Chapter 6

Dispersion

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The waves we've looked at so far in this book have been "dispersionless" waves, that is, waves whose speed is independent of ω and k. In all of the systems we've studied (longitudinal spring/mass, transverse string, longitudinal sound), we ended up with a wave equation of the form,

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}\,, \tag{1}$$

where c depends on various parameters in the setup. The solutions to this equation can be built up from exponential functions, $\psi(x,t) = Ae^{i(kx-\omega t)}$. Plugging this function into Eq. (1) gives

$$\omega^2 = c^2 k^2. \tag{2}$$

This is the so-called *dispersion relation* for the above wave equation. But as we'll see, it is somewhat of a trivial dispersion relation, in the sense that there is no dispersion. We'll explain what we mean by this below.

The velocity of the wave is $\omega/k = \pm c$, which is independent of ω and k. More precisely, this is the *phase velocity* of the wave, to distinguish it from the *group velocity* which we'll define below. The qualifier "phase" is used here, because the speed of a sinusoidal wave $\sin(kx - \omega t)$ is found by seeing how fast a point with constant phase, $kx - \omega t$, moves. So the phase velocity is given by

$$kx - \omega t = \text{Constant} \quad \Longrightarrow \quad \frac{d(kx-\omega t)}{dt} = 0 \quad \Longrightarrow \quad k\frac{dx}{dt} - \omega = 0 \quad \Longrightarrow \quad \frac{dx}{dt} = \frac{\omega}{k}\,, \tag{3}$$

as desired.

As we've noted many times, a more general solution to the wave equation in Eq. (1) is *any* function of the form $f(x - ct)$; see Eq. (2.97). So the phase velocity could reasonably be called the "argument velocity," because c is the speed with which a point with constant argument, $x - ct$, of the function f moves.

However, not all systems have the property that the phase velocity ω/k is constant (that is, independent of ω and k). It's just that we've been lucky so far. We'll now look at a so-called *dispersive* system, in which the phase velocity isn't constant. We'll see that things get more complicated for a number of reasons. In particular, a new feature that arises is the *group velocity*.

The outline of this chapter is as follows. In Section 6.1 we discuss a classic example of a dispersive system: transverse waves in a setup consisting of a massless string with discrete point masses attached to it. We will find that ω/k is not constant. That is, the speed of a wave depends on its ω (or k) value. In Section 6.2 we discuss *evanescent waves*. Certain dispersive systems support sinusoidal waves only if the frequency is above or below a certain cutoff value. We will determine what happens when these bounds are crossed. In Section 6.3 we discuss the *group velocity*, which is the speed with which a wave packet (basically, a bump in the wave) moves. We will find that this speed is *not* the same as the phase velocity. The fact that these two velocities are different is a consequence of the fact that in a dispersive system, waves with different frequencies move with different speeds. The two velocities are the same in a non-dispersive system, which is why there was never any need to introduce the group velocity in earlier chapters.

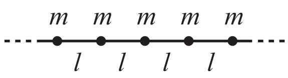

Figure 1

6.1 Beads on a string

Consider a system that is made up of beads on a massless string. The beads have mass m and are glued to the string with separation ℓ, as shown in Fig. 1. The tension is T. We'll assume for now that the system extends infinitely in both directions. The goal of this section is to determine what transverse waves on this string look like. We'll find that they behave fundamentally different from the waves on the continuous string that we discussed in Chapter 4. However, we'll see that in a certain limit they behave the same.

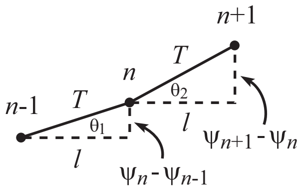

Figure 2

We'll derive the wave equation for the beaded string by writing down the transverse $F = ma$ equation on a given bead. Consider three adjacent beads, label by $n-1$, n, and $n+1$, as shown in Fig. 2. For small transverse displacements, ψ, we can assume (as we did in Chapter 4) that the beads move essentially perpendicular to the equilibrium line of the string. And as in Chapter 4, the tension is essentially constant, for small displacements. So the transverse $F = ma$ equation on the middle mass in Fig. 2 is (using $\sin\theta \approx \tan\theta$ for small angles)

$$\begin{aligned} m\ddot{\psi}_n &= -T\sin\theta_1 + T\sin\theta_2 \\ &= -T\left(\frac{\psi_n - \psi_{n-1}}{\ell}\right) + T\left(\frac{\psi_{n+1} - \psi_n}{\ell}\right) \\ \Longrightarrow \quad \ddot{\psi}_n &= \omega_0^2(\psi_{n+1} - 2\psi_n + \psi_{n-1}), \qquad \text{where } \omega_0^2 = \frac{T}{m\ell}. \end{aligned} \tag{4}$$

This has *exactly* the same form as the $F = ma$ equation for the longitudinal spring/mass system we discussed in Section 2.3; see Eq. (2.41). The only difference is that ω_0^2 now equals $T/m\ell$ instead of k/m. We can therefore carry over *all* of the results from Section 2.3. You should therefore reread that section before continuing onward here.

We will, however, make one change in notation. The results in Section 2.3 were written in terms of n, which labeled the different beads. But for various reasons, we'll now find it more convenient to work in terms of the position, x, along the string (as we did in Section 2.4). The solutions in Eq. (2.55) are linear combinations of functions of the form,

$$\psi_n(t) = \text{trig}(n\theta)\,\text{trig}(\omega t), \tag{5}$$

where each "trig" means either sine or cosine, and where we are now using ψ to label the displacement (which is now transverse). θ can take on a continuous set of values, because we're assuming for now that the string extends infinitely in both directions, so there's aren't any boundary conditions that restrict θ. ω can also take on a continuous set of values, but it must be related to θ by Eq. (2.56):

$$2\cos\theta \equiv \frac{2\omega_0^2 - \omega^2}{\omega_0^2} \quad \Longrightarrow \quad \omega^2 = 4\omega_0^2\left(\frac{1-\cos\theta}{2}\right) \quad \Longrightarrow \quad \omega = 2\omega_0 \sin\left(\frac{\theta}{2}\right). \tag{6}$$

Let's now switch from the $n\theta$ notation in Eq. (5) to the more common kx notation. But remember that we only care about x when it is a multiple of ℓ, because these are the locations of the beads. We define k by

$$kx \equiv n\theta \implies k(n\ell) = n\theta \implies k\ell = \theta. \quad (7)$$

We have chosen the $x = 0$ point on the string to correspond to the $n = 0$ bead. The $\psi_n(t)$ in Eq. (5) now becomes

$$\psi(x,t) = \mathrm{trig}(kx)\,\mathrm{trig}(\omega t). \quad (8)$$

In the old notation, θ gave a measure of how fast the wave oscillated as a function of n, In the new notation, k gives a measure of how fast the wave oscillates as a function of x. k and θ differ simply by a factor of the bead spacing, ℓ. Plugging $\theta = k\ell$ into Eq. (6) gives the relation between ω and k:

$$\boxed{\omega(k) = 2\omega_0 \sin\left(\frac{k\ell}{2}\right)} \qquad \text{(dispersion relation)} \quad (9)$$

where $\omega_0 = \sqrt{T/m\ell}$. This is known as the *dispersion relation* for our beaded-string system. It tells us how ω and k are related. It looks quite different from the $\omega(k) = ck$ dispersion relation for a continuous string (technically $\omega(k) = \pm ck$, but we generally don't bother with the sign). However, we'll see below that they agree in a certain limit.

What is the velocity of a wave with wavenumber k? (Just the phase velocity for now. We'll introduce the group velocity in Section 6.3.) The velocity is still ω/k (the reasoning in Eq. (3) is still valid), so we have

$$c(k) = \frac{\omega}{k} = \frac{2\omega_0 \sin(k\ell/2)}{k}\,. \quad (10)$$

The main point to note here is that this velocity depends on k, unlike in the dispersionless systems in earlier chapters. In the present system, ω isn't proportional to k.

We can perform a double check on the velocity $c(k)$. In the limit of very small ℓ (technically, in the limit of very small $k\ell$), we essentially have a continuous string. So Eq. (10) had better reduce to the $c = \sqrt{T/\mu}$ result we found in Eq. (4.5) for transverse waves on a continuous string. Or said in another way, the velocity in Eq. (10) had better not depend on k in this limit. And indeed, using $\sin\epsilon \approx \epsilon$, we have

$$c(k) = \frac{2\omega_0 \sin(k\ell/2)}{k} \approx \frac{2\omega_0 (k\ell/2)}{k} = \omega_0 \ell = \sqrt{\frac{T}{m\ell}}\,\ell = \sqrt{\frac{T}{m/\ell}} \equiv \sqrt{\frac{T}{\mu}}\,, \quad (11)$$

where μ is the mass density per unit length. So it does reduce properly to the constant value of $\sqrt{T/\mu}$. Note that the condition $k\ell \ll 1$ can be written as $(2\pi/\lambda)\ell \ll 1 \Longrightarrow \ell \ll \lambda$. In other words, if the spacing between the beads is much shorter than the wavelength of the wave in question, then the string acts like a continuous string. This makes sense. And it makes sense that the condition should involve these two lengths, because they are the only two length scales in the system.

If the $\ell \ll \lambda$ condition doesn't hold, then the value of ω/k in Eq. (10) isn't independent of k, so the beaded string apparently doesn't behave like a continuous string. What does it behave like? Well, the exact expression for ω in terms of k given in Eq. (9) yields the plot shown in Fig. 3.

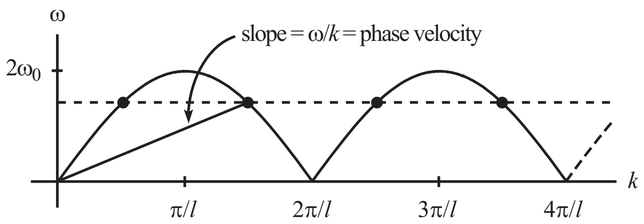

Figure 3

There are various things to note about this figure:

1. Given a value of k and its associated value of ω, the phase velocity of the wave is ω/k. But ω/k is the slope from the origin to the point (k, ω) in the figure, as shown. So the phase velocity has this very simple graphical interpretation. As we saw in Eq. (11), the slope starts off with a value of $\omega_0 \ell = \sqrt{T/\mu}$ near the origin, but then it decreases. It then repeatedly increases and decreases as the point (k, ω) runs over the successive bumps in the figure. As $k \to \infty$, the slope ω/k goes to zero (we'll talk more about this in the third comment below).

