Chapter 4

F=ma

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4.1 Introduction

Newton's laws

In the preceding two chapters, we dealt with *kinematics*. We took the motions of objects as given and then looked at positions, velocities, and accelerations as functions of time. We weren't concerned with the forces that caused the objects' motions. We will now deal with *dynamics*, where the goal is to understand *why* objects move the way they do. This chapter and the following ones will therefore be concerned with force, mass, energy, momentum, etc.

The motion of any object is governed by Newton's three laws:

- First Law: "A body moves with constant velocity (which may be zero) unless acted on by a force."

  If you think hard about this law, it seems a bit circular because we haven't defined what a force is. But if you think harder, there is in fact some content there. See Section 3.1 in Morin (2008) for a discussion of this.

- Second Law: "The rate of change of the momentum of a body equals the force acting on the body."

  In cases where the mass of the body doesn't change (we'll deal with the more general case in Chapter 6), this law becomes

  $$\mathbf{F} = m\mathbf{a}. \tag{4.1}$$

  This is a vector equation, so it is really three equations, namely $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$.

- Third Law: "Given two bodies A and B, if A exerts a force on B, then B exerts an equal and opposite force on A."

  As we'll see in Chapter 6, this law basically postulates conservation of momentum. There is a great deal of physical content in this law; it says that things don't happen in isolation by magic. Instead, if an object feels a force, then there must be another object somewhere feeling the opposite force.

The second law, $\mathbf{F} = m\mathbf{a}$, is the one we'll get the most mileage out of. The unit of force is called a newton (N), and from $F = ma$ we see that $1\,\text{N} = 1\,\text{kg}\,\text{m/s}^2$. A few types of forces (gravitational, tension, normal, friction, and spring) come up again and again, so let's take a look at each of these in turn.

Gravity

The gravitational force on an object near the surface of the earth is proportional to the mass of the object. More precisely, the force is mg downward, where $g = 9.8\,\text{m/s}^2$. This mg force is a special case of the more general gravitational force we will encounter in Chapter 11. Substituting mg for the force F in Newton's second law quickly gives $mg = ma \implies a = g$. That is, all objects fall with the same acceleration (in the absence of air resistance).

Tension

If you pull on a rope, the rope pulls back on you with the same force, by Newton's third law. The magnitude of this force is called the tension in the rope. As far as the direction of the tension goes, the question, "At a given interior point in the rope, which way does the tension point?" can't be answered. What we *can* say is that the tension at the given point pulls leftward on the point just to its right, and pulls rightward on the point just to its left. Equivalently, the tension pulls leftward on someone holding the right end of the rope, and it pulls rightward on someone holding the left end of the rope.

Normal force

Whereas a tension arises from a material resisting being stretched, a normal force arises from a material resisting being compressed. If you push leftward on the right face of a wooden block, then the normal force from the block pushes rightward on your hand. And similarly, the leftward force from your hand is itself a normal force pushing on the block. In the case where you push inward on the ends of an object shaped like a rod, people sometimes say that there is a "negative tension" in the rod, instead of calling it a normal force at the ends. But whatever name you want to use, the rod pushes back on you at the ends. If instead of a rigid rod we have a flexible rope, then the rope can support a tension, but not a normal force.

Friction

The friction force between two objects is extremely complicated on a microscopic scale. But fortunately we don't need to understand what is going on at that level to get a rough handle on friction forces. To a good approximation under most circumstances, we can say the following things about kinetic friction (where two objects are moving with respect to each other) and static friction (where two objects are at rest with respect to each other).

- Kinetic friction: If there is slipping between two objects, then to a good approximation under non-extreme conditions, the friction force is proportional to the normal force between the objects, with the constant of proportionality (called the *coefficient of kinetic friction*) labeled as μ_k:

  $$F_k = \mu_k N. \tag{4.2}$$

  The friction force is independent of the contact area and relative speed. The direction of the friction force on a given object is opposite to the direction of the velocity of that object relative to the other object.

- Static friction: If there is no slipping between two objects, then to a good approximation under non-extreme conditions, the *maximum value* of the friction force is proportional to the normal force between the objects, with the constant of proportionality (called the *coefficient of static friction*) labeled as μ_s:

  $$F_s \le \mu_s N. \tag{4.3}$$

  As in the kinetic case, the friction force is independent of the contact area. Note well the *equality* in Eq. (4.2) and the *inequality* in Eq. (4.3). Equation (4.3) gives only an *upper limit* on the static friction force. If you push on an object with a force that is smaller than $\mu_s N$, then the static friction force is exactly equal and opposite to your force, and the object stays at rest.[1] But if you increase your force so that it exceeds $\mu_s N$, then the maximum friction force isn't enough to keep the object at rest. So it will move, and the friction force will abruptly drop to the kinetic value of $\mu_k N$. (It turns out that μ_k is always less than or equal to μ_s; see Problem 4.1 for an explanation why.)

The expressions in Eqs. (4.2) and (4.3) will of course break down under extreme conditions (large normal force, high relative speed, pointy shapes). But they work fairly well under normal conditions. Note that the coefficients of kinetic and static friction, μ_k and μ_s, are properties of *both* surfaces together. A single surface doesn't have a coefficient of friction. What matters is how two surfaces interact.

Spring force

To a good approximation for small displacements, the restoring force from a spring is proportional to the stretching distance. That is, $F = -kx$, where k is the *spring constant.* A large value of k means a stiff spring; a small value means a weak spring. The reason why the $F = -kx$ relation is a good approximation for small displacements in virtually any system is explained in Problem 10.1.

If x is positive then the force F is negative; and if x is negative then F is positive. So $F = -kx$ does indeed describe a restoring force, where the spring always tries to bring x back to zero. The $F = -kx$ relation, known as *Hooke's law,* breaks down if x is too large, but we'll assume that it holds for the setups we're concerned with.

The tension and normal forces discussed above are actually just special cases of spring forces. If you stand on a floor, the floor acts like a very stiff spring. The matter in the floor compresses a tiny amount, exactly the amount that makes the upward "spring" force (which we call a normal force in this case) be equal to your weight.

We'll always assume that our springs are massless. Massive springs can get very complicated because the force (the tension) will in general vary throughout the spring. In a massless spring, the force is the same everywhere in it. This follows from the reasoning in Problem 4.3(a).

Centripetal force

The centripetal force is the force that keeps an object moving in a circle. Since we know from Eq. (3.7) that the acceleration for uniform (constant speed) circular motion points radially inward with magnitude $a = v^2/r$, the centripetal force likewise points radially inward with magnitude $F = ma = mv^2/r$. This force might be due to the tension in a string, or the friction force acting on a car's tires as it rounds a corner, etc. Since $v = r\omega$, we can also write F as $mr\omega^2$.

