# Chapter 6

# Appendices



## 6.1 Appendix A: Qualitative relativity questions

### Basic principles

1. QUESTION: You are in a spaceship sailing along in outer space. Is there any way you can measure your speed without looking outside?

   ANSWER: There are two points to be made here. First, the question is meaningless, because absolute speed doesn't exist. The spaceship doesn't have a speed; it only has a speed relative to something else. Second, even if the question asked for the speed with respect to, say, a given piece of stellar dust, the answer would be "no." Uniform speed is not measurable from within the spaceship. Acceleration, on the other hand, is measurable (assuming there is no gravity around to confuse it with).

2. QUESTION: Two people, $A$ and $B$, are moving with respect to each other in one dimension. Is the speed of $B$ as viewed by $A$ equal to the speed of $A$ as viewed by $B$?

   ANSWER: Yes. The first postulate of relativity states that all inertial frames are equivalent, which implies that there is no preferred location or direction in space. If the relative speed measured by the left person were larger than the relative speed measured by the right person, then there would be a preferred direction in space. Apparently people on the left always measure a larger speed. This violates the first postulate. Likewise if the left person measures a smaller speed. The two speeds must therefore be equal.

3. QUESTION: Is the second postulate of relativity (that the speed of light is the same in all inertial frames) really necessary? Or is it already implied by the first postulate (that the laws of physics are the same in all inertial frames)?

   ANSWER: It is necessary. The speed-of-light postulate is not implied by the laws-of-physics postulate. The latter doesn't imply that baseballs have the same speed in all inertial frames, so it likewise doesn't imply that light has the same speed.

   It turns out that nearly all the results in special relativity can be deduced by using only the laws-of-physics postulate. What you can find (with some work) is that there is some limiting speed, which may be finite or infinite; see Section 2.7. The speed-of-light postulate fills in the last bit of information by telling us what the limiting speed is.

4. Question: Is the speed of light equal to $c$, under all circumstances?

   Answer: No. In Section 1.2 we stated the speed-of-light postulate as, "The speed of light in vacuum has the same value $c$ in any inertial frame." There are two key words here: "vacuum" and "inertial." If we are dealing with a medium (such as glass or water) instead of vacuum, then the speed of light is smaller than $c$. And in an accelerating (noninertial) reference frame, the speed of light can be larger or smaller than $c$.

5. Question: In a given frame, a clock reads noon when a ball hits it. Do observers in all other frames agree that the clock reads noon when the ball hits is?

   Answer: Yes. Some statements, such as the one at hand, are frame independent. The relevant point is that everything happens at one location, so we don't have to worry about any $Lv/c^2$ loss-of-simultaneity effects. The $L$ here is zero. Equivalently, the ball-hits-clock event and the clock-reads-noon event are actually the same event. They are described by the same space and time coordinates. Therefore, since we have only one event, there is nothing for different observers to disagree on.

6. Question: Can information travel faster than the speed of light?

   Answer: No. If it could, we would be able to generate not only causality-violating setups, but also genuine contradictions. See the discussions at the ends of Sections 2.3 and 2.4.

7. Question: Is there such a thing as a perfectly rigid object?

   Answer: No. From the discussions of causality in Sections 2.3 and 2.4, we know that information can't travel infinitely fast (in fact, it can't exceed the speed of light). So it takes time for the atoms in an object to communicate with each other. If you push on one end of a stick, the other end won't move right away.

