (1998) proof of Theorem 2.

Over half the players strictly prefer z to y, so $z \succ y$. However, $x_j > z_j$ and since $z_i = 0$ for $\frac{N-1}{2}$ players, it follows that $x \succeq z$. Thus, y cannot cover x, a contradiction. $\square$

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