The term "centripetal" isn't the same type of term as the above "gravity," "tension," etc. descriptors, because the latter terms describe the *type* of force, whereas "centripetal" simply describes the *direction* of the force (radially inward). The word "centripetal" is therefore more like the words "eastward" or "downward," etc. For example, we might say, "The downward force is due to gravity," or "The centripetal force is due to the tension in a string." The centripetal force is *not* a magical special new kind of force. It is simply one of the standard forces (or a combination of them) that points radially inward, and whose magnitude we know always equals mv^2/r.

Free-body diagrams

The "**F**" in Newton's second law in Eq. (4.1) is the *total* force on an object, so it is important to determine what all the various forces are. The best way to do this is to draw a picture. The picture of an object that shows all of the forces acting on it is called a *free-body diagram.* More precisely,

[1] The friction force certainly can't be *equal* to $\mu_s N$ in this case, because if you push rightward with a very small force, then the leftward (incorrect) $\mu_s N$ friction force would cause there to be a nonzero net force, which would hurl the object leftward back toward you!

a free-body diagram shows all of the *external* forces (that is, forces due to other objects) acting on a given object. There are undoubtedly also *internal* forces acting within the object; each atom might be pushing or pulling on the atom next to it. But these internal forces cancel in pairs (by Newton's third law), so they don't produce any acceleration of the object. Only external forces can do that. (We'll assume we have a rigid object, so that the distance between any two given points remains fixed.)

In simple cases (for example, ones involving only one force), you can get away with not drawing a free-body diagram. But in more complicated cases (for example, ones involving forces pointing in various directions), a diagram is absolutely critical. A problem is often hopeless without a diagram, but trivial with one.

The length of a force vector is technically a measure of the magnitude of the force. But when drawing a free-body diagram, the main point is just to indicate all the forces that exist. In general we don't yet know the relative sizes, so it's fine to give all the vectors the same length (unless it's obvious that a certain force is larger than another). Also, the exact location of each force vector isn't critical (at least in this chapter), although the most sensible thing to do is to draw the vector near the place where the force acts. However, when we discuss torque in Chapter 7, the location of the force *will* be important.

Note that due to Newton's third law, for every force vector that appears in the free-body diagram for one object, there is an opposite force vector that appears in the free-body diagram for another object. An example involving two blocks on a table is shown in Fig. 4.1. If a person applies a force F to the left block, then the two free-body diagrams are shown (assume there is no friction from the table). Note that the force pushing the right block rightward is *only* the normal force between the blocks, and *not* the applied force F. True, the N force wouldn't exist without the F force, but the right block feels only the N force; it doesn't care about the original cause of N.

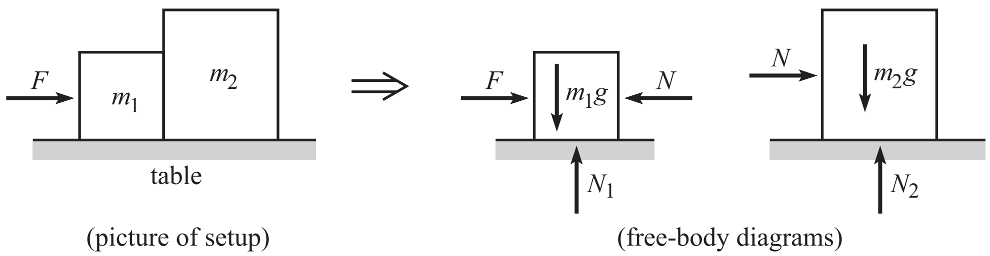


Figure 4.1

If we happened to be concerned also with the free-body diagram for the person applying the force F, then we would draw a force F acting leftward, along with a downward mg gravitational force and an upward normal force from the ground (and also probably a rightward friction force from the ground). Likewise, the free-body diagram for the table would involve gravity and the N_1 and N_2 normal forces pointing downward, along with upward normal forces from the ground at the bases of the legs. But if we're concerned only with the blocks, then all of this other information is irrelevant.

If the direction of the acceleration of an object is known, it is often helpful to draw the acceleration vector in the free-body diagram. But you should be careful to draw this vector with a dotted line or something similar, so that you don't mistake it for a force. Remember that although F equals ma, the quantity ma is *not* a force; see the last section of this introduction.

Atwood's machines

The name *Atwood's machine* is the term used for any system of pulleys, strings, and masses. Although a subset of these systems is certainly very useful in everyday life (a "block and tackle" enables you to lift heavy objects; see Problem 4.4), the main reason for all the Atwood's problems in this chapter is simply that they're good practice for drawing free-body diagrams and applying $F = ma$. An additional ingredient in solving any Atwood's problem is the so-called "conservation of string" relation. This is the condition that the length of any given string doesn't change. This constrains the motion of the various masses and pulleys. A few useful Atwood's facts that come up again and again are derived in Problem 4.3. In this chapter we will assume that all strings and pulleys are massless.

The four forces

Having discussed many of the forces we see in everyday life, we should make at least a brief mention of where all these forces actually come from. There are four known fundamental forces in nature:

- Gravitational: Any two masses attract each other gravitationally. We are quite familiar with the gravitational force due to the earth. The gravitational force between everyday-sized objects is too small to observe without sensitive equipment. But on the planetary scale and larger, the gravitational force dominates the other three forces.

- Electromagnetic: The single word "electromagnetic" is indeed the proper word to use here, because the electric and magnetic forces are two aspects of the same underlying force. (In some sense, the magnetic force can be viewed as a result of combining the electric force with special relativity.) Virtually all everyday forces have their origin in the electric force. For example, a tension in a string is due to the electric forces holding the molecules together in the string.

- Weak: The weak force is responsible for various nuclear processes; it isn't too important in everyday life.

- Strong: The strong force is responsible for holding the protons and neutrons together in a nucleus. Without the strong force, matter as we know it wouldn't exist. But taking the existence of matter for granted, the strong force doesn't show up much in everyday life.

ma is not a force!

Newton's second law is "F equals ma," which says that ma equals a force. Does this imply that *ma is* a force? Absolutely not. What the law says is this: Write down the sum of all the forces on an object, and also write down the mass times the acceleration of the object. The law then says that these two quantities have the same value. This is what a physical law does; it says to take two things that aren't obviously related, and then demand that they are equal. In a simple freefall setup, the $F = ma$ equation is $mg = ma$, which tells us that a equals g. But just because $a = g$, this doesn't mean that the two side of $mg = ma$ represent the same type of thing. The left side is a force, the right side is a mass times an acceleration. So when drawing a free-body diagram, you should *not* include ma as one of the forces. If you do, you will end up double counting things. However, as mentioned above, it is often helpful to indicate the acceleration of the object in the free-body diagram. Just be careful to distinguish this from the forces by drawing it with a dotted line.

4.2 Multiple-choice questions

4.1. Two people pull on opposite ends of a rope, each with a force F. The tension in the rope is

(a) $F/2$ (b) F (c) $2F$

4.2. You accelerate the two blocks in Fig. 4.2 by pushing on the bottom block with a force F. The top block moves along with the bottom block. What force directly causes the top block to accelerate?