8. Question: Can an object (with nonzero mass) move at the speed of light?

   Answer: No. Here are four reasons why. (1) Energy: $v = c$ implies $\gamma = \infty$, which implies that $E = \gamma mc^2$ is infinite. The object must therefore have an infinite amount of energy (unless $m = 0$, as for a photon). All the energy in the universe, let alone all the king's horses and all the king's men, can't accelerate something to speed $c$. (2) Momentum: again, $v = c$ implies $\gamma = \infty$, which implies that $p = \gamma mv$ is infinite. The object must therefore have an infinite amount of momentum (unless $m = 0$, as for a photon). (3) Force: since $F = \gamma^3 ma$, we have $a = F/m\gamma^3$. And since $\gamma \to \infty$ as $v \to c$, the acceleration $a$ becomes smaller and smaller (for a given $F$) as $v$ approaches $c$. It can be shown that with the $\gamma^3$ factor in the denominator, the acceleration drops off quickly enough so that the speed $c$ is never reached in a finite time. (4) Velocity-addition formula: no matter what speed you give an object with respect to the frame it was just in (that is, no matter how you accelerate it), the velocity-addition formula always yields a speed that is less than $c$. The only way the resulting speed can equal $c$ is if one of the two speeds in the formula is $c$.

9. Question: Imagine closing a very large pair of scissors. If arranged properly, it is possible for the point of intersection of the blades to move faster than the speed of light. Does this violate anything in relativity?

   Answer: No. If the angle between the blades is small enough, then the tips of the blades (and all the other atoms in the scissors) can move at a speed well below $c$, while the intersection point moves faster than $c$. But this doesn't violate anything in relativity. The intersection point isn't an actual object, so there is nothing wrong with it moving faster than $c$.

You might be worried that this result allows you to send a signal down the scissors at a speed faster than $c$. However, since there is no such thing as a perfectly rigid object, it is impossible to get the far end of the scissors to move right away, when you apply a force at the handle. The scissors would have to already be moving, in which case the motion is independent (at least for a little while) of any decision you make at the handle to change the motion of the blades.

10. QUESTION: A mirror moves toward you at speed $v$. You shine a light toward it, and the light beam bounces back at you. What is the speed of the reflected beam?

    ANSWER: The speed is $c$, as always. You will observe the light (which is a wave) having a higher wave frequency due to the Doppler effect. But the speed is still $c$.

11. QUESTION: Person $A$ chases person $B$. As measured in the ground frame, they have speeds $v_A$ and $v_B$. If they start a distance $L$ apart (as measured in the ground frame), how much time will it take (as measured in the ground frame) for $A$ to catch $B$?

    ANSWER: In the ground frame, the relative speed is $v_A - v_B$. Person $A$ must close the initial gap of $L$, so the time it takes is $L/(v_A - v_B)$. There is no need to use any fancy velocity-addition or length-contraction formulas, because all quantities in this problem are measured with respect to the *same* frame. So it quickly reduces to a simple "(rate)(time) = (distance)" problem. Alternatively, the two positions in the ground frame are given by $x_A = v_A t$ and $x_B = L + v_B t$. Setting these positions equal to each other gives $t = L/(v_A - v_B)$.

    Note that no object in this setup moves with speed $v_A - v_B$. This is simply the rate at which the gap between $A$ and $B$ closes, and a gap isn't an actual thing.

12. QUESTION: How do you synchronize two clocks that are at rest with respect to each other?

    ANSWER: One way is to put a light source midway between the two clocks and send out signals, and then set the clocks to a certain value when the signals hit them. Another way is to put a watch right next to one of the clocks and synchronize it with that clock, and then move the watch very slowly over to the other clock and synchronize that clock with it. Any time-dilation effects can be made arbitrarily small by moving the watch sufficiently slowly, because the time-dilation effect is second order in $v$ (and because the travel time is only first order in $1/v$).

### The fundamental effects

13. QUESTION: Two clocks at the ends of a train are synchronized with respect to the train. If the train moves past you, which clock shows a higher time?

    ANSWER: The rear clock shows a higher time. It shows $Lv/c^2$ more than the front clock, where $L$ is the proper length of the train.

14. QUESTION: Does the rear-clock-ahead effect imply that the rear clock runs faster than the front clock?

    ANSWER: No. Both clocks run at the same rate in the ground frame. It's just that the rear clock is always a fixed time of $Lv/c^2$ ahead of the front clock.

15. QUESTION: Moving clocks run slow. Does this result have anything to do with the time it takes light to travel from the clock to your eye?