2. Note that $\omega_0 \equiv \sqrt{T/m\ell}$ can be written as $\omega_0 = (1/\ell)\sqrt{T/\mu}$, where $\mu = m/\ell$ is the mass density. So if we hold T and μ constant and decrease ℓ (for example, if we keep subdividing the masses, thereby making the string more and more continuous), then ω_0 grows like $1/\ell$. So the maximum height of the bumps in Fig. 3, which is $2\omega_0$, behaves like $1/\ell$. But the width of the bumps also behaves like $1/\ell$. So if we decrease ℓ while keeping T and μ constant, the whole figure simply expands uniformly. The linear approximation in Eq. (11) near the origin is therefore relevant for a larger range of k values. This means that the string behaves like a continuous string for more k values, which makes sense.

3. From Fig. 3, we see that many different values of k give the same ω value. In particular, k_1, $2\pi/\ell - k_1$, $2\pi/\ell + k_1$, etc., all give the same ω. However, it turns out that only the first half-bump of the curve (between $k = 0$ and $k = \pi/\ell$) is necessary for describing any possible motion of the beads. The rest of the curve gives repetitions of the first half-bump. The reason for this is that we care only about the movement of the beads, which are located at positions of the form $x = n\ell$. We don't care about the rest of the string. Consider, for example, the case where the wavenumber is $k_2 = 2\pi/\ell - k_1$. A rightward-traveling wave with this wavenumber takes the form,

$$\begin{aligned} A\cos(k_2 x - \omega t) &= A\cos\left(\left(\frac{2\pi}{\ell} - k_1\right)(n\ell) - \omega t\right) \\ &= A\cos\Big(2n\pi - k_1(n\ell) - \omega t\Big) \\ &= A\cos(-k_1 x - \omega t). \end{aligned} \tag{12}$$

We therefore conclude that a rightward-moving wave with wavenumber $2\pi/\ell - k_1$ and frequency ω gives exactly the same positions of the beads as a *leftward*-moving wave with wavenumber k_1 and frequency ω. (A similar statement holds with "right" and "left" reversed.) If we had instead picked the wavenumber to be $2\pi/\ell + k_1$, then a quick sign change in Eq. (12) shows that this would yield the same positions as a *rightward*-moving wave with wavenumber k_1 and frequency ω. The rightward/leftward correspondence alternates as we run through the class of equivalent k's.

It is worth emphasizing that although the waves have the same values at the positions of the beads, the waves look quite different at other locations on the string. Fig. 4 shows the case where $k_1 = \pi/2\ell$, and so $k_2 = 2\pi/\ell - k_1 = 3\pi/2\ell$. The two waves have common values at positions of the form $x = n\ell$ (we have arbitrarily chosen $\ell = 1$). The k values are in the ratio of 1 to 3, so the speeds ω/k are in the ratio of 3 to 1 (because the ω values are the same). The k_2 wave moves slower. From the previous paragraph, if the k_2 wave has speed v to the right, then the k_1 wave has speed $3v$ to the left. If we look at slightly later times when the waves have moved distances $3d$ to the left and d to the right, we see that they still have have common values at positions of the form $x = n\ell$. This is what Eq. (12) says in equations. The redundancy of the k values is simply the Nyquist effect we discussed at the end of Section 2.3, so you should reread that subsection now if you haven't already done so.

In comment "1" above, we mentioned that as $k \to \infty$, the phase velocity ω/k goes to zero. It is easy to see this graphically. Fig. 5 shows waves with wavenumbers $k_1 = \pi/2\ell$, and $k_2 = 6\pi/\ell - k_1 = 11\pi/2\ell$. The wave speed of the latter is small; it is only 1/11 times the speed of the former. This makes sense, because the latter wave (the very wiggly one) has to move only a small distance horizontally in order for the dots (which always have integral values of x here) to move appreciable distances vertically. A small movement in the wiggly wave will cause a dot to undergo, say, a full oscillation cycle. At the location of any of the dots, the slope of the k_2 wave is always -11 times the slope of the k_1 wave. So the k_2 wave has to move only 1/11 as far as the k_1 wave, to give the same change in height of a given dot. This slope ratio of -11 (at x values of the form $n\ell$) is evident from taking the $\partial/\partial x$ derivative of Eq. (12); the derivatives (the slopes) are in the ratio of the k's.

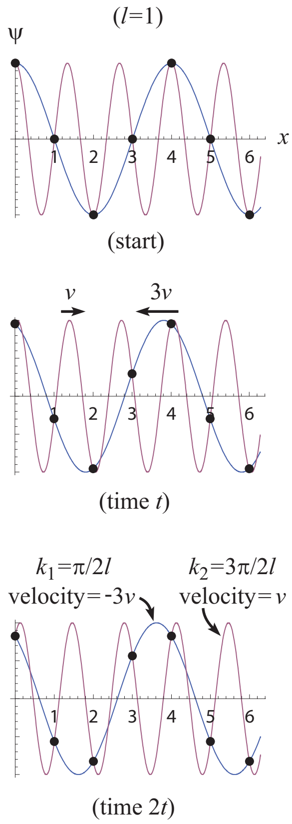

Figure 4

4. How would the waves in Fig. 4 behave if we were instead dealing with the dispersionless system of transverse waves on a continuous string, which we discussed in Chapter 4? (The length ℓ now doesn't exist, so we'll just consider two waves with wavenumbers k and $3k$, for some value of k.) In the dispersionless case, all waves move with the *same* speed. (We would have to be given more information to determine the direction, because $\omega/k = c$ only up to a $\pm$ sign.) The transverse-oscillation frequency of a given point described by the k_2 wave is therefore 3 times what the frequency would be if the point were instead described by the k_1 wave, as indicated in the straight-line dispersion relation in Fig. 6. Physically, this fact is evident if you imagine shifting both of the waves horizontally by, say, 1 cm on the paper. Since the k_2 wavelength is 1/3 the k_1 wavelength, a point on the k_2 curve goes through 3 times as much oscillation phase as a point on the k_2 wave. In contrast, in a dispersion*ful* system, the speeds of the waves don't have to all be equal. And furthermore, for the k values associated with the points on the horizontal line in Fig. 3, the speeds work out in just the right way so that the oscillation frequencies of the points don't depend on which wave they're considered to be on.

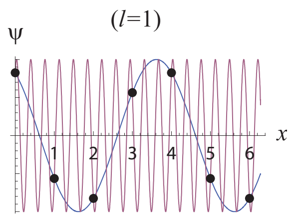

Figure 5

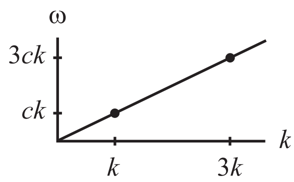

Figure 6

6.2 Evanescent waves

6.2.1 High-frequency cutoff

The dispersion relation in Eq. (9) follows from Eq. (6), which we derived back in Section 2.3.1 (although we didn't use the term "dispersion" there). The bulk of the derivation was contained in Claim 2.1. But recall that this claim assumed that $\omega \le 2\omega_0$. This is consistent with the fact that the largest value of ω in Fig. 3 is $2\omega_0$? However, what if we grab the end of a string and wiggle it sinusoidally with a frequency ω that is larger than $2\omega_0$. We're free to pick any ω we want, and the string will certainly undergo *some* kind of motion. But apparently this motion, whatever it is, isn't described by the above sinusoidal waves that we found for the $\omega \le 2\omega_0$ case.

If $\omega > 2\omega_0$, then the math in Section 2.3.1 that eventually led to the $\omega = 2\omega_0 \sin(k\ell/2)$ result in Eq. (9) is still perfectly valid. So if $\omega > 2\omega_0$, we conclude that $\sin(k\ell/2)$ must be greater than 1. This isn't possible if k is real, but it *is* possible if k is complex. So let's plug $k \equiv K + i\kappa$ into $\omega = 2\omega_0 \sin(k\ell/2)$, and see what we get. We obtain (the trig sum formula works fine for imaginary arguments)

$$\begin{aligned}\frac{\omega}{2\omega_0} &= \sin\left(\frac{K\ell}{2} - \frac{i\kappa\ell}{2}\right) \\ &= \sin\left(\frac{K\ell}{2}\right)\cos\left(\frac{i\kappa\ell}{2}\right) - \cos\left(\frac{K\ell}{2}\right)\sin\left(\frac{i\kappa\ell}{2}\right). \end{aligned} \tag{13}$$

By looking at the Taylor series for cosine and sine, the $\cos(i\kappa\ell/2)$ function is real because the series has only even exponents, while the $\sin(i\kappa\ell/2)$ function is imaginary (and nonzero) because the series has only odd exponents. But we need the righthand side of Eq. (13) to be real, because it equals the real quantity $\omega/2\omega_0$. The only way for this to be the case is for the $\cos(K\ell/2)$ coefficient of $\sin(i\kappa\ell/2)$ to be zero. Therefore, we must have $K\ell = \pi, 3\pi, 5\pi, \ldots$. However, along the same lines as the redundancies in the k values we discussed in the third comment in the previous section, the $3\pi, 5\pi, \ldots$ values for $K\ell$ simply reproduce the motions (at least at the locations of the beads) that are already described by the $Ka = \pi$ value. So we need only consider the $K\ell = \pi \Longrightarrow K = \pi/\ell$ value. Said in another way, if we're ignoring all the Nyquist redundancies, then we know that $k = \pi/\ell$ when $\omega = 2\omega_0$ (see Fig. 3). And since the real part of k should be continuous at $\omega = 2\omega_0$ (imagine increasing ω gradually across this threshold), we conclude that $K = \pi/\ell$ for all $\omega > 2\omega_0$. So $k \equiv K + i\kappa$ becomes

$$k = \frac{\pi}{\ell} + i\kappa. \tag{14}$$

Plugging $K = \pi/\ell$ into Eq. (13) yields

$$\frac{\omega}{2\omega_0} = (1)\cos\left(\frac{i\kappa\ell}{2}\right) - 0 \quad\Longrightarrow\quad \cosh\left(\frac{\kappa\ell}{2}\right) = \frac{\omega}{2\omega_0}. \tag{15}$$

This equation determines κ. You van verify the conversion to the hyperbolic cosh function by writing out the Taylor series for both $\cos(iy)$ and $\cosh(y)$. We'll keep writing things in terms of κ, but it's understood that we can solve for it by using Eq. (15).