(a) the normal force between the blocks
(b) the friction force between the blocks
(c) the gravitational force on the top block
(d) the force you apply to the bottom block

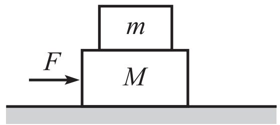


Figure 4.2

4.3. Three boxes are pushed with a force F across a frictionless table, as shown in Fig. 4.3. Let N_1 be the normal force between the left two boxes, and let N_2 be the normal force between the right two boxes. Then

(a) $F = N_1 = N_2$
(b) $F + N_1 = N_2$
(c) $F > N_1 = N_2$
(d) $F < N_1 < N_2$
(e) $F > N_1 > N_2$

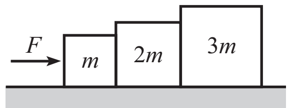


Figure 4.3

4.4. Two blocks with masses 2 kg and 1 kg lie on a frictionless table. A force of 3 N is applied as shown in Fig. 4.4. What is the normal force between the blocks?

(a) 0 (b) 0.5 N (c) 1 N (d) 2 N (e) 3 N

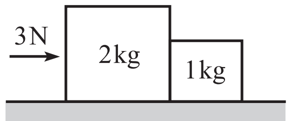


Figure 4.4

4.5. In the system shown in Fig. 4.5, the ground is frictionless, the blocks have mass m and $2m$, and the string connecting them is massless. If you accelerate the system to the right, as shown, the tension is the same everywhere throughout the string connecting the masses because

(a) the string is massless
(b) the ground is frictionless
(c) the ratio of the masses is 2 to 1
(d) the acceleration of the system is nonzero
(e) The tension is the same throughout any string; no conditions are necessary.

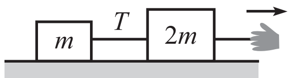


Figure 4.5

4.6. You are in a plane accelerating down a runway during takeoff, and you are holding a pendulum (say, a shoe hanging from a shoelace). The string of the pendulum

(a) hangs straight downward
(b) hangs downward and forward, because the net force on the pendulum must be zero
(c) hangs downward and forward, because the net force must be nonzero
(d) hangs downward and backward, because the net force must be zero
(e) hangs downward and backward, because the net force must be nonzero

4.7. When you stand at rest on a floor, you exert a downward normal force on the floor. Does this force cause the earth to accelerate in the downward direction?

(a) Yes, but the earth is very massive, so you don't notice the motion.
(b) Yes, but you accelerate along with the earth, so you don't notice the motion.
(c) No, because the normal force isn't a real force.
(d) No, because you are also pulling on the earth gravitationally.
(e) No, because there is also friction at your feet.

4.8. If you stand at rest on a bench, the bench exerts a normal force on you, equal and opposite to your weight. Which force is related to this normal force by Newton's third law?

(a) the gravitational force from the earth on you

(b) the gravitational force from you on the earth

(c) the normal force from you on the bench

(d) none of the above

4.9. The driver of a car steps on the gas, and the car accelerates with acceleration a. When writing down the horizontal $F = ma$ equation for the car, the "F" acting on the car is

(a) the normal force between the tires and the ground

(b) the friction force between the tires and the ground

(c) the force between the driver's foot and the pedal

(d) the energy obtained by burning the gasoline

(e) the backward friction force that balances the forward ma force

4.10. The static friction force between a car's tires and the ground can do all of the following *except*

(a) speed the car up

(b) slow the car down

(c) change the car's direction

(d) It can do all of the above things.

4.11. A car is traveling forward along a road. The driver wants to arrange for the car's acceleration to point diagonally backward and leftward. The driver should

(a) turn right and accelerate

(b) turn right and brake

(c) turn left and accelerate

(d) turn left and brake

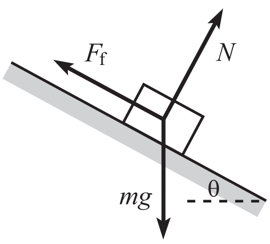


Figure 4.6

4.12. A block is at rest on a plane inclined at angle θ. The forces on it are the gravitational, normal, and friction forces, as shown in Fig. 4.6. These are *not* drawn to scale. Which of the following statements is *always* true, for any θ?

(a) $mg \le N$ and $mg \le F_f$

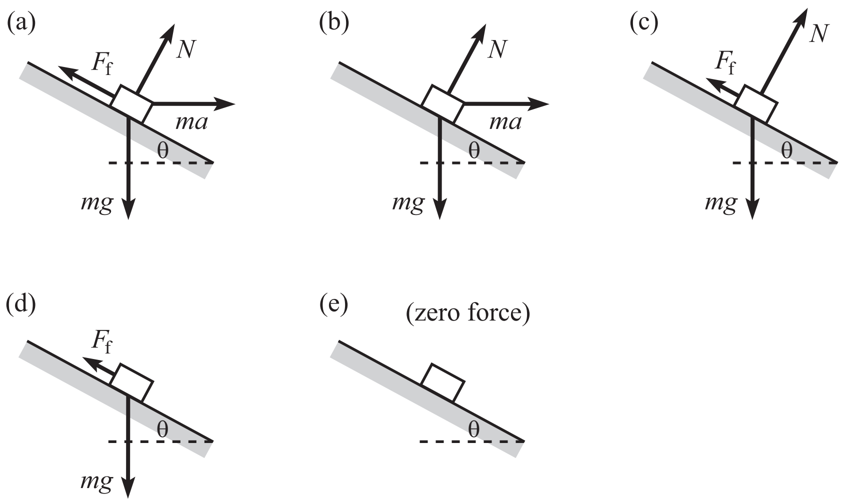
(b) $mg \ge N$ and $mg \ge F_f$

(c) $F_f = N$

(d) $F_f + N = mg$

(e) $F_f > N$ if $\mu_s > 1$

4.13. A block sits on a plane, and there is friction between the block and the plane. The plane is accelerated to the right. If the block remains at the same position on the plane, which of the following pictures might show the free-body diagram for the block? (All of the vectors shown are forces.)



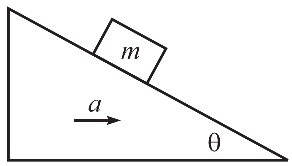


Figure 4.7

4.14. A block with mass m sits on a frictionless plane inclined at angle θ, as shown in Fig. 4.7. If the plane is accelerated to the right with the proper acceleration that causes the block to remain at the same position with respect to the plane, what is the normal force between the block and the plane?