    ANSWER: No. When we talk about how fast a clock is running in a given frame, we are referring to what the clock actually reads in that frame. It will certainly take time for the light from the clock to reach your eye, but it is understood that you subtract off this transit time in order to calculate the time (in your frame) at which the clock actually shows a particular reading. Likewise, other relativistic effects, such as length contraction and the loss of simultaneity, have nothing to do with the time it takes light to reach your eye. They deal only with what really *is*, in your frame. One way to avoid the complication of the travel time of light is to use the lattice of clocks and meter sticks described in Section 1.3.4.

16. QUESTION: A clock on a moving train reads $T$. Does the clock read $\gamma T$ or $T/\gamma$ in the ground frame?

    ANSWER: Neither. It reads $T$. Clock *readings* don't get dilated. *Elapsed times* are what get dilated. If the clock advances from $T_1$ to $T_2$, then the time between these readings, as measured in the train frame, is just $T_2 - T_1$. But the time between the readings, as measured in the ground frame, is $\gamma(T_2 - T_1)$, due to time dilation.

17. QUESTION: Does time dilation depend on whether a clock is moving across your vision or directly away from you?

    ANSWER: No. A moving clock runs slow, no matter which way it is moving. This is clearer if you think in terms of the lattice of clocks and meter sticks in Section 1.3.4. If you imagine a million people standing at the points of the lattice, then they all observe the clock running slow. Time dilation is an effect that depends on the *frame* and the speed of a clock with respect to it. It doesn't matter where you are in the frame (as long as you're at rest in it), as you look at a moving clock.

18. QUESTION: Does special-relativistic time dilation depend on the acceleration of the moving clock you are looking at?

    ANSWER: No. The time-dilation factor is $\gamma = 1/\sqrt{1 - v^2/c^2}$, which doesn't depend on the acceleration $a$. The only relevant quantity is the $v$ at a given instant; it doesn't matter if $v$ is changing. As long as you represent an inertial frame, then the clock you are viewing can undergo whatever motion it wants, and you will observe it running slow by the simple factor of $\gamma$. See the third remark in the solution to Problem 2.12.

    However, if *you* are accelerating, then you can't naively apply the results of special relativity. To do things correctly, it is easiest to think in terms of general relativity (or at least the Equivalence Principle). This is discussed in Chapter 5.

19. QUESTION: Two twins travel away from each other at relativistic speed. The time-dilation result says that each twin sees the other twin's clock running slow. So each says the other has aged less. How would you reply to someone who asks, "But which twin really *is* younger?"

    ANSWER: It makes no sense to ask which twin really is younger, because the two twins aren't in the same reference frame; they are using different coordinates to measure time. It's as silly as having two people run away from each other into the distance (so that each person sees the other become small), and then asking: Who is really smaller?

20. QUESTION: A train moves at speed $v$. A ball is thrown from the back to the front. In the train frame, the time of flight is $T$. Is it correct to use time dilation to say that the time of flight in the ground frame is $\gamma T$?

    ANSWER: No. The time-dilation result holds only for two events that happen at the *same place* in the relevant reference frame (the train, here). Equivalently, it holds if you are looking at a *single* moving clock. The given information tells us that the reading on the front clock (when the ball arrives) minus the reading on the back clock (when the ball is thrown) is $T$. It makes no sense to apply time dilation to the difference in these readings, because they come from two different clocks.

Another way of seeing why simple time dilation is incorrect is to use the Lorentz transformation. If the proper length of the train is $L$, then the correct time on the ground (between the ball-leaving-back and ball-hitting-front events) is given by $\Delta t_g = \gamma(\Delta t_t + v\,\Delta x_t/c^2) = \gamma T + \gamma v L/c^2$, which isn't equal to $\gamma T$. Equivalently, if you look at a *single* clock on the train, for example the back clock, then it starts at zero but ends up at $T + Lv/c^2$ due to the rear-clock-ahead effect (because the front clock shows $T$ when the ball arrives). Applying time dilation to this elapsed time on a *single* clock gives the correct time in the ground frame. This is the argument we used when deriving $\Delta t_g$ in Section 2.1.2.