What does the general exponential solution, $Be^{i(kx-\omega t)}$, for ψ look like when k takes on the value in Eq. (14)? (We could work in terms of trig functions, but it's *much* easier to use exponentials; we'll take the real part in the end.) Remembering that we care only about the position of the string at the locations of the masses, the exponential solution at positions of the form $x = n\ell$ becomes

$$\begin{aligned} \psi(x,t) &= Be^{i(kx-\omega t)} \\ \Longrightarrow\quad \psi(n\ell,t) &= Be^{i((\pi/\ell+i\kappa)(n\ell)-\omega t)} \\ &= Be^{-\kappa n\ell}e^{i(n\pi-\omega t)} \\ &= Be^{-\kappa n\ell}(-1)^n e^{-i\omega t} \\ &\to \boxed{Ae^{-\kappa n\ell}(-1)^n\cos(\omega t+\phi)} \end{aligned} \tag{16}$$

where we have taken the real part. The phase ϕ comes from a possible phase in B.[1] If we

[1] If you're worried about the legality of going from real k values to complex ones, and if you have your doubts that this ψ function actually does satisfy Eq. (4), you should plug it in and explicitly verify that it works, provided that κ is given by Eq. (15). This is the task of Problem [to be added].

want to write this as a function of $x = n\ell$, then it equals $\psi(x,t) = Ae^{-\kappa x}(-1)^{x/\ell}\cos(\omega t+\phi)$. But it is understood that this is valid only for x values that are integral multiples of ℓ. Adjacent beads are 180° out of phase, due to the $(-1)^n$ factor in ψ. As a function of position, the wave is an alternating function that is suppressed by an exponential. A wave that dies out exponentially like this is called an *evanescent wave*. Two snapshots of the wave at a time of maximal displacement (that is, when $\cos(\omega t+\phi) = 1$) are shown in Fig. 7, for the values $A = 1$, $\ell = 1$. The first plot has $\kappa = 0.03$, and the second has $\kappa = 0.3$. From Eq. (15), we then have $\omega \approx (2.02)(2\omega_0)$ in the latter, and ω is essentially equal to $2\omega_0$ in the former.

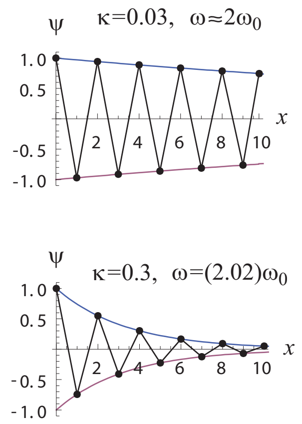

Figure 7

Since the time and position dependences in the wave appear in separate factors (and not in the form of a $kx - \omega t$ argument), the wave is a standing wave, not a traveling wave. As time goes on, each wave in Fig. 7 simply expands and contracts (and inverts), with frequency ω. All the beads in each wave pass through equilibrium at the same time.

REMARK: A note on terminology. In Section 4.6 we discussed *attenuated* waves, which are also waves that die out. The word "attenuation" is used in the case of actual sinusoidal waves that decrease to zero due to an envelope curve (which is usually an exponential); see Fig. 4.26. The word "evanescent" is used when there is no oscillatory motion; the function simply decreases to zero. The alternating signs in Eq. (16) make things a little ambiguous. It's semantics as to whether Eq. (16) represents two separate exponential curves going to zero, or a very fast oscillation within an exponential envelope. But we'll choose to call it an evanescent wave. At any rate, the system in Section 6.2.2 below will support waves that are unambiguously evanescent. ♣

We see that $2\omega_0$ is the frequency above which the system doesn't support traveling waves. Hence the "High-frequency cutoff" name of this subsection. We can have nice traveling-wave motion below this cutoff, but not above. Fig. 8 shows the $\psi(x,t) = A\cos(kx-\omega t+\phi)$ waves that arise if we wiggle the end of the string with frequencies ω equal to $(0.1)\omega_0$, ω_0, $(1.995)\omega_0$, and $2\omega_0$. The corresponding values of k are determined from Eq. (9). We have arbitrarily picked $A = 1$ and $\ell = 1$. The sine waves are shown for convenience, but they aren't really there. We haven't shown the actual straight-line string segments between the masses. At the $\omega = 2\omega_0$ cutoff between the traveling waves in Fig. 8 and the evanescent waves in Fig. 7, the masses form two horizontal lines that simply rise and fall.

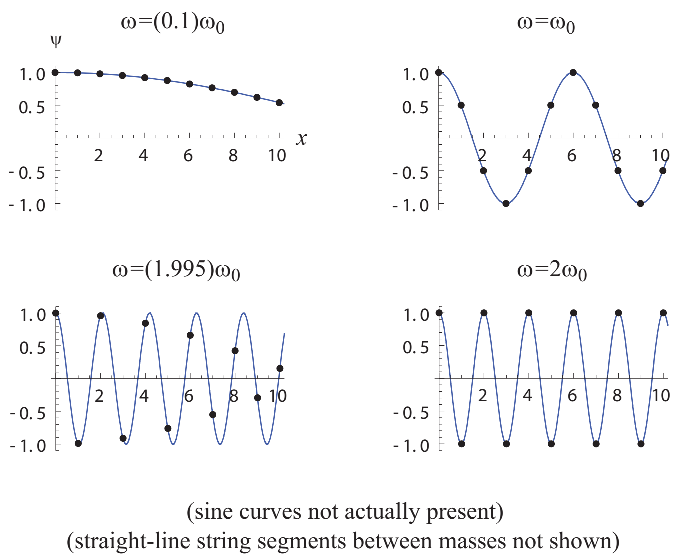

Figure 8

The cutoff case of $\omega = 2\omega_0$ can be considered to be either an evanescent wave or a traveling wave. So both of these cases must reduce to the same motion when $\omega = \omega_0$. Let's verify this. If we consider the wave to be an evanescent wave, then with $\omega = 2\omega_0$, Eq. (15) gives $\kappa = 0$. So there is no exponential decay, and the beads' positions in Eq. (16) simply alternate indefinitely to the right, in two horizontal lines. If we instead consider the wave to be a traveling wave, then with $\omega = 2\omega_0$, Eq. (9) gives $k\ell = \pi \Longrightarrow k = \pi/\ell$, which means that the wavelength approaches 2ℓ. The traveling wave at positions of the form $x = n\ell$ looks like

$$\begin{aligned} A\cos(kx - \omega t + \phi) &= A\cos\left(\frac{\pi}{\ell}\cdot n\ell - \omega t + \phi\right) \\ &= A\cos(n\pi - \omega t + \phi) \\ &= A\cos(n\pi)\cos(\omega t - \phi) \\ &= A(-1)^n\cos(\omega t - \phi), \end{aligned} \tag{17}$$

which agrees with Eq. (16) when $\kappa = 0$ (with a different definition of ϕ).

In the extreme case where $\omega \gg \omega_0$, Eq. (15) tells us that κ is large, which means that the exponential factor $e^{-\kappa n\ell}$ goes to zero very quickly. This makes sense. If you wiggle the end of the string very quickly (as always, we're assuming that the amplitude is small, in particular much smaller than the bead spacing), then the bead that you're holding onto will move with your hand, but all the other beads will hardly move at all. This is because they have essentially zero time to accelerate in one direction before you move your hand to the other side and change the direction of the force.

In practice, ω doesn't have to be much larger than $2\omega_0$ for this lack of motion to arise. Even if ω is only $4\omega_0$, then Eq. (15) gives the value of $\kappa\ell$ as 2.6. The amplitude of the $n = 1$ and $n = 2$ masses are then suppressed by a factors of $e^{-\kappa n\ell} = e^{-1(2.6)} \approx 1/14$, and $e^{-\kappa n\ell} = e^{-2(2.6)} \approx 1/200$. So if you grab the $n = 0$ mass at the end of the string and move it back and forth at frequency $4\omega_0$, you'll end up moving the $n = 1$ mass a little bit, but all the other masses will essentially not move.

Remark: For given values of T and μ, the relation $\omega_0 = (1/\ell)\sqrt{T/\mu}$ (see Eq. (11)) implies that if ℓ is very small, you need to wiggle the string very fast to get into the $\omega > 2\omega_0$ evanescent regime. In the limit of a continuous string ($\ell \to 0$), ω_0 is infinite, so you can never get to the evanescent regime. In other words, any wiggling that you do will produce a normal traveling wave. This makes intuitive sense. It also makes dimensional-analysis sense, for the following reason. Since a continuous string is completely defined in terms of the two parameters T (with units of kg m/s^2) and μ (with units of kg/m), there is no way to combine these parameters to form a quantity with the dimensions of frequency (that is, s^{-1}). So for a continuous string, there is therefore no possible frequency value that can provide the cutoff between traveling and evanescent waves. All waves must therefore fall into one of these two categories, and it happens to be the traveling waves. If, however, a length scale ℓ is introduced, then it *is* possible to generate a frequency (namely ω_0), which can provide the scale of the cutoff (which happens to be $2\omega_0$). ♣

Power

If you wiggle the end of a beaded string with a frequency larger than $2\omega_0$, so that an evanescent wave of the form in Eq. (16) arises, are you transmitting any net power to the string? Since the wave dies off exponentially, there is essentially no motion far to the right. Therefore, there is essentially no power being transmitted across a given point far to the right. So it had better be the case that you are transmitting zero net power to the string, because otherwise energy would be piling up indefinitely somewhere between your hand and

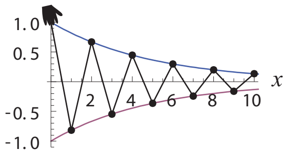

Figure 9

the given point far to the right. This is impossible, because there is no place for this energy to go.[2]

It is easy to see directly why you transmit zero net power over a full cycle (actually over each half cycle). Let's start with the position shown in Fig. 9, which shows a snapshot when the masses all have maximal displacement. The $n = 0$ mass is removed, and you grab the end of the string where that mass was. As you move your hand downward, you do negative work on the string, because you are pulling upward (and also horizontally, but there is no work associated with this force because your hand is moving only vertically), but your hand is moving in the opposite direction, downward. However, after your hand passes through equilibrium, it is still moving downward but now pulling downward too (because the string you're holding is now angled up to the right), so you are doing positive work on the string. The situation is symmetric (except for signs) on each side of equilibrium, so the positive and negative works cancel, yielding zero net work, as expected.