(a) mg (b) $mg\sin\theta$ (c) $mg/\sin\theta$ (d) $mg\cos\theta$ (e) $mg/\cos\theta$

4.15. A bead is arranged to move with *constant* speed around a hoop that lies in a vertical plane. The magnitude of the net force on the bead is

(a) largest at the bottom

(b) largest at the top

(c) largest at the side points

(d) the same at all points

4.16. A toy race car travels through a loop-the-loop (a circle in a vertical plane) on a track. Assuming that the speed at the top of the loop is above the threshold to remain in contact with the track, the car's acceleration at the top is

(a) downward and larger than g

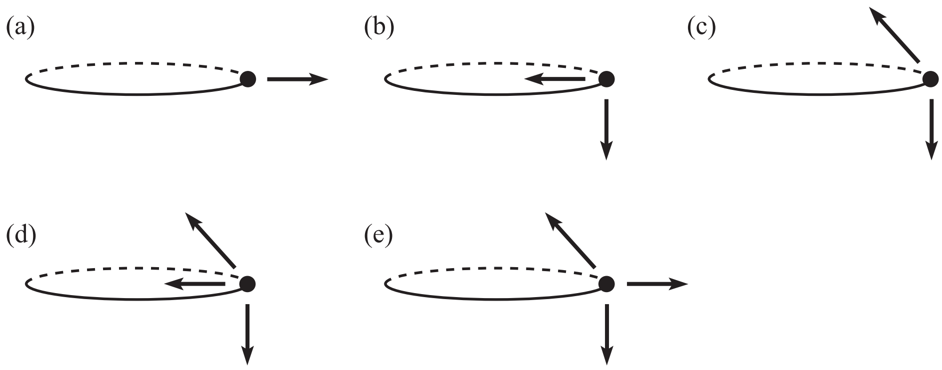
(b) downward and smaller than g

(c) zero

(d) upward and smaller than g

(e) upward and larger than g

4.17. A plane in a holding pattern is flying in a horizontal circle at constant speed. Which of the following free-body diagrams best illustrates the various forces acting on the plane at the instant shown? (See Problem 4.20 for a quantitative treatment of this setup.)



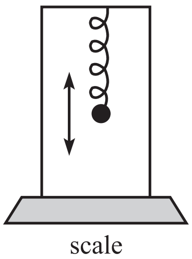


Figure 4.8

4.18. A mass hangs from a spring and oscillates vertically. The top end of the spring is attached to the top of a box, and the box is placed on a scale, as shown in Fig. 4.8. The reading on the scale is largest when the mass is

(a) at its maximum height
(b) at its minimum height
(c) at the midpoint of its motion
(d) All points give the same reading.

4.19. A spring with spring constant k hangs vertically from a ceiling, initially at its relaxed length. You attach a mass m to the end and bring it down to a position that is $3mg/k$ below the initial position. You then let go. What is the upward acceleration of the mass right after you let go?

(a) 0 (b) g (c) $2g$ (d) $3g$ (e) $4g$

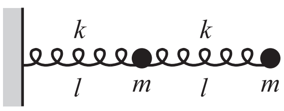


Figure 4.9

4.20. Two springs both have spring constant k and relaxed length zero. They are each stretched to a length ℓ and then attached to two masses and a wall, as shown in Fig. 4.9. The masses are simultaneously released. Immediately afterward, the magnitudes of the accelerations of the left and right masses are, respectively,

(a) $2k\ell/m$ and $k\ell/m$
(b) $k\ell/m$ and $2k\ell/m$
(c) $k\ell/m$ and $k\ell/m$
(d) 0 and $2k\ell/m$
(e) 0 and $k\ell/m$

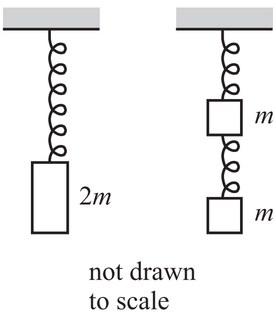


Figure 4.10

4.21. A mass $2m$ suspended from a given spring causes it to stretch relative to its relaxed length. The mass and the spring are then each cut into two identical pieces and connected as shown in Fig. 4.10. Is the bottom of the lower mass higher than, lower than, or at the same height as the bottom of the original mass? (This one takes a little thought.)

(a) higher
(b) lower
(c) same height

4.22. What is the conservation-of-string relation for the Atwood's machine shown in Fig. 4.11? All of the accelerations are defined to be positive upward.

(a) $a_3 = -2(a_1 + a_2)$

(b) $a_3 = -(a_1 + a_2)$

(c) $a_3 = -(a_1 + a_2)/2$

(d) $a_3 = -(a_1 + a_2)/4$

(e) $a_3 = -2a_2$

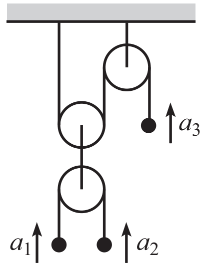


Figure 4.11

4.23. What is the conservation-of-string relation for the Atwood's machine shown in Fig. 4.12? All of the accelerations are defined to be positive upward.

(a) $a_3 = -a_1 - a_2$

(b) $a_3 = -2a_1 - 2a_2$

(c) $a_3 = -4a_2$

(d) $2a_3 = -a_1 - a_2$

(e) $4a_3 = -a_1 - a_2$

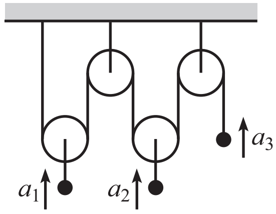


Figure 4.12

4.24. What is the conservation-of-string relation for the Atwood's machine shown in Fig. 4.13? All of the accelerations are defined to be positive upward.

(a) $a_1 = a_3$

(b) $a_1 = -a_3$

(c) $a_2 = -(a_1 + a_3)/2$

(d) $a_2 = -(a_1 + a_3)$

(e) $a_2 = -2(a_1 + a_3)$

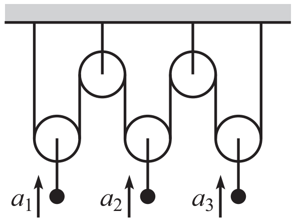


Figure 4.13

4.3 Problems

The first five problems are foundational problems.

4.1. **Coefficients of friction**

Explain why the coefficient of static friction, μ_s, must always be at least as large as the coefficient of kinetic friction, μ_k.

4.2. **Cutting a spring in half**

A spring has spring constant k. If it is cut in half, what is the spring constant of each of the resulting shorter springs?

4.3. **Useful Atwood's facts**

In the Atwood's machine shown in Fig. 4.14(a), the pulleys and strings are massless (as we will assume in all of the Atwood's problems in this chapter). Explain why (a) the tension is the same throughout the long string, as indicated, (b) the tension in the bottom string is twice the tension in the long string, as indicated, and (c) the acceleration of the right mass is negative twice the acceleration of the left mass.

Also, in Fig. 4.14(b), explain why (d) the acceleration of the left mass equals negative the average of the accelerations of the right two masses.