21. Question: Someone says, "A stick that is length-contracted isn't *really* shorter, it just *looks* shorter." How do you respond?

    Answer: The stick really *is* shorter in your frame. Length contraction has nothing to do with how the stick looks, because light takes time to travel to your eye. It has to do with where the ends of the stick are at simultaneous times in your frame. This is, after all, how you measure the length of something. At a given instant in your frame, the distance between the ends of the stick is genuinely less than the proper length of the stick. If a green sheet of paper slides with a relativistic speed $v$ over a purple sheet (of the same proper size), and if you take a photo when the centers coincide, then the photo will show some of the purple sheet (or essentially all of it in the $v \to c$ limit).

22. Question: Consider a stick that moves in the direction in which it points. Does its length contraction depend on whether this direction is across your vision or directly away from you?

    Answer: No. The stick is length-contracted in both cases. Of course, if you look at the stick in the latter case, then all you see is the end, which is just a dot. But the stick is indeed shorter in your reference frame. As in Question 17 above concerning time dilation, length contraction depends on the frame, not where you are in it.

23. Question: If you move at the speed of light, what shape does the universe take in your frame?

    Answer: The question is meaningless, because it's impossible for you to move at the speed of light. A meaningful question to ask is: What shape does the universe take if you move at a speed very close to $c$ (with respect to, say, the average velocity of all the stars)? The answer is that in your frame everything will be squashed along the direction of your motion, due to length contraction. Any given region of the universe will be squashed down to a pancake.

24. Question: Eq. (1.14) says that the time in the observer's frame is *longer* than the proper time, while Eq. (1.20) says that the length in the observer's frame is *shorter* than the proper length. Why does this asymmetry exist?

    Answer: The asymmetry arises from the different assumptions that lead to time dilation and length contraction. If a clock and a stick are at rest on a train moving with speed $v$ relative to you, then time dilation is based on the assumption that $\Delta x_{\text{train}} = 0$ (this holds for two ticks on a train clock), while length contraction is based on the assumption that $\Delta t_{\text{you}} = 0$ (you measure a length by observing where the ends are at simultaneous times in your frame). These conditions deal with *different frames*, and this causes the asymmetry. Mathematically, time dilation follows from the second equation in Eq. (2.2):

$$\Delta t_{\text{you}} = \gamma\left(\Delta t_{\text{train}} + (v/c^2)\,\Delta x_{\text{train}}\right) \implies \Delta t_{\text{you}} = \gamma\,\Delta t_{\text{train}} \quad (\text{if } \Delta x_{\text{train}} = 0). \quad (6.1)$$

Length contraction follows from the first equation in Eq. (2.4):

$$\Delta x_{\text{train}} = \gamma\left(\Delta x_{\text{you}} - v\,\Delta t_{\text{you}}\right) \implies \Delta x_{\text{train}} = \gamma\,\Delta x_{\text{you}} \quad (\text{if } \Delta t_{\text{you}} = 0). \quad (6.2)$$

These equations *are* symmetric with respect to the $\gamma$ *factors*. The $\gamma$ goes on the side of the equation associated with the frame in which the given space or time interval is zero. However, the equations *aren't* symmetric with respect to *you*. You appear on different sides of the equations because for length contraction your frame is the one where the given interval is zero, whereas for time dilation it isn't. Since the end goal is to solve for the "you" quantity, we must divide Eq. (6.2) by $\gamma$ to isolate $\Delta x_{\text{you}}$. This is why Eqs. (1.14) and (1.20) aren't symmetric with respect to the "observer" (that is, "you") label.

25. Question: When relating times via time dilation, or lengths via length contraction, how do you know where to put the $\gamma$ factor?

    Answer: There are various answers to this, but probably the safest method is to (1) remember that moving clocks run slow and moving sticks are short, then (2) identify which times or lengths are larger or smaller, and then (3) put the $\gamma$ factor where it needs to be so that the relative size of the times or lengths is correct.