In short, your force is in "quadrature" with the velocity of your hand. That is, it is 90° out of phase with the velocity (behind it). So the product of the force and the velocity (which is the power) cancels over each half cycle. This is exactly the same situation that arises in a simple harmonic oscillator with a mass on a spring. You can verify that the force and velocity are in quadrature (the force is ahead now), and there is no net work done by the spring (consistent with the fact that the average motion of the mass does't change over time).

How do traveling waves ($\omega < 2\omega_0$) differ from evanescent waves ($\omega > 2\omega_0$), with regard to power? For traveling waves, when you wiggle the end, your force isn't in quadrature with the velocity of your hand, so you end up doing net positive work. In the small-ω limit (equivalently, the continuous-string limit), your force is exactly *in phase* with the velocity, so you're always doing positive work. However, as ω increases (with a beaded string), your force gradually shifts from being in phase with the velocity at $\omega \approx 0$, to being in quadrature with it at $\omega = 2\omega_0$, at which point no net work is being done. The task of Problem [to be added] is to be quantitative about this.

6.2.2 Low-frequency cutoff

The above beaded string supported traveling waves below a certain cutoff frequency but not above it. Let's now consider the opposite situation. We'll look at a system that supports traveling waves *above* a certain cutoff frequency but not below it. Hence the "Low-frequency cutoff" name of this subsection.

Consider a *continuous* string (not a beaded one) with tension T and density μ. Let the (infinite) string be connected to a wall by an essentially continuous set of springs, initially at their relaxed length, as shown in Fig. 10.

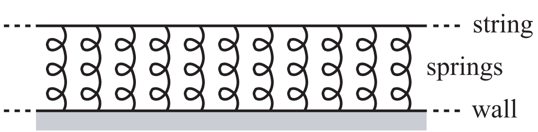

Figure 10

If the springs weren't present, then the return force (for transverse waves) on a little piece of the string with length Δx would be $T\Delta x(\partial^2\psi/\partial x^2)$; see Eq. (4.2). But we now also

[2]We are assuming steady-state motion. At the start, when you get the string going, you *are* doing net work on the string. Energy piles up at the start, because the string goes from being in equilibrium to moving back and forth. But in steady state, the average motion of the string doesn't change in time.

have the spring force, $-(\sigma\Delta x)\psi$, where σ is the spring constant per unit length. (The larger the piece, the more springs that touch it, so the larger the force.) The transverse $F = ma$ equation on the little piece is therefore modified from Eq. (4.3) to

$$\begin{aligned}(\mu\Delta x)\frac{\partial^2\psi}{\partial t^2} &= T\Delta x\frac{\partial^2\psi}{\partial x^2} - (\sigma\Delta x)\psi \\ \Longrightarrow \frac{\partial^2\psi}{\partial t^2} &= c^2\frac{\partial^2\psi}{\partial x^2} - \omega_s^2\psi, \qquad \text{where} \quad c^2 \equiv \frac{T}{\mu} \quad \text{and} \quad \omega_s^2 \equiv \frac{\sigma}{\mu}. \qquad (18)\end{aligned}$$

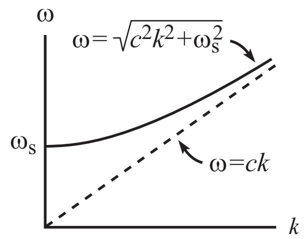

Figure 11

We will find that c is *not* the wave speed, as it was in the simple string system with no springs. To determine the dispersion relation associated with Eq. (18), we can plug in our standard exponential solution, $\psi(x,t) = Ae^{i(kx-\omega t)}$. This tells us that ω and k are related by

$$\boxed{\omega^2 = c^2k^2 + \omega_s^2} \qquad \text{(dispersion relation)} \qquad (19)$$

This is the dispersion relation for the string-spring system. The plot of ω vs. k is shown in Fig. 11. There is no (real) value of k that yields a ω smaller than ω_s. However, there is an *imaginary* value of k that does. If $\omega < \omega_s$, then Eq. (19) gives

$$k = \frac{\sqrt{\omega^2 - \omega_s^2}}{c} \equiv i\kappa, \qquad \text{where} \quad \kappa \equiv \frac{\sqrt{\omega_s^2 - \omega^2}}{c}. \qquad (20)$$

Another solution for κ is the negative of this one, but we'll be considering below the case where the string extends to $x = +\infty$, so this other solution would cause ψ to diverge, given our sign convention in the exponent of $e^{i(kx-\omega t)}$. Substituting $k \equiv i\kappa$ into $\psi(x,t) = Ae^{i(kx-\omega t)}$ gives

$$\psi(x,t) = Ae^{i((i\kappa)x-\omega t)} = Ae^{-\kappa x}e^{-i\omega t} \to \boxed{Be^{-\kappa x}\cos(\omega t + \phi)} \qquad (21)$$

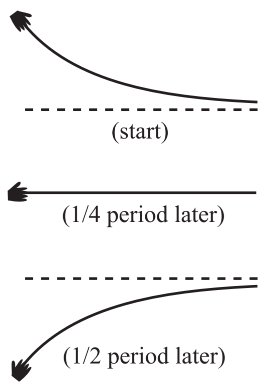

Figure 12

where as usual we have taken the real part. We see that $\psi(x,t)$ decays as a function of x, and that all points on the string oscillate with the *same* phase as a function of t. (This is in contrast with adjacent points having opposite phases in the above beaded-string setup. Opposite phases wouldn't make any sense here, because we don't have discrete adjacent points on a continuous string.) So we have an evanescent standing wave. If we wiggle the left end up and down sinusoidally with a frequency $\omega < \omega_s$, then snapshots of the motion take the general form shown in Fig. 12. The rate of the exponential decrease (as a function of x) depends on ω. If ω is only slightly smaller than ω_s, then the κ in Eq. (20) is small, so the exponential curve dies out very slowly to zero. In the limit where $\omega \approx 0$, we're basically holding the string at rest in the first position in Fig. 12, and you can show from scratch by balancing transverse forces in this static setup that the string does indeed take the shape of a decreasing exponential with $\kappa \approx \omega_s/c = \sqrt{\sigma/T}$; see Problem [to be added]. This static case yields the quickest spatial decay to zero.

It makes sense that the system doesn't support traveling waves for $\omega < \omega_s$, because even without any tension force, the springs would still make the atoms in the string oscillate with a frequency of at least $\sqrt{\sigma\Delta x/\mu\Delta x} = \omega_s$. Adding on a tension force will only increase the restoring force (at least in a traveling wave, where the curvature is always toward the x axis), and thus also the frequency. In the cutoff case where $\omega = \omega_s$, the string remains straight, and it oscillates back and forth as a whole, just as a infinite set of independent adjacent masses on springs would oscillate.

We have been talking about evanescent waves where $\omega < \omega_s$, but we can still have normal traveling waves if $\omega > \omega_s$, because the k in Eq. (20) is then real. If $\omega \gg \omega_s$, then Eq. (20) tells us that $\omega \approx ck$. In other words, the system acts like a simple string with no springs attached to it. This makes sense; on the short time scale of the oscillations, the spring doesn't have time to act, so it's effectively like it's not there. Equivalently, the transverse force from the springs completely dominates the spring force. If on the other hand ω is only slightly larger than ω_s, then Eq. (20) says that k is very small, which means that the wavelength is very large. In the $\omega \to \omega_s$ limit, we have $\lambda \to \infty$. The speed of the wave is then $\omega/k \approx \omega_s/k \approx \infty$. This can be seen graphically in Fig. 13, where the slope from the origin is ω/k, which is the phase velocity (just as it was in Fig. 3). This slope can be made arbitrarily large by making k be arbitrarily small. We'll talk more about excessively large phase velocities (in particular, larger than the speed of light) in Section 6.3.3.

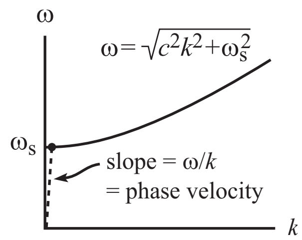

Figure 13

Note that the straight-line shape of the string in the $\omega = \omega_s$ case that we mentioned above can be considered to be the limit of a traveling wave with an infinitely long wavelength, and also an evanescent wave with an infinitely slow decay.

Power

As with the evanescent wave on the beaded string in Section 6.2.1, no net power can be transmitted in the present evanescent wave, because otherwise there would be energy piling up somewhere (because the wave dies out). But there is no place for it to pile up, because we are assuming steady-state motion. This can be verified with the same reasoning as in the beaded-string case; the net power you transmit to the string as you wiggle the left end alternates sign each quarter cycle, so there is complete cancelation over a full cycle.

Let's now consider the modified setup shown in Fig. 14. To the left of $x = 0$, we have a normal string with no springs. What happens if we have a rightward-traveling wave that comes in from the left, with a frequency $\omega < \omega_s$. (Or there could even be weak springs in the left region, as long as we have $\omega_{s,\text{left}} < \omega < \omega_{s,\text{right}}$. This would still allow a traveling wave in the left region.)