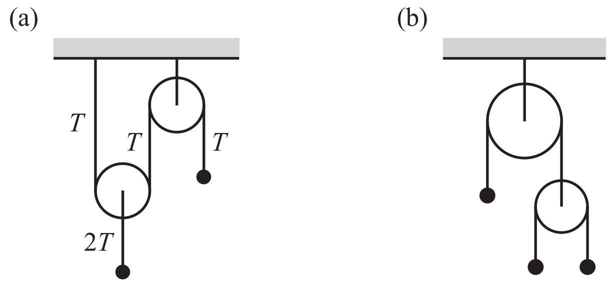


Figure 4.14

4.4. **Block and tackle**

(a) What force on the rope must be exerted by the person in Fig. 4.15(a) in order to hold up the block, or equivalently to move it upward at constant speed? The rope wraps twice around the top of the top pulley and the bottom of the bottom pulley. (Assume that the segment of rope attached to the center of the top pulley is essentially vertical.)

(b) Now consider the case where the person (with mass m) stands on the block, as shown in Fig. 4.15(b). What force is now required?

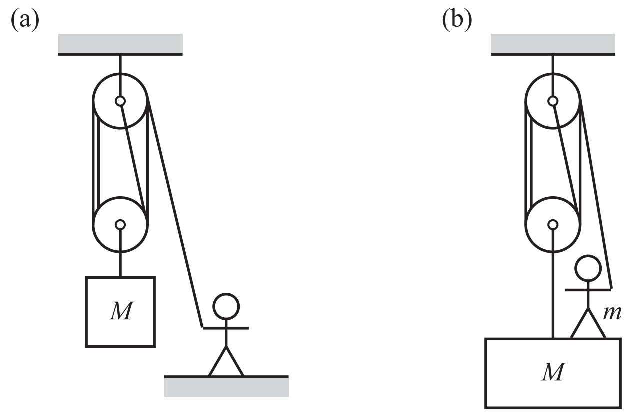


Figure 4.15

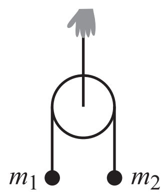


Figure 4.16

4.5. **Equivalent mass**

In Fig. 4.16 you support the pulley system, with your hand at rest. If you have your eyes closed and think that you are instead supporting a single mass M at rest, what is M in terms of m_1 and m_2? Is M simply equal to $m_1 + m_2$?

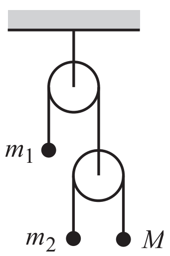


Figure 4.17

The following eight problems involve Atwood's machines. This large number of Atwood's problems shouldn't be taken to imply that they're terribly important in physics (they're not). Rather, they are included here because they provide good practice with $F = ma$.

4.6. **Atwood's 1**

Consider the Atwood's machine shown in Fig. 4.17. The masses are held at rest and then released. In terms of m_1 and m_2, what should M be so that m_1 doesn't move? What relation must hold between m_1 and m_2 so that such an M exists?

4.7. **Atwood's 2**

Consider the Atwood's machine shown in Fig. 4.18. Masses of m and $2m$ lie on a frictionless table, connected by a string that passes around a pulley. The pulley is connected to another mass of $2m$ that hangs down over another pulley, as shown. Find the accelerations of all three masses.

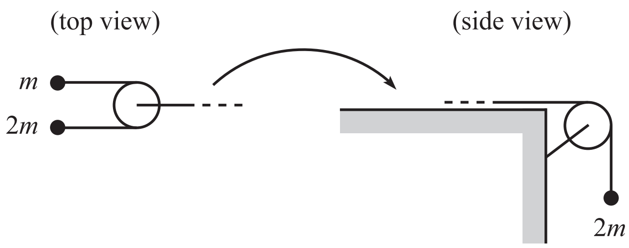


Figure 4.18

4.8. **Atwood's 3**

Consider the Atwood's machine shown in Fig. 4.19, with masses m, m, and $2m$. Find the acceleration of the mass $2m$.

4.9. **Atwood's 4**

In the Atwood's machine shown in Fig. 4.20, both masses are m. Find their accelerations.

4.10. **Atwood's 5**

Consider the triple Atwood's machine shown in Fig. 4.21. What is the acceleration of the rightmost mass? *Note*: The math isn't as bad as it might seem at first. You should take advantage of the fact that many of your $F = ma$ equations look very similar.

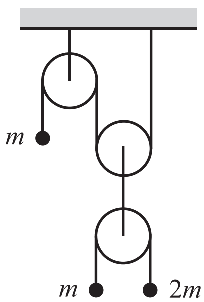


Figure 4.19

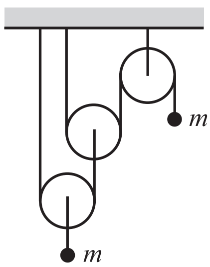


Figure 4.20

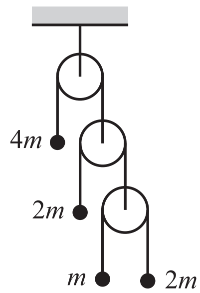


Figure 4.21

4.11. **Atwood's 6**

Consider the Atwood's machine shown in Fig. 4.22. The middle mass is glued to the long string. Find the accelerations of all three masses, and also the tension everywhere in the long string.

4.12. **Atwood's 7**

In the Atwood's machine shown in Fig. 4.23, both masses are m. Find their accelerations.

4.13. **Atwood's 8**

In the Atwood's machine shown in Fig. 4.24, both masses are m. The two strings that touch the center of the left pulley are both attached to its axle. Find the accelerations of the masses.

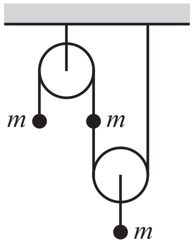


Figure 4.22

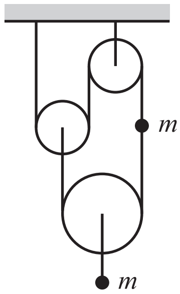


Figure 4.23

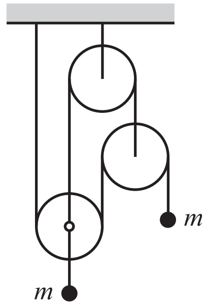
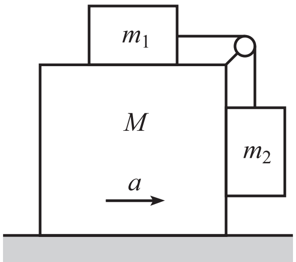


Figure 4.24



(side view)

Figure 4.25

4.14. **No relative motion**

All of the surfaces in the setup in Fig. 4.25 are frictionless. You push on the large block and give it an acceleration a. For what value of a is there no relative motion among the masses?

4.15. **Slipping blocks**

A block with mass m sits on top of a block with mass $2m$ which sits on a table. The coefficients of friction (both static and kinetic) between all surfaces are $\mu_s = \mu_k = 1$. A string is connected to each mass and wraps halfway around a pulley, as shown in Fig. 4.26. You pull on the pulley with a force of $6mg$.

(a) Explain why the bottom block must slip with respect to the table. *Hint*: Assume that it doesn't slip, and show that this leads to a contradiction.