### Other kinematics topics

26. Question: Two objects fly toward you, one from the east with speed $u$, and the other from the west with speed $v$. Is it correct that their relative speed, as measured by you, is $u + v$? Or should you use the velocity-addition formula, $V = (u + v)/(1 + uv/c^2)$? Is it possible for their relative speed, as measured by you, to exceed $c$?

    Answer: Yes, no, yes, to the three questions. It is legal to simply add the two speeds to obtain $u + v$. There is no need to use the velocity-addition formula, because both speeds are measured with respect to the *same thing* (namely you), and because we are asking for the relative speed as measured by that thing. It is perfectly legal for the result to be greater than $c$, but it must be less than (or equal to, for photons) $2c$.

27. Question: In what situations is the velocity-addition formula relevant?

    Answer: The formula is relevant in the two scenarios in Fig. 1.41, assuming that the goal is to find the speed of $A$ as viewed by $C$ (or vice versa). The second scenario is the same as the first scenario, but from $B$'s point of view. In the first scenario, the two speeds are measured with respect to different frames, so it isn't legal to simply add them. In the second scenario, although the speeds are measured with respect to the same frame ($B$'s frame), the goal is to find the relative speed as viewed by someone else ($A$ or $C$). So the simple sum $v_1 + v_2$ isn't relevant (as it was in Question 26).

28. Question: Two objects fly toward you, one from the east with speed $u$, and the other from the west with speed $v$. What is their relative speed?

    Answer: The question isn't answerable. It needs to be finished with, ". . . as viewed by so-and-so." The relative speed as viewed by you is $u + v$, and the relative speed as viewed by either object is $(u + v)/(1 + uv/c^2)$.

    However, if someone says, "$A$ and $B$ move with relative speed $v$," and if there is no mention of a third entity, then it is understood that $v$ is the relative speed as viewed by either $A$ or $B$.

29. Question: A particular event has coordinates $(x, t)$ in one frame. How do you use a Lorentz transformation (L.T.) to find the coordinates of this event in another frame?

    Answer: You don't. L.T.'s have nothing to do with single events. They deal only with *pairs* of events and the *separation* between them. As far as a single event goes, its coordinates in another frame can be anything you want, simply by defining your origin to be wherever and whenever you please. But for pairs of events, the separation is a well-defined quantity, independent of your choice of origin. It is therefore a meaningful question to ask how the separations in two different frames are related, and the L.T.'s answer this question.

    This question is similar to Question 16, where we noted that clock *readings* (that is, time *coordinates*) don't get dilated. Rather, *elapsed times* (the *separations* between time coordinates) are what get dilated.

30. Question: When using the L.T.'s, how do you tell which frame is the moving "primed" frame?

    Answer: You don't. There is no preferred frame, so it doesn't make sense to ask which frame is moving. We used the primed/unprimed notation in the derivation in Section 2.1.1 for ease of notation, but don't take this to imply that there is a fundamental frame $S$ and a less fundamental frame $S'$. In general, a better strategy is to use subscripts that describe the two frames, such as "g" for ground and "t" for train, as we did in Section 2.1.2. For example, if you know the values of $\Delta t_{\text{t}}$ and $\Delta x_{\text{t}}$ on a train (which we'll assume is moving in the positive $x$ direction with respect to the ground), and if you want to find the values of $\Delta t_{\text{g}}$ and $\Delta x_{\text{g}}$ on the ground, then you can write down:

$$\begin{aligned} \Delta x_{\text{g}} &= \gamma(\Delta x_{\text{t}} + v\,\Delta t_{\text{t}}), \\ \Delta t_{\text{g}} &= \gamma(\Delta t_{\text{t}} + v\,\Delta x_{\text{t}}/c^2). \end{aligned} \tag{6.3}$$

    If instead you know the intervals on the ground and you want to find them on the train, then you just need to switch the subscripts "g" and "t" and change both signs to "−" (see the following question).