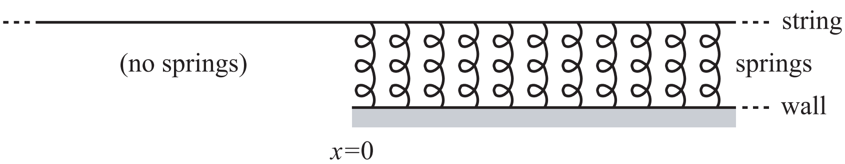

Figure 14

From the same reasoning as above, the fact that the wave dies out on the right side implies that no net power is transmitted along the string (in the steady state). However, there certainly *is* power transmitted in the incoming traveling wave. Where does it go? Apparently, there must be *complete* reflection at $x = 0$, so that all the power gets reflected back. The spring region therefore behaves effectively like a brick wall, as far as reflection goes. But the behavior isn't exactly like a brick wall, because ψ isn't constrained to be zero at the boundary in the present case.

To figure out what the complete wave looks like, we must apply the boundary conditions (continuity of the function and the slope) at $x = 0$.[3] If we work with exponential solutions, then the incoming, reflected, and transmitted waves take the form of $Ae^{i(\omega t-kx)}$, $Be^{i(\omega t+kx)}$, and $Ce^{i\omega t}e^{-\kappa x}$, respectively. The goal is to solve for B and C (which may be complex) in terms of A, that is, to solve for the ratios B/A and C/A. For ease of computation, it is

[3] As usual, the continuity of the slope follows from the fact that there can be no net force on the essentially massless atom at the boundary. The existence of the springs in the right region doesn't affect this fact.

customary to divide all of the functions by A, in which case the total waves on the two sides of the boundary can be written as

$$\begin{aligned} \psi_{\rm L} &= (e^{-ikx} + Re^{ikx})e^{i\omega t}, \\ \psi_{\rm R} &= De^{-\kappa x}e^{i\omega t}, \end{aligned} \tag{22}$$

where $R \equiv B/A$ and $D \equiv C/A$. R is the complex reflection coefficient. Its magnitude is what we normally call the "reflection coefficient," and its phase yields the phase of the reflected wave when we eventually take the real part to obtain the physical solution. The task of Problem [to be added] is to apply the boundary conditions and solve for R and D.

Tunneling

What happens if we have a setup in which the region with strings is finite, as shown in Fig. 15? If a rightward-traveling wave with $\omega < \omega_{\rm s}$ comes in from the left, some of it will make it through, and some of it will be reflected. That is, there will be a nonzero rightward-traveling wave in the right region, and there will be the usual combination of a traveling and standing wave in the left region that arises from partial reflection.

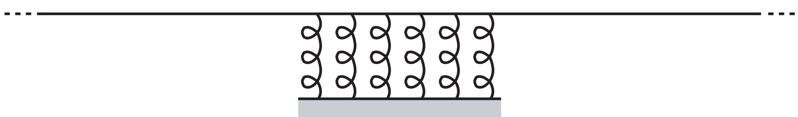

Figure 15

The nonzero wave in the right region implies that power is transmitted. This is consistent with the fact that we *cannot* use the reasoning that zero power follows from the fact that the wave dies out to zero; the wave doesn't die out to zero, because the middle region has finite length. We can't rule out the $e^{+\kappa x}$ solution, because the $x = +\infty$ boundary condition isn't relevant now. So the general (steady state) solution in the left, middle, and right regions takes the form,

$$\begin{aligned} \psi_{\rm L} &= (e^{-ikx} + Re^{ikx})e^{i\omega t}, \\ \psi_{\rm M} &= (Be^{-\kappa x} + Ce^{\kappa x})e^{i\omega t}, \\ \psi_{\rm R} &= Te^{-ikx}e^{i\omega t}, \end{aligned} \tag{23}$$

where R and T are the complex reflection and transmission coefficients, defined to be the ratio of the reflected and transmitted (complex) amplitudes to the incident (complex) amplitude. As in Eq. (22), we have written the waves in their complex forms, with the understanding that we will take the real part to find the actual wave. The four boundary conditions (value and slope at each of the two boundaries) allow us to solve for the four unknowns (R, T, B, C). This is the task of Problem [to be added]; the math gets a bit messy.

The effect where some of the wave makes it through the "forbidden" region where traveling waves don't exist is known as *tunneling*. The calculation in Problem [to be added] is exactly the same as in a *quantum mechanical* system involving tunneling through a classically forbidden region (a region where the total energy is less than the potential energy). In quantum mechanics, the waves are probability waves instead of transverse string waves, so the interpretation of the waves is different. But all the math is exactly the same as in the above string-spring system. We'll talk much more about quantum-mechanical waves in Chapter 11.

6.3 Group velocity

6.3.1 Derivation

Whether a system is dispersionless (with a linear relationship between ω and k) or dispersionful (with a nonlinear relationship between ω and k), the phase velocity in both cases is $v_{\rm p} = \omega/k$. The phase velocity gives the speed of a *single* sinusoidal traveling wave. But what if we have, say, a wave in the form of a lone bump, which (from Fourier analysis) can be written as the sum of many (or an infinite number of) sinusoidal waves. How fast does this bump move? If the system is dispersionless, then all of the wave components move with the same speed $v_{\rm p}$, so the bump also moves with this speed. We discussed this effect in Section 2.4 (see Eq. (2.97)), where we noted that any function of the form $f(x-ct)$ is a solution to a *dispersionless* wave equation, that is, an equation of the form $\partial^2\psi/\partial t^2 = c^2(\partial^2\psi/\partial x^2)$. This equation leads to the relation $\omega = ck$, where c takes on a single value that is independent of ω and k.

However, if the system is dispersionful, then ω/k depends on k (and ω), so the different sinusoidal waves that make up the bump travel at different speeds. So it's unclear what the speed of the bump is, or even if the bump *has* a well-defined speed. It turns out that it does in fact have a well-defined speed, and it is given by the slope of the $\omega(k)$ curve:

$$\boxed{v_{\rm g} = \frac{d\omega}{dk}} \tag{24}$$

This is called the *group velocity*, which is a sensible name considering that a bump is made up of a group of Fourier components, as opposed to a single sinusoidal wave. Although the components travel at different speeds, we will find below that they conspire in such a way as to make their sum (the bump) move with speed $v_{\rm g} = d\omega/dk$. However, an unavoidable consequence of the differing speeds of the components is the fact that as time goes on, the bump will shrink in height and spread out in width, until you can hardly tell that it's a bump. In other words, it will disperse. Hence the name dispersion.

Since the bump consists of wave components with many different values of k, there is an ambiguity about which value of k is the one where we should evaluate $v_{\rm g} = d\omega/dk$. The general rule is that it is evaluated at the value of k that dominates the bump. That is, it is evaluated at the peak of the Fourier transform of the bump.

We'll now derive the result for the group velocity in Eq. (24). And because it is so important, we derive it in three ways.

First derivation

Although we just introduced the group velocity by talking about the speed of a bump, which consists of many Fourier components, we can actually understand what's going on by considering just two waves. Such a system has all the properties needed to derive the group velocity. So consider the two waves:

$$\begin{aligned}\psi_1(x,t) &= A\cos(\omega_1 t - k_1 x),\\ \psi_2(x,t) &= A\cos(\omega_2 t - k_2 x).\end{aligned} \tag{25}$$

It isn't necessary that they have equal amplitudes, but it simplifies the discussion. Let's see what the sum of these two waves looks like. It will be advantageous to write the ω's and

k's in terms of their averages and differences.[4] So let's define:

$$\omega_+ \equiv \frac{\omega_1+\omega_2}{2}, \qquad \omega_- \equiv \frac{\omega_1-\omega_2}{2}, \qquad k_+ \equiv \frac{k_1+k_2}{2}, \qquad k_- \equiv \frac{k_1-k_2}{2}. \tag{26}$$

Then $\omega_1 = \omega_+ + \omega_-$ and $\omega_2 = \omega_+ - \omega_-$. And likewise for the k's. So the sum of the two waves can be written as

$$\begin{aligned} \psi_1(x,t)+\psi_2(x,t) &= A\cos\Big((\omega_+ + \omega_-)t - (k_+ + k_-)x\Big) \\ &\quad +A\cos\Big((\omega_+ - \omega_-)t - (k_+ - k_-)x\Big) \\ &= A\cos\Big((\omega_+ t - k_+ x) + (\omega_- t - k_- x)\Big) \\ &\quad +A\cos\Big((\omega_+ t - k_+ x) - (\omega_- t - k_- x)\Big) \\ &= 2A\cos\left(\omega_+ t - k_+ x\right)\cos\left(\omega_- t - k_- x\right), \end{aligned} \tag{27}$$

where we have used $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$. We see that the sum of the two original traveling waves can be written as the product of two other traveling waves.

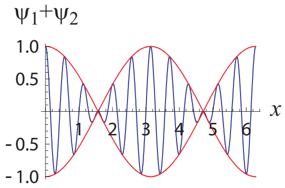

Figure 16

This is a general result for any values of ω_1, ω_2, k_1, and k_2. But let's now assume that ω_1 is very close to ω_2 (more precisely, that their difference is small compared with their sum). And likewise that k_1 is very close to k_2. We then have $\omega_- \ll \omega_+$ and $k_- \ll k_+$. Under these conditions, the sum $\psi_1+\psi_2$ in Eq. (27) is the product of the quickly-varying (in both space and time) wave, $\cos\left(\omega_+ t - k_+ x\right)$, and the slowly-varying wave, $\cos\left(\omega_- t - k_- x\right)$. A snapshot (at the arbitrarily-chosen time of $t = 0$) of $\psi_1+\psi_2$ is shown in Fig. 16. The quickly-varying wave is the actual sum, while the slowly-varying envelope is the function $2A\cos\left(\omega_- t - k_- x\right)$. We have arbitrarily picked $2A = 1$ in the figure. And we have chosen $k_1 = 10$ and $k_2 = 12$, which yield $k_+ = 11$ and $k_- = 1$. So the envelope function is $\cos(x)$, and the wiggly function (which equals $\psi_1+\psi_2$) is $\cos(11x)\cos(x)$.