(b) Explain why the top block must slip with respect to the bottom block. (Same hint.)

(c) What is the acceleration of your hand?

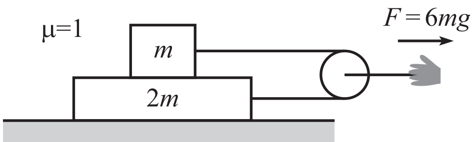


Figure 4.26

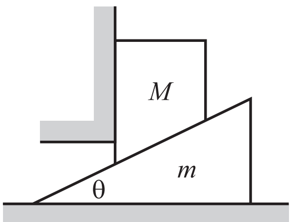


Figure 4.27

4.16. **Block and wedge**

A block with mass M rests on a wedge with mass m and angle θ, which lies on a table, as shown in Fig. 4.27. All surfaces are frictionless. The block is constrained to move vertically by means of a wall on its left side. What is the acceleration of the wedge?

4.17. **Up and down a plane**

A block with mass m is projected up along the surface of a plane inclined at angle θ. The initial speed is v_0, and the coefficients of both static and kinetic friction are equal to 1. The block reaches a highest point and then slides back down to the starting point.

(a) Show that in order for the block to in fact slide back down (instead of remaining at rest at the highest point), θ must be greater than 45°.

(b) Assuming that $\theta > 45°$, find the times of the up and down motions.

(c) Assuming that $\theta > 45°$, is the total up and down time longer or shorter than the total time it would take (with the same initial v_0) if the plane were frictionless? Or does the answer to this question depend on what θ is? (The solution to this gets a little messy.)

4.18. **Rope in a tube**

A rope is free to slide frictionlessly inside a circular tube that lies flat on a horizontal table. In part (a) of Fig. 4.28, the rope moves at constant speed. In part (b) of the figure, the rope is at rest, and you pull on its right end to give it a tangential acceleration. What is the direction of the net force on the rope in each case?

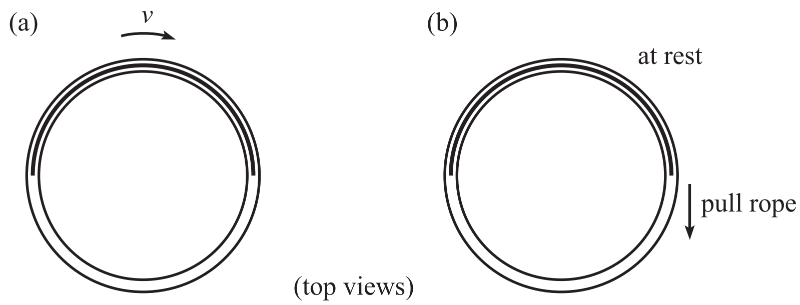


Figure 4.28

4.19. **Circling bucket**

You hold the handle of a bucket of water and swing it around in a vertical circle, keeping your arm straight. If you swing it around fast enough, the water will stay inside the bucket, even at the highest point where the bucket is upside down. What, roughly, does "fast enough" here mean? (You can specify the maximum time of each revolution.) Make whatever reasonable assumptions you want to make for the various parameters involved. You can work in the approximation where the speed of the bucket is roughly constant throughout the motion.

4.20. **Banking an airplane**

A plane in a holding patter flies at speed v in a horizontal circle of radius R. At what angle should the plane be banked so that you don't feel like you are getting flung to the side in your seat? At this angle, what is your apparent weight (that is, what is the normal force from the seat)?

4.21. **Breaking and turning**

You are driving along a horizontal straight road that has a coefficient of static friction μ with your tires. If you step on the brakes, what is your maximum possible deceleration? What is it if you are instead traveling with speed v around a bend with radius of curvature R?

4.22. **Circle of rope**

A circular loop of rope with radius R and mass density λ (kg/m) lies on a frictionless table and rotates around its center, with all points moving at speed v. What is the tension in the rope? *Hint*: Consider the net force on a small piece of rope that subtends an angle $d\theta$.

4.23. **Cutting the string**

A mass m is connected to the end of a massless string of length ℓ. The top end of the string is attached to a ceiling that is a distance ℓ above the floor. Initial conditions have been set up so that the mass swings around in a horizontal circle, with the string always making an angle θ with respect to the vertical, as shown in Fig. 4.29. If the string is cut, what horizontal distance does the mass cover between the time the string is cut and the time the mass hits the floor?

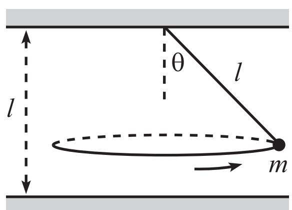


Figure 4.29

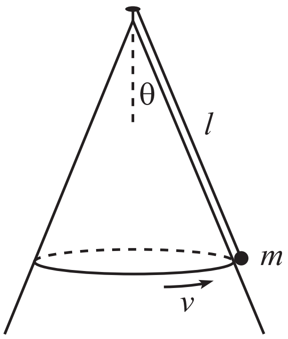
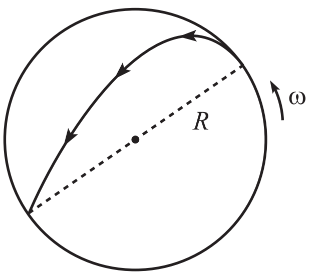


Figure 4.30



(side view)

Figure 4.31

4.24. **Circling around a cone**

A mass m is attached by a massless string of length ℓ to the tip of a frictionless cone, as shown in Fig. 4.30. The half-angle at the vertex of the cone is θ. If the mass moves around in a horizontal circle at speed v on the cone, find (a) the tension in the string, (b) the normal force from the cone, and (c) the maximum speed v for which the mass stays in contact with the cone.

4.25. **Penny in a dryer**

Consider a clothes dryer with radius R, which spins with angular frequency ω. (In other words, consider a cylinder that spins with angular frequency ω around its axis, which is oriented horizontally.) A small object, such as a penny, is in the dryer. The penny rotates along with the dryer and gets carried upward, but eventually loses contact with the dryer and sails through the air (as clothes do in a dryer), eventually coming in contact with the dryer again. Assume for simplicity that the coefficient of friction is very large, so that the penny doesn't slip with respect to the dryer as long as the normal force is nonzero.

Let's assume that you want the trajectory of the penny to look like the one shown in Fig. 4.31, starting and ending at diametrically opposite points. In order for this to happen, where must the penny lose contact with the dryer? (Give the angle θ with respect to the vertical.) What must ω be, in terms of g and R?

4.4 Multiple-choice answers

4.1. [b] The tension is simply F. A common error is to double F (because there are two people pulling) and say that the tension is $2F$. This is incorrect, because every piece of the rope pulls on the piece to its right with a force F, and also on the piece to its left with a force F. This common value is the tension.