31. Question: How do you determine the sign in the L.T.'s in Eq. (6.3)?

    Answer: The sign is a "+" if the frame associated with the left side of the equation (the ground, in Eq. (6.3)) sees the frame associated with the right side (the train) moving in the positive direction. The sign is a "−" if the motion is in the negative direction. This rule follows from looking at the motion of a specific point in the train frame. Since $\Delta x_{\text{t}} = 0$ for two events located at a specific point in the train, the L.T. for $x$ becomes $\Delta x_{\text{g}} = \pm\gamma v\,\Delta t_{\text{t}}$. So if the point moves in the positive (or negative) direction in the ground frame, then the sign must be "+" (or "−") so that $\Delta x_{\text{g}}$ is positive (or negative).

32. Question: In relativity, the temporal order of two events in one frame may be reversed in another frame. Does this imply that there exists a frame in which I get off a bus before I get on it?

    Answer: No. The order of two events can be reversed in another frame only if the events are spacelike separated, that is, if $\Delta x > c\,\Delta t$ (which means that the events are too far apart for even light to go from one to the other). The two relevant events here (getting on the bus, and getting off the bus) are not spacelike separated, because the bus travels at a speed less than $c$, of course. They are timelike separated. Therefore, in all frames it is the case that I get off the bus after I get on it.

There would be causality problems if there existed a frame in which I got off the bus before I got on it. If I break my ankle getting off a bus, then I wouldn't be able to make the mad dash that I made to catch the bus in the first place, in which case I wouldn't have the opportunity to break my ankle getting off the bus, in which case I could have made the mad dash to catch the bus and get on, and, well, you get the idea.

33. QUESTION: Does the longitudinal Doppler effect depend on whether the source or the observer is the one that is moving in a given frame?

    ANSWER: No. Since there is no preferred reference frame, only the relative motion matters. If light needed an "ether" to propagate in, then there would be a preferred frame. But there is no ether. This should be contrasted with the everyday Doppler effect for sound. Sounds needs air (or some other medium) to propagate in. So in this case there *is* a preferred frame – the rest frame of the air.

### Dynamics

34. QUESTION: How can you prove that $E = \gamma mc^2$ and $p = \gamma mv$ are conserved?

    ANSWER: You can't. Although there are strong theoretical reasons why the $E$ and $p$ given by these expressions should be conserved, in the end it comes down to experiment. And every experiment that has been done so far is consistent with these $E$ and $p$ being conserved. But this is no proof, of course. As is invariably the case, these expressions are undoubtedly just the limiting expressions of a more correct theory.

35. QUESTION: The energy of an object with mass $m$ and speed $v$ is $E = \gamma mc^2$. Is the statement, "A photon has zero mass, so it must have zero energy," correct or incorrect?

    ANSWER: It is incorrect. Although $m$ is zero, the $\gamma$ factor is infinite because $v = c$ for a photon. And infinity times zero is undefined. A photon does indeed have energy, and it happens to equal $hf$, where $h$ is Planck's constant and $f$ is the frequency of the light.

36. QUESTION: A particle has mass $m$. Is its relativistic mass equal to $\gamma m$?

    ANSWER: Maybe. Or more precisely: if you want it to be. You can define the quantity $\gamma m$ to be whatever you want. But calling it "relativistic mass" isn't the most productive definition, because $\gamma m$ already goes by another name. It's just the energy, up to factors of $c$. The use of the word "mass" for this quantity, although quite permissible, is certainly not needed. See the discussion on page 129.

37. QUESTION: When using conservation of energy in a relativistic collision, do you need to worry about possible heat generated, as you do for nonrelativistic collisions?

    ANSWER: No. The energy $\gamma mc^2$ is conserved in relativistic collisions, period. Any heat that is generated in a particle shows up as an increase in mass. Of course, energy is also conserved in nonrelativistic collisions, period. But if heat is generated, then the energy isn't all in the form of $mv^2/2$ kinetic energies of macroscopic particles.

38. QUESTION: How does the relativistic energy $\gamma mc^2$ reduce to the nonrelativistic kinetic energy $mv^2/2$?

    ANSWER: The Taylor approximation $1/\sqrt{1 - v^2/c^2} \approx 1 + v^2/2c^2$ turns $\gamma mc^2$ into $mc^2 + mv^2/2$. The first term is the rest energy. If we assume that a collision is elastic, which means that the masses don't change, then conservation of $\gamma mc^2$ reduces to conservation of $mv^2/2$.