At t increases, the quickly- and slowly-varying waves will move horizontally. What are the velocities of these two waves? The velocity of the quickly-wiggling wave is ω_+/k_+, which is essentially equal to either of ω_1/k_1 and ω_2/k_2, because we are assuming $\omega_1 \approx \omega_2$ and $k_1 \approx k_2$. So the phase velocity of the quickly-wiggling wave is essentially equal to the phase velocity of either wave.

The velocity of the slowly-varying wave (the envelope) is

$$\frac{\omega_-}{k_-} = \frac{\omega_1-\omega_2}{k_1-k_2}. \tag{28}$$

(Note that this may be negative, even if the phase velocities of the original two waves are both positive.) If we have a linear dispersion relation, $\omega = ck$, then this speed equals $c(k_1-k_2)/(k_1-k_2) = c$. So the group velocity equals the phase velocity, and this common velocity is constant, independent of k. But what if ω and k aren't related linearly? Well, if ω is given by the function $\omega(k)$, and if k_1 is close to k_2, then $(\omega_1-\omega_2)/(k_1-k_2)$ is essentially equal to the derivative, $d\omega/dk$. This is the velocity of the envelope formed by the two waves, and it is called the the *group velocity.* To summarize:

$$\boxed{v_{\rm p} = \frac{\omega}{k} \qquad \text{and} \qquad v_{\rm g} = \frac{d\omega}{dk}} \tag{29}$$

In general, both of these velocities are functions of k. And in general they are not equal. (The exception to both of these statements occurs in the case of linear dispersion.) In the

[4]We did something similar to this when we talked about beats in Section 2.1.4. But things are a little different here because the functions are now functions of both x and t, as opposed to just t.

general case where $v_g \neq v_p$, the fast wiggles in Fig. 16 move with respect to the envelope. If $v_p > v_g$, then the little wiggles pop into existence at the left end of an envelope bump (or the right end if $v_p < v_g$). They grow and then shrink as they move through the bump, until finally they disappear when they reach the right end of the bump.

In the case of the beaded-string system discussed in Section 6.1, the plot of $\omega(k)$ was shown in Fig. 3. So the phase and group velocities are shown graphically in Fig. 17. For $0 < k < \pi/\ell$ (which is generally the part of the graph we're concerned with), the slope of the curve at any point is less than the slope of the line from the origin to the point, so we see that v_g is always less than v_p.

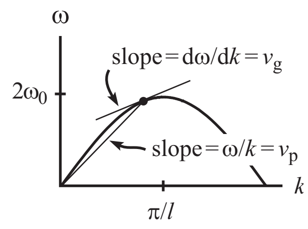

Figure 17

In the case of the string/spring system discussed in Section 6.2.2, the plot of $\omega(k)$ was shown in Fig. 11. So the phase and group velocities are shown graphically in Fig. 18. We again see that v_g is always less than v_p. However, this need not be true in general, as we'll see in the examples in Section 6.3.2 below. For now, we'll just note that in the particular case where the plot of $\omega(k)$ passes through the origin, there are two basic possibilities of what the $\omega(k)$ curve can look like, depending on whether it is concave up or down. These are shown in Fig. 19. The first case always has $v_g > v_p$, while the second always has $v_g < v_p$. In the first case, sinusoidal waves with small k (large λ) travel slower (that is, they have a smaller v_p) than waves with large k (small λ). The opposite is true in the second case.

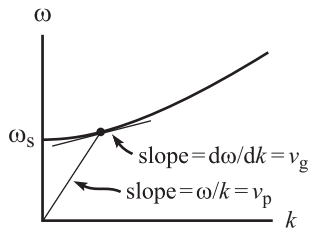

Figure 18

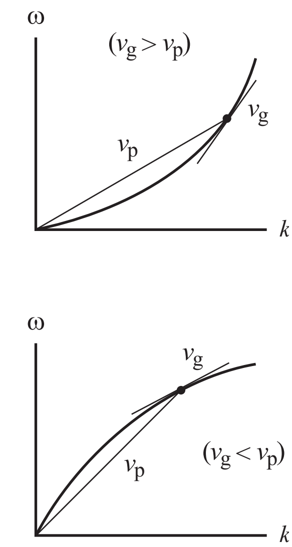

Figure 19

Second derivation

Consider two waves with different values of ω and k, as shown in the first pair of waves in Fig. 20. These two waves constructively interfere at the dots, so there will be a bump there. When and where does the next bump occur? If we can answer these questions, then we can find the effective velocity of the bump.

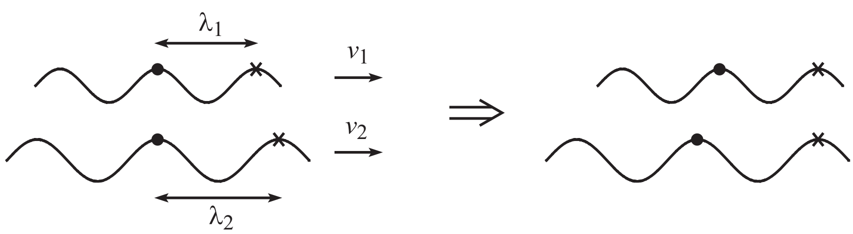

Figure 20

If $v_1 = v_2$ (that is, if $\omega_1/k_1 = \omega_2/k_2$), then both waves travel at the same speed, so the bump simply travels along with the waves, at their common speed. But if $v_1 \neq v_2$, then the dots will become unaligned. If we assume that $v_1 > v_2$ (the $v_1 < v_2$ case is similar), then at some later time the next two peaks will line up, as shown in the second pair of waves in Fig. 20. These peaks are marked with x's. There will then be a bump at this new location. (If $v_1 < v_2$, the next alignment will occur to the left of the initial one.)

When do these next peaks line up? The initial distance between the x's is $\lambda_2 - \lambda_1$, and the top wave must close this gap at a relative speed of $v_1 - v_2$, so $t = (\lambda_2 - \lambda_1)/(v_1 - v_2)$. Equivalently, just set $\lambda_1 + v_1 t = \lambda_2 + v_2 t$, because these two quantities represent the positions of the two x's, relative to the initial dots. Having found the time t, the position of the next alignment is given by $x = \lambda_1 + v_1 t$ (and also $\lambda_2 + v_2 t$). The velocity at which the bump effectively travels is therefore

$$\frac{x}{t} = \frac{\lambda_1 + v_1 t}{t} = \frac{\lambda_1}{t} + v_1 = \lambda_1 \left(\frac{v_1 - v_2}{\lambda_2 - \lambda_1}\right) + v_1 = \frac{\lambda_1 v_1 - \lambda_1 v_2}{\lambda_2 - \lambda_1} + \frac{\lambda_2 v_1 - \lambda_1 v_1}{\lambda_2 - \lambda_1}$$

$$= \quad \frac{\lambda_2 v_1 - \lambda_1 v_2}{\lambda_2 - \lambda_1} = \frac{\frac{2\pi}{k_2}\frac{\omega_1}{k_1} - \frac{2\pi}{k_1}\frac{\omega_2}{k_2}}{\frac{2\pi}{k_2} - \frac{2\pi}{k_1}} = \frac{\frac{\omega_1-\omega_2}{k_1 k_2}}{\frac{k_1-k_2}{k_1 k_2}} = \frac{\omega_1 - \omega_2}{k_1 - k_2} \equiv v_{\rm g}. \tag{30}$$

This is the same speed we obtained in Eq. (28). This is no surprise, because we basically did the same calculation here. In the previous derivation, we assumed that the waves were nearly identical, whereas we didn't assume that here. This assumption isn't needed for the $v_{\rm g} = (\omega_1 - \omega_2)/(k_1 - k_2)$ result. Whenever and wherever a bump in the present derivation exists, it touches the top of the envelope curve (if we had drawn it). So what we effectively did in this derivation is find the speed of the envelope curve. But this is exactly what we did in the previous derivation.

In between the alignments of the dot and the x in Fig. 20, the bump disappears, then appears in the negative direction, then disappears again before reappearing at the x. This is consistent with the fact that the wiggly wave in Fig. 16 doesn't always (in fact, rarely does) touch the midpoint (the highest point) of the envelope bump. But on average, the bump effectively moves with velocity $v_{\rm g} = (\omega_1 - \omega_2)/(k_1 - k_2)$.

Note that if k_1 is very close to k_2, and if ω_1 is *not* very close to ω_2, then $v_{\rm g} = (\omega_1 - \omega_2)/(k_1 - k_2)$ is large. It is easy to see intuitively why this is true. We may equivalently describe this scenario by saying that λ_1 is very close to λ_2, and that v_1 is not very close to v_2 (because $v = \omega k$). The nearly equal wavelengths imply that in Fig. 20 the two x's are very close together. This means that it takes essentially no time for them to align (because the velocities aren't close to each other). The location of the alignment therefore jumps ahead by a distance of one wavelength in essentially no time, which means that the effective speed is large (at least as large as the λ_1/t term in Eq. (30)).

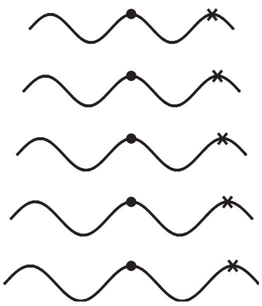

Figure 21

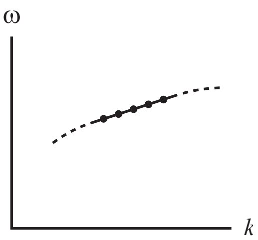

Figure 22

What if we have a large number of waves with roughly the same values of k (and hence ω), with a peak of each wave lining up, as shown by the dots in Fig. 21? Since the plot of $\omega(k)$ at any point is locally approximately a straight line, the quotient $(\omega_1 - \omega_2)/(k_1 - k_2)$, which is essentially equal to the derivative $d\omega/dk$, is the same for all nearby points, as shown in Fig. 22. This means that the next bumps (the x's in Fig. 21) will all line up at the same time and at the same place, because the location of all of the alignments is given by $x = v_{\rm g}t$, by Eq. (30). In other words, the group velocity $v_{\rm g}$ is well defined. The various waves all travel with different phase velocities $v_{\rm p} = \omega/k$, but this is irrelevant as far as the group velocity goes, because $v_{\rm g}$ depends on the *differences* in ω and k through Eq. (30), and not on the actual values of ω and k.