REMARK: If you try to change the setup by replacing one person with a wall, then the wall still pulls on the rope with a force F (assuming that the other person still does), so the setup hasn't actually changed at all. If you instead remove one person (say, the left one) and replace her with nothing, then the setup has certainly changed. The remaining (right) person will accelerate the rope rightward, and the tension will vary over the length (assuming that the rope has mass). Points closer to the left end don't have as much mass to their left that they need to accelerate, so the tension is smaller there.

4.2. [b] Friction is the horizontal force that acts on the top block.

REMARK: The friction force wouldn't exist without the normal force between the blocks, which in turn wouldn't exist without the gravitational force on the top block. But these forces are vertical and therefore can't directly cause the horizontal acceleration of the top block. Likewise, the friction force wouldn't exist if you weren't pushing on the bottom block. But your force isn't what directly causes the acceleration of the top block; if the surfaces are greased down, then you can push on the bottom block all you want, and the top block won't move.

4.3. [e] The three boxes all have the same acceleration; call it a. Then the force F equals $F = (6m)a$ because this is the force that accelerates all three boxes, which have a total mass of $6m$. Similarly, $N_1 = (5m)a$ because N_1 is the force that accelerates the right two boxes. And $N_2 = (3m)a$ because N_2 is the force that accelerates only the right box. Therefore, $F > N_1 > N_2$. As a double check, the net force on the middle block is $N_1 - N_2 = 5ma - 3ma = 2ma$, which is correctly $(2m)a$.

4.4. [c] The acceleration of the system is $a = F/m = (3\,\text{N})/(3\,\text{kg}) = 1\,\text{m/s}^2$. The normal force N on the 1 kg block is what causes this block to accelerate at $1\,\text{m/s}^2$, so N must be given by $N = ma = (1\,\text{kg})(1\,\text{m/s}^2) = 1\,\text{N}$.

4.5. [a] The massless nature of the string implies that the tension is the same everywhere throughout it, because if the tension varied along the length, then there would exist a massless piece that had a net force acting on it, yielding infinite acceleration. Conversely, if the string had mass, then the tension would have to vary along it, so that there would be a net force on each little massive piece, to cause the acceleration (assuming the acceleration is nonzero). So none of the other answers can be the reason why the tension is the same everywhere along the string.

4.6. [e] The pendulum is accelerating forward (as is everything else in the plane), so there must be a forward net force on it. If the pendulum hangs *downward* and *backward*, then the tension force on the pendulum's mass is *upward* and *forward*. The upward component cancels the gravitational force (the weight), and the forward (uncanceled) component is what causes the forward acceleration.

REMARK: If the pendulum's mass is m, then from the above reasoning, the vertical component of the tension is mg, and the horizontal component is ma (where a is the acceleration of the plane and everything in it). So if the string makes an angle θ with the vertical, then $\tan\theta = ma/mg \implies a = g\tan\theta$. You can use this relation to deduce your acceleration from a measurement of the angle θ. Going in the other direction, a typical plane might have a takeoff acceleration of around $2.5\,\text{m/s}^2$, in which case we can deduce what θ is: $\tan\theta = a/g \approx 1/4 \implies \theta \approx 15^\circ$. For comparison, a typical car might be able to go from 0 to 60 mph (27 m/s) in 7 seconds, which implies an acceleration of about $4\,\text{m/s}^2$ and an angle of $\theta = 22^\circ$.

A more extreme case is a fighter jet taking off from an aircraft carrier. With the help of a catapult, the acceleration can be as large as $3g \approx 30\,\text{m/s}$. A pendulum in the jet would therefore hang at an angle of $\tan^{-1}(3) \approx 72^\circ$, which is more horizontal than vertical. In the accelerating reference frame of the jet (accelerating frames are the subject of Chapter 12, so we're getting ahead of ourselves here), the direction of the hanging pendulum defines "downward." So the pilot effectively lives in a world where gravity points diagonally downward and backward at an angle of 72° with respect to the vertical. The direction of the jet's forward motion along the runway is therefore nearly *opposite* to this "downward" direction (as opposed to being roughly perpendicular to downward in the case of a passenger airplane). The jet pilot will therefore have the sensation that he is flying *upward*, even though he is actually moving horizontally along the runway. A possible dangerous consequence of this sensation is that if it is nighttime and there are minimal visual cues, the pilot may mistakenly try to correct this "error" by turning downward, causing the jet to crash into the ocean.

4.7. [d] The normal force from you on the earth is equal and opposite to the gravitational force from you on the earth. (Yes, you pull on the earth, just as it pulls on you.) So the net force on the earth is zero, and it therefore doesn't accelerate.

REMARK: Similarly, *you* also don't accelerate, because the normal force from the earth on you is equal and opposite to the gravitational force from the earth on you. So the net force on you is zero. This wouldn't be the case if instead of standing at rest, you jump upward. You are now accelerating upward (while your feet are in contact with the ground). And consistent with this, the upward normal force from the earth on you is *larger* than the downward gravitational force from the earth on you. So the net force on you is upward. In contrast, after you leave the ground the normal force drops to zero, so the downward gravitational force is all there is, and you accelerate downward.

4.8. [c] Newton's third law says that the forces that *two* bodies exert on *each other* are equal in magnitude and opposite in direction. The given force is the (upward) normal force from the bench on you. The two objects here are the *bench* and *you*, so the force that is related by the third law must be the (downward) normal force from you on the bench. Similarly, choices (a) and (b) are a third-law pair.

REMARK: The earth isn't relevant at all in the third-law statement concerning you and the bench. Of course, the gravitational force from the earth on you (that is, your weight) *is* related to the given normal force from the bench on you, but this relation does *not* involve the third law. It involves the *second* law. More precisely, the second law says that since you are at rest (and hence not accelerating), the total force on you must be zero. So the downward force from the earth on you (your weight) must be equal and opposite to the upward normal force from the bench on you. Note that since three objects (earth, you, bench) were mentioned in the preceding statement, there is no way that it can be a third-law statement, because such a statement must involve only *two* objects. Just because two forces are equal and opposite, this doesn't mean they are related by the third law.

4.9. [b] The friction force is the horizontal force that makes the car accelerate; you won't go anywhere on ice. The other choices are incorrect because: (a) the normal force is vertical, (c) the force applied to the pedal is an internal force within the car, and only external forces appear in $F = ma$ (and besides, the force on the pedal is far smaller than the friction between the tires and the ground), (d) energy isn't a force, and (e) the friction force doesn't point backward, and ma isn't a force! (See the discussion on page 72.)

4.10. [d] When you step on the gas, the friction force speeds you up (if you are on ice, you won't go anywhere). When you hit the brakes, the friction force slows you down (if you are on ice, you won't slow down). And when you turn the steering wheel, the friction force is the centripetal force that causes you to move in the arc of a circle as you change your direction (if you are on ice, your direction won't change).