39. QUESTION: Given the energy $E$ and momentum $p$ of a particle, what is the quickest way to obtain the mass $m$?

    ANSWER: The quickest way is to use the "Very Important Relation" in Eq. (3.12). Whenever you know two of the three quantities $E$, $p$, and $m$, this equation gives you the third. This isn't the only way to obtain $m$, of course. For example, you can use $v/c^2 = p/E$ to get $v$, and then plug the result into $E = \gamma mc^2$. But the nice thing about Eq. (3.12) is that you never have to deal with $v$.

40. QUESTION: For a collection of particles, why is the value of $E_{\text{total}}^2 - p_{\text{total}}^2 c^2$ invariant, as Eq. (3.26) states? What is the invariant value?

    ANSWER: $E_{\text{total}}^2 - p_{\text{total}}^2 c^2$ is invariant because $E_{\text{total}}$ and $p_{\text{total}}$ transform according to the L.T.'s (see Eq. (3.25)), due to the fact that the single-particle transformations in Eq. (3.20) are *linear*. Given that $E_{\text{total}}$ and $p_{\text{total}}$ do indeed transform via the L.T.'s, the invariance of $E_{\text{total}}^2 - p_{\text{total}}^2 c^2$ follows from a calculation similar to the one for $c^2(\Delta t)^2 - (\Delta x)^2$ in Eq. (2.25).

    In the center-of-mass (or really center-of-momentum) frame, the total momentum is zero. So the invariant value of $E_{\text{total}}^2 - p_{\text{total}}^2 c^2$ equals $(E_{\text{total}}^{\text{CM}})^2$. For a single particle, this is simply $(mc^2)^2 = m^2c^4$, as we know from Eq. (3.12).

41. QUESTION: Why do $E$ and $p$ transform the same way $\Delta t$ and $\Delta x$ do, via the L.T.'s?

    ANSWER: In Section 3.2 we used the velocity-addition formula to show that $E$ and $p$ transform via the L.T.'s. This derivation, however, doesn't make it intuitively clear why $E$ and $p$ should transform like $\Delta t$ and $\Delta x$. In contrast, the 4-vector approach in Section 4.2 makes it quite clear. To obtain the energy-momentum 4-vector $(E, \mathbf{p})$ from the displacement 4-vector $(dt, d\mathbf{r})$, we simply need to divide by the proper time $d\tau$ (which is an invariant) and then multiply by the mass $m$ (which is again an invariant). The result is therefore still a 4-vector (which is by definition a 4-tuple that transforms according to the L.T.'s).

42. QUESTION: Why do the *differences* in the coordinates, $\Delta x$ and $\Delta t$, transform via the L.T.'s, while it is the actual *values* of $E$ and $p$ that transform via the L.T.'s?

    ANSWER: First, note that it wouldn't make any sense for the $x$ and $t$ coordinates themselves to transform via the L.T.'s, as we saw in Question 29. Second, as we noted in the preceding question, the 4-vector approach in Section 4.2 shows that $E$ and $p$ are proportional to the differences $\Delta t$ and $\Delta x$. So $E$ and $p$ have these differences built into them. (Of course, due to the linearity of the L.T.'s, differences of $E$'s and $p$'s also transform via the L.T.'s.)

43. QUESTION: In a nutshell, why isn't $F$ equal to $ma$ (or even $\gamma ma$) in relativity?

    ANSWER: $F$ equals $dp/dt$ in relativity, as it does in Newtonian physics. But the relativistic momentum is $p = \gamma mv$, and $\gamma$ changes with time. So $dp/dt = m(\gamma\dot{v} + \dot{\gamma}v) = \gamma ma + \dot{\gamma}mv$. The second term here isn't present in the Newtonian case.

44. QUESTION: In a given frame, does the acceleration vector $\mathbf{a}$ necessarily point along the force vector $\mathbf{F}$?