Third derivation

By definition, $v_{\rm g}$ is the velocity at which a bump in a wave travels. From Fourier analysis, we know that in general a wave consists of components with many different frequencies. If these components are to "work together" to form a bump at a certain location, then the phases $\omega_i t - k_i t + \phi_i$ of the different components (or at least many of them) must be equal at the bump, if they are to add constructively to form the bump.

Assume that we have a bump at a particular value of x and t. We are free to pick the origins of x and t to be wherever and whenever we want, so let's pick the bump to be located at $x = 0$ and $t = 0$. Since the phases $\omega_i t - k_i t + \phi_i$ are all equal (or at least many of them) in a region around some particular k value, we conclude that the ϕ_i are all equal, because $x = t = 0$. In other words, ϕ is independent of k.[5]

At what other values of x and t, besides $(x, t) = (0, 0)$, is there a bump? That is, at what other values of x and t are the phases still all equal? Well, we want $\omega t - kx + \phi$ to be independent of k near some particular k value, because then the phases of the waves for

[5]You can check that the following derivation still works in the case of general initial coordinates (x_0, t_0); see Problem [to be added]. But it's less messy if we choose $(0, 0)$.

all the different k values will be equal, which means that the waves will add constructively and therefore produce another bump. (We have dropped the index i on ω and k, and it is understood that ω is a function $\omega(k)$ of k.) Demanding that the phase be independent of k gives

$$0 = \frac{d(\omega t - kx + \phi)}{dk} \quad \Longrightarrow \quad 0 = \frac{d\omega}{dk}t - x \quad \Longrightarrow \quad \frac{x}{t} = \frac{d\omega}{dk}\,, \tag{31}$$

where we have used $d\phi/dk = 0$. So we have a bump at any values of x and t satisfying this relation. In other words, the speed of the bump is

$$v_{\rm g} = \frac{d\omega}{dk}\,, \tag{32}$$

in agreement with the result from the previous derivations.

Note that the *phase* velocity (of single traveling wave) is obtained by demanding that the phase $\omega t - kx + \phi$ of the wave be independent of *time*:

$$0 = \frac{d(\omega t - kx + \phi)}{dt} \quad \Longrightarrow \quad 0 = \omega - k\frac{dx}{dt} \quad \Longrightarrow \quad v_{\rm p} \equiv \frac{dx}{dt} = \frac{\omega}{k}\,. \tag{33}$$

But the *group* velocity (of a group of traveling waves) is obtained by demanding that the phase $\omega t - kx + \phi$ of all the different waves be independent of the *wavenumber* k:

$$0 = \frac{d(\omega t - kx + \phi)}{dk} \quad \Longrightarrow \quad 0 = \frac{d\omega}{dk}t - x \quad \Longrightarrow \quad v_{\rm g} \equiv \frac{x}{t} = \frac{d\omega}{dk}\,. \tag{34}$$

Just because the quantity $d\omega/dk$ exists, there's no guarantee that there actually *will* be a noticeable bump traveling at the group velocity $v_{\rm g}$. It's quite possible (and highly likely if things are random) that there is no constructive interference anywhere. But what we showed above was that *if* there is a bump at a given time and location, then it travels with velocity $v_{\rm g} = d\omega/dk$, evaluated at the k value that dominates the bump. This value can be found by calculating the Fourier transform of the bump.

6.3.2 Examples

Beaded string

We discussed the beaded string in Section 6.1. Eq. (9) gives the dispersion relation as $\omega(k) = 2\omega_0 \sin(k\ell/2)$, where $\omega_0 \equiv \sqrt{T/m\ell}$. Therefore,

$$v_{\rm p} = \frac{\omega}{k} = \frac{2\omega_0 \sin(k\ell/2)}{k}\,, \qquad \text{and} \qquad v_{\rm g} = \frac{d\omega}{dk} = \omega_0 \ell \cos(k\ell/2). \tag{35}$$

For small k (more precisely, for $k\ell \ll 1$), we can use $\sin\epsilon \approx \epsilon$ and $\cos\epsilon \approx 1$, to quickly show that

$$\begin{aligned} v_{\rm p} &\approx \frac{2\omega_0(k\ell/2)}{k} = \omega_0\ell = \sqrt{\frac{T}{m\ell}}\,\ell = \sqrt{\frac{T}{m/\ell}} = \sqrt{\frac{T}{\mu}}\,, \\ v_{\rm g} &\approx \omega_0\ell(1) = \sqrt{\frac{T}{\mu}}\,. \end{aligned} \tag{36}$$

As expected, these both agree with the (equal) phase and group velocities for a continuous string, because $k\ell \ll 1$ implies $\ell \ll \lambda$, which means that the string is essentially continuous on a length scale of the wavelength.

String/spring system

We discussed the string/spring system in Section 6.2.2. Eq. (19) gives the dispersion relation as $\omega^2 = c^2k^2 + \omega_s^2$, where $c^2 \equiv T/\mu$ and $\omega_s^2 \equiv \sigma/\mu$. Therefore,

$$v_p = \frac{\omega}{k} = \frac{\sqrt{c^2k^2 + \omega_s^2}}{k}, \qquad \text{and} \qquad v_g = \frac{d\omega}{dk} = \frac{c^2 k}{\sqrt{c^2k^2 + \omega_s^2}}. \tag{37}$$

If $\omega_s \approx 0$, then these reduce to $v_p \approx c$ and $v_g \approx c$, as expected. If ω_s is large (more precisely, if $\omega_s \gg ck$), then v_p is large and v_g is small (more precisely, $v_p \gg c$ and $v_g \ll c$). These facts are consistent with the slopes in Fig. 18.

Stiff string

When dealing with uniform strings, we generally assume that they are perfectly flexible. That is, we assume that they don't bounce back when they are bent. But if we have a "stiff string" that offers resistance when bent, it can be shown that the wave equation picks up an extra term and now takes the form,

$$\frac{\partial^2\psi}{\partial t^2} = c^2\left[\frac{\partial^2\psi}{\partial x^2} - \alpha\left(\frac{\partial^4\psi}{\partial x^4}\right)\right],$$

where α depends on various things (the cross-sectional area, Young's modulus, etc.).[6] Plugging in $\psi(x,t) = Ae^{i(\omega t - kx)}$ yields the dispersion relation,

$$\omega^2 = c^2k^2 + \alpha c^2k^4 \quad \Longrightarrow \quad \omega = ck\sqrt{1+\alpha k^2}. \tag{38}$$

This yields

$$v_p = \frac{\omega}{k} = c\sqrt{1+\alpha k^2}, \qquad \text{and} \qquad v_g = \frac{d\omega}{dk} = \frac{c(1+2\alpha k^2)}{\sqrt{1+\alpha k^2}}. \tag{39}$$

The dispersion relation in Eq. (38) has implications in piano tuning, because although the strings in a piano are reasonably flexible, they aren't perfectly so. They are slightly stiff, with a small value of α. If they were perfectly flexible ($\alpha = 0$), then the linear dispersion relation, $\omega = ck$, would imply that the standing-wave frequencies are simply proportional to the mode number, n, because the wavenumbers take the usual form of $k = n\pi/L$. So the "first harmonic" mode ($n = 2$) would have twice the frequency of the fundamental mode ($n = 1$). In other words, it would be an octave higher.

However, for a stiff string ($\alpha \neq 0$), Eq. (38) tells us that the frequency of the first harmonic is larger than twice the frequency of the fundamental. (The k values still take the form of $k = n\pi/L$. This is a consequence of the boundary conditions and is independent of the dispersion relation.)

Consider two notes that are an "octave" apart on the piano (the reason for the quotes will soon be clear). These notes are in tune with each other if the *first harmonic* of the lower string equals the *fundamental* of the higher string. Your ear then won't hear any beats between these two modes when the strings are played simultaneously, so things will sound nice.[7] A piano is therefore tuned to make the first harmonic of the lower string equal to

[6] In a nutshell, the fourth derivative comes from the facts that (1) the resistance to bending (the so-called "bending moment") is proportional to the curvature, which is the second derivative of ψ, and (2) the resulting net transverse force can be shown to be proportional to the second derivative of the bending moment.

[7] Your ear only cares about beats between nearby frequencies. The relation between the two fundamentals is irrelevant because they are so far apart. Beats don't result from widely-different frequencies.

the fundamental of the higher string. But since the dispersion relation tells us that the first harmonic (of any string) has more than twice the frequency of the fundamental, we conclude that the spacing between the fundamentals of the two strings is larger than an octave. But this is fine because it's what your ear wants to hear. The (equal) relation between the first harmonic of the lower string and the fundamental of the higher string is what's important. The relation between the two fundamentals doesn't matter.[8]

Power law

If a dispersion relation takes the form of a power law, $\omega = Ak^r$, then

$$v_{\rm p} = \frac{\omega}{k} = Ak^{r-1}, \qquad \text{and} \qquad v_{\rm g} = \frac{d\omega}{dk} = rAk^{r-1}. \tag{40}$$

We see that $v_{\rm g} = rv_{\rm p}$ for any value of k. If $r = 1$, then we have a dispersionless system. If $r > 1$, then the dispersion curve is concave up, so it looks like the first plot we showed in Fig. 19, with $v_{\rm g} > v_{\rm p}$. Sinusoidal waves with small k travel slower than waves with large k. If $r < 1$, then we have the second plot in Fig. 19, and these statements are reversed.