REMARK: Note that unless you are skidding (which rarely happens in everyday driving), the friction force between the tires and the ground is static and not kinetic. The point on a tire that is instantaneously in contact with the ground is instantaneously at rest. (The path traced out by a point on a rolling wheel is known as a *cycloid*, and the speed of the point is zero where it touches the ground.) If this weren't the case (that is, if we were perpetually skidding in our cars), then we would need to buy new tires every week, and we would be listening constantly to the sound of screeching tires.

4.11. [d] Braking will yield a backward component of the acceleration, and turning left will yield a leftward component. This leftward component is the centripetal acceleration for the circular arc (at least locally) that the car is now traveling in.

REMARK: Depending on what you do with the gas pedal, brake, and steering wheel, the total acceleration vector (or equivalently, the total force vector) can point in any horizontal direction. The acceleration can have a forward or backward component, depending on whether you are stepping on the gas or the brake. And it can have a component to either side if the car is turning. The relative size of these components is arbitrary, so the total acceleration vector can point in any horizontal direction.

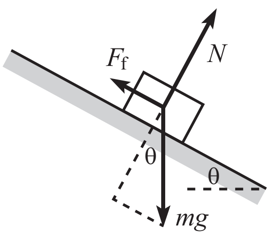


Figure 4.32

4.12. [b] The total acceleration (and hence total force) is zero. In Fig. 4.32 we have broken the $m\mathbf{g}$ force into its components parallel are perpendicular to the plane, and we have drawn the forces to scale. Zero net force perpendicular to the plane gives $N = mg\cos\theta$, which implies $mg \geq N$. And zero net force along the plane gives $F_f = mg\sin\theta$, which implies $mg \geq F_f$. The $\theta \to 0$ limit gives a counterexample to choices (a), (c), and (e), because $F_f \to 0$ when $\theta \to 0$. Choice (d) would be almost true if we were talking about vectors (the correct statement would be $\mathbf{F}_f + \mathbf{N} + m\mathbf{g} = 0$). But we're dealing with magnitudes here, so choice (d) is equivalent to $\sin\theta + \cos\theta = 1$, which isn't true.

4.13. [c] The gravitational force exists, as does the normal force (to keep the block from falling though the plane). The friction force might or might not exist (and it might point in either direction). So the correct answer must be (a), (b), or (c). But ma isn't a force, so we're left with choice (c) as the only possibility. If you want to draw the acceleration vector in a free-body diagram, you must draw it differently (say, with a dotted line) to signify that it isn't a force!

REMARK: Since we are assuming that the block remains at the same position on the plane, it has a nonzero a_x but a zero a_y. Given a_x, the (positive) horizontal component of $\mathbf{N}$ plus the (positive or negative or zero) horizontal component of $\mathbf{F}_f$ must equal ma_x. ($\mathbf{F}_f$ will point down along the plane if a_x is larger than a certain value; as an exercise, you can determine this value.) And zero a_y means that the upward components of $\mathbf{N}$ and $\mathbf{F}_f$ must balance the downward $m\mathbf{g}$ force.

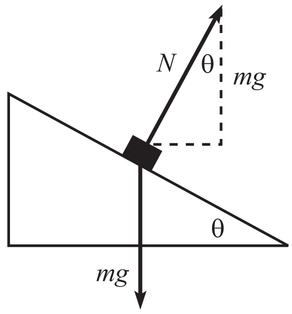


Figure 4.33

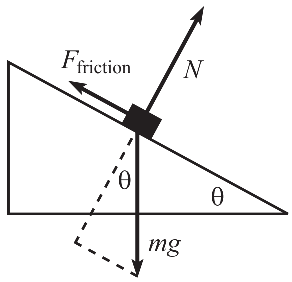


Figure 4.34

4.14. [e] Since the plane is frictionless, the only forces acting on the block are gravity and the normal force, as shown in Fig. 4.33. Since the block's acceleration is horizontal, the vertical component of the normal force must equal mg, to yield zero net vertical force. The right triangle then implies that $N = mg/\cos\theta$.

Limits: If $\theta = 0$ then $N = mg$. This makes sense, because the plane is horizontal and the normal force simply needs to balance the weight mg. If $\theta \to 90°$ then $N \to \infty$. This makes sense, because a needs to be huge to keep the block from falling.

Remark: Consider instead the case where there is friction between the block and the plane, and where the entire system is *static*. Then the normal force takes on the standard value of $N = mg\cos\theta$. This is most easily derived from the fact that there is no acceleration perpendicular to the plane, which then implies that N is equal to the component of gravity perpendicular to the plane; see Fig. 4.34. This should be contrasted with the original setup, where mg was equal to a component (the vertical component) of N. In any case, the normal force doesn't just magically turn out to be $mg\cos\theta$ or $mg/\cos\theta$ or whatever. It (or any other force) is determined by applying $F = ma$.

4.15. [d] The magnitude of the radial acceleration is v^2/R, which is the same at all points because v is constant. And the tangential acceleration is always zero because again v is constant. So the net force always points radially inward with constant magnitude mv^2/R. The vertical orientation of the hoop is irrelevant in this question, given that the speed is constant. The answer would be the same if the hoop were horizontal.

Remark: Be careful not to confuse the *total* force on the bead with the *normal* force from the hoop. The normal force *does* depend on the position of the bead. It is largest at the bottom of the hoop, because there the radial $F = ma$ equation (with inward taken to be positive for both N and a) is

$$N_{\text{bot}} - mg = \frac{mv^2}{R} \implies N_{\text{bot}} = \frac{mv^2}{R} + mg. \quad (4.4)$$

At the top of the hoop, the radial $F = ma$ equation (with inward again taken to be positive for both N and a) is

$$N_{\text{top}} + mg = \frac{mv^2}{R} \implies N_{\text{top}} = \frac{mv^2}{R} - mg. \quad (4.5)$$

If this is negative (if v is small), it just means that the normal force actually points radially outward (that is, upward).

4.16. [a] In the threshold case where the car barely doesn't stay in contact with the track at the top, the normal force N is zero, so the car is in freefall (while moving sideways). The downward acceleration is therefore g. If the speed is above the threshold value, then N is nonzero. The total downward force, $F = mg + N$, is therefore larger than the mg due to gravity. The downward acceleration, which is $F/m = g + N/m$, is therefore larger than g.

Remark: It isn't necessary to mention the v^2/R expression for the centripetal acceleration in this problem. But if you want to write down the radial $F = ma$ equation at the top of the loop, you can show as an exercise that the minimum speed required to barely maintain contact with the track at the top of the loop (that is, to make $N \geq 0$) is $v = \sqrt{gR}$.

4.17. [c] Gravity acts downward. The force from the air must have an upward component to balance gravity (because there is zero vertical acceleration), and also a radially inward component to provide the nonzero centripetal acceleration. So the net force from the air points in a diagonal direction, upward and leftward. The gravitational and air forces are the only two forces, so the correct answer is (c).

Remark: Choices (d) and (e) are incorrect because although they have the correct gravitational and air forces, they have an incorrect additional horizontal force