    ANSWER: No. We showed in Eq. (3.65) that $\mathbf{F} = m(\gamma^3 a_x, \gamma a_y)$. This isn't proportional to $(a_x, a_y)$. The different powers of $\gamma$ come from the facts that $\mathbf{F} = d\mathbf{p}/dt = d(\gamma m\mathbf{v})/dt$, and that $\gamma$ has a first-order change if $v_x$ changes, but not if $v_y$ changes, assuming that $v_y$ is initially zero. The particle therefore responds differently to forces in the $x$ and $y$ directions. It is easier to accelerate something in the transverse direction.

### General relativity

45. Question: How would the non equality (or non proportionality) of gravitational and inertial mass be inconsistent with the Equivalence Principle?

    Answer: In a box floating freely in space, two different masses that start at rest with respect to each other will remain that way. But in a freefalling box near the earth, two different masses that start at rest with respect to each other will remain that way if and only if their accelerations are equal. And since $F = ma \implies m_g g = m_i a \implies a = (m_g/m_i)g$, we see that the $m_g/m_i$ ratios must be equal if the masses are to remain at rest with respect to each other. That is, $m_g$ must be proportional to $m_i$. If this isn't the case, then the masses will diverge, which means that it is possible to distinguish between the two settings, in contradiction to the Equivalence Principle.

46. Question: You are in either a large box accelerating at $g$ in outer space or a large box on the surface of the earth. Is there any experiment you can do that will tell you which box you are in?

    Answer: Yes. The Equivalence Principle involves the word "local," or equivalently the words "small box." Under this assumption, you can't tell which box you are in. However, if the box is large, you can imagine letting go of two balls separated by a nonnegligible "vertical" distance. In the box in outer space, the balls will keep the same distance as they "fall." But near the surface of the earth, the gravitational force decreases with height, due to the $1/r^2$ dependence. The top ball will therefore fall slower, making the balls diverge. Alternatively, you can let go of two balls separated by a nonnegligible "horizontal" distance. In the box on the earth, the balls will head toward each other because the gravitational field lines converge to the center of the earth.

47. Question: In a gravitational field, if a low clock sees a high clock run fast by a factor $f_1$, and if a high clock sees a low clock run slow by a factor $f_2$, then $f_1 f_2 = 1$. But in a special-relativistic setup, both clocks see the other clock running slow by the factor $f_1 = f_2 = 1/\gamma$. So we have $f_1 f_2 = 1/\gamma^2 \neq 1$. Why is the product $f_1 f_2$ equal to 1 in the GR case but not in the SR case?

    Answer: In the GR case, both clocks are in the same frame. After a long time, the two clocks can be slowly moved together without anything exciting or drastic happening to their readings. And when the clocks are finally sitting next to each other, it is certainly true that if $B$'s clock reads, say, twice what $A$'s reads, then $A$'s must read half of what $B$'s reads.

    In contrast, the clocks in the SR case are *not* in the same frame. If we want to finally compare the clocks by bringing them together, then something drastic *does* happen with the clocks. The necessary acceleration that must take place leads to the accelerating clock seeing the other clock whip ahead; see Exercise 1.30. It is still certainly true (as it was in the GR case) that when the clocks are finally sitting next to each other, if $B$'s clock reads twice what $A$'s reads, then $A$'s reads half of what $B$'s reads. But this fact implies nothing about the product $f_1 f_2$ while the clocks are sailing past each other, because the clock readings (or at least one of them) necessarily change in a drastic manner by the time the clocks end up sitting next to each other.

48. QUESTION: How is the maximal-proper-time principle consistent with the result of the standard twin paradox?

    ANSWER: If twin $A$ floats freely in outer space, and twin $B$ travels to a distant star and back, then a simple time-dilation argument from $A$'s point of view tells us that $B$ is younger when he returns. This is consistent with the maximal-proper-time principle, because $A$ is under the influence of only gravity (zero gravity, in fact), whereas $B$ feels a normal force from the spaceship during the turning-around period. So $A$ ends up older.