Quantum mechanics

In nonrelativistic quantum mechanics, particles are replaced by probability waves. The wave equation (known as the Schrodinger equation) for a free particle moving in one dimension happens to be

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\,, \tag{41}$$

where $\hbar = 1.05 \cdot 16^{-34}\,{\rm J}\cdot{\rm s}$ is Planck's constant. Plugging in $\psi(x,t) = Ae^{i(\omega t - kx)}$ yields the dispersion relation,

$$\omega = \frac{\hbar k^2}{2m}\,. \tag{42}$$

We'll give an introduction to quantum mechanics in Chapter 12, but for now we'll just note that the motivation for the dispersion relation (and hence the wave equation) comes from the substitutions of $E = \hbar\omega$ and $p = \hbar k$ into the standard classical relation, $E = p^2/2m$. We'll discuss the origins of these forms of E and p in Chapter 12.

The dispersion relation gives

$$v_{\rm p} = \frac{\omega}{k} = \frac{\hbar k}{2m}\,, \qquad \text{and} \qquad v_{\rm g} = \frac{d\omega}{dk} = \frac{\hbar k}{m}\,. \tag{43}$$

Classically, the velocity of a particle is given by $v = p/m$. So if $p = \hbar k$, then we see that $v_{\rm g}$, and not $v_{\rm p}$, corresponds to the classical velocity of a particle. This is consistent with the fact that a particle can be thought of as a localized bump in the probability wave, and this bump moves with the group velocity $v_{\rm g}$. A single sinusoidal wave moving with velocity $v_{\rm p}$ doesn't correspond to a localized particle, because the wave (which represents the probability) extends over all space. So we shouldn't expect $v_{\rm p}$ to correspond to the standard classical velocity of p/m.

[8] A nice article on piano tuning is: *Physics Today*, December 2009, pp 46-49. It's based on a letter from Richard Feynman to his piano tuner. See in particular the "How to tune a piano" box on page 48.

Water waves

We'll discuss water waves in detail in Chapter 11, but we'll invoke some results here so that we can see what a few phase and group velocities look like. There are three common types of waves:

- *Small ripples*: If the wavelength is short enough so that the effects of surface tension dominate the effects of gravity, then the dispersion relation takes the form, $\omega = \sqrt{\sigma k^3/\rho}$, where σ is the surface tension and ρ is the mass density. The surface tension dominates if the wavelength be small compared with about 2 cm. The dispersion relation then gives

$$v_{\rm p} = \frac{\omega}{k} = \sqrt{\frac{\sigma k}{\rho}}, \qquad \text{and} \qquad v_{\rm g} = \frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{\sigma k}{\rho}}. \tag{44}$$

  So $v_{\rm g} = 3v_{\rm p}/2$. The smaller the wavelength (the larger the k), then the larger the $v_{\rm p}$. Very small ripples travel fast.

- *Long wavelengths in deep water*: If the wavelength is large compared with 2 cm, then the effects of gravity dominate. If we further assume that the wavelength is small compared with the depth of the water, then the dispersion relation takes the form, $\omega = \sqrt{gk}$. This gives

$$v_{\rm p} = \frac{\omega}{k} = \sqrt{\frac{g}{k}}, \qquad \text{and} \qquad v_{\rm g} = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}}. \tag{45}$$

  So $v_{\rm g} = v_{\rm p}/2$. The larger the wavelength (the smaller the k), then the larger the $v_{\rm p}$. Long waves travel fast.

- *Long wavelengths, compared with depth*: If the wavelength is large compared with the depth of the water, then the dispersion relation takes the form, $\omega = \sqrt{gH}\,k$, where H is the depth. This is a dispersionless system, with

$$v_{\rm p} = v_{\rm g} = \sqrt{gH}. \tag{46}$$

  So all waves travel with the same speed (provided that the wavelength is large compared with H). This has dramatic consequences with regard to tsunamis.

  Consider a huge wave that is created in the ocean, most commonly by an earthquake. If the wave has, say, an amplitude of 100 ft and a wavelength of half a mile (which is indeed a huge wave), what will happen as it travels across the ocean? The depth of the ocean is on the order of a few miles, so we're in the regime of "long wavelengths in deep water." From above, this is a *dispersive* system. Different wavelengths therefore travel with different speeds, and the wave disperses. It grows shallower and wider, until there is hardly anything left. When it reaches the other side of the ocean, it will be barely distinguishable from no wave at all. The fast Fourier components of the initial bump (the ones with long wavelengths) will arrive much sooner than the slower components, so the energy of the wave will be diluted over a long period of time.

  However, consider instead a wave with an amplitude of only 5 ft, but with a wavelength of 10 miles. (Assuming roughly the same shape, this wave has the same volume as the one above.) What will happen to this wave as it travels across the ocean? We're now in the "long wavelengths, compared with depth" regime. This is a *nondispersive* system, so all of the different Fourier components of the initial "bump" travel with the same speed. The wave therefore keeps the same shape as it travels across the ocean.

Now, a 5 ft wave might not seem severe, but when the wave reaches shallower water near the shore, its energy gets concentrated into a smaller region, so the amplitude grows. If the boundary between the ocean and land were a hypothetical vertical wall extending miles downward, then the waves would simply reflect off the wall and travel back out to sea. But in reality the boundary is sloped. In short, the very long wavelength allows the wave to travel intact all the way across the ocean, and the sloped shore causes the amplitude to grow once the wave arrives.

What is the speed of a tsunami wave in deep water? The average depth of the Pacific Ocean is about 4000 m, so we have $v = \sqrt{gH} \approx 200\,\text{m/s} \approx 450\,\text{mph}$, which is quite fast. It takes only a little over a minute for all of the 10-mile wave to hit the shore. So the energy is deposited in a short amount of time. It isn't diluted over a large time as it was with the half-mile wave above. Note that in contrast with the dramatic effects at the shore, the wave is quite unremarkable far out to sea. It rises to a height of 5 ft over the course of many miles, so the slope at any point is extremely small. It is impossible to spot such a wave visually, but fortunately deep-sea pressure sensors on the ocean floor can measure changes in the water level with extreme precision.

6.3.3 Faster than c?

Group velocity

In the first derivation of the group velocity in Section 6.3.1, we found the velocity of the envelope to be $v_\text{g} = (\omega_2 - \omega_1)/(k_2 - k_1)$. If $\omega_1 \neq \omega_2$, and if the k values are close together, then v_g is large. In fact, v_g can be made arbitrarily large by making k_2 be arbitrarily close to k_1. This means that it is possible for v_g to exceed the speed of light. Is this bad? Does it mean that we can send a signal faster than the speed of light (commonly denoted by "c"), which would violate the theory of relativity? Answers: No, No.

In order for this scenario to be possible, the two individual waves we used in the derivation of Eq. (28) already needed to be in existence over a very wide range of x values. So the envelope in Fig. 16 is going to travel with velocity v_g, independent of what anyone does. Two people therefore can't use this effect to send information. To communicate something, you need to *change* the wave, and it can be demonstrated that the *leading edge* of this change can never travel faster than c.

Demonstrating this fact requires invoking some facts about relativity. It certainly can't be demonstrated *without* invoking anything about relativity, because there is nothing at all special about the speed of light if you haven't invoked the postulates of relativity. One line of reasoning is to say that if the leading edge travels faster than c, then there exists a frame in which causality is violated. This in turn violates innumerable laws of physics ($F = ma$ type laws, for example). The fact of the matter is that a line of atoms can't talk to each other faster than c, independent of whether they're part of a wave.

At any rate, the point of the present discussion isn't so much to say what's right about the relativistic reasoning (because we haven't introduced relativity in this book), but rather to say what's wrong about the reasoning that $v_\text{g} > c$ implies a contradiction with relativity. As we stated above, the error is that no information is contained in a wave that is formed from the superposition of two waves that already existed over a wide range of x values. If you had set up these waves, you must have set them up a while ago.

Another way to have $v_\text{g} > c$ is shown in Fig. 23. Because the bump creeps forward, it might be possible for v_g to exceed c even if the speed of the leading edge doesn't. But that's fine. The leading edge is what is telling someone that something happened, and this speed never exceeds c.

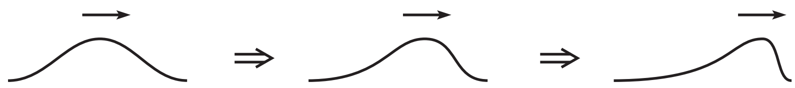

Figure 23

Phase velocity

The phase velocity can also exceed c. For the string/spring system in Section 6.2.2, we derived the dispersion relation, $\omega^2 = v^2k^2 + \omega_s^2$, where $v^2 \equiv T/\mu$ and $\omega_s^2 \equiv \sigma/\mu$. (We're using v here, to save c for the speed of light.) The phase velocity, $v_p = \omega/k$, is the slope of the line from the origin to a point on the $\omega(k)$ curve. By making k as small as we want, we can make the slope be as large as we want, as we noted in Fig. 13. Is this bad? No. Again, we need to make a *change* in the wave if we want to convey information. And any signal can travel only as fast as the leading edge of the change.

It is quite easy to create a system whose phase velocity v_p is arbitrarily large. Just put a bunch of people in a long line, and have them stand up and sit down at *prearranged* times. If the person a zillion meters away stands up 1 second after the person at the front of the line stands up, then the phase velocity is $v_p = 1\,\text{zillion}\,\text{m/s}$. But no information is contained in this "wave" because the actions of the people were already decided.

This is sort of like scissors. If you have a huge pair of scissors held at a very small angle, and if you close them, then it seems like the intersection point can travel faster than c. You might argue that this doesn't cause a conflict with relativity, because there is no actual object in this system that is traveling faster than c (the intersection point isn't an actual object). However, although it is correct that there isn't a conflict, this reasoning isn't valid. Information need not be carried by an actual object.

The correct reasoning is that the intersection point will travel faster than c only if you *prearrange* for the blades to move at a given instant very far away. If you were to simply apply forces at the handles, then the parts of the blades very far away wouldn't know right away that they should start moving. So the blades would bend, even if they were made out of the most rigid material possible.

Said in another way, when we guess a solution of the form $e^{i(\omega t - kx)}$ in our various wave equations, it is assumed that this is the solution for *all* space and time. These waves always were there, and they always will be there, so they don't convey any information by themselves. We have to make a *change* in them to send a signal. And the leading edge of the change can travel no faster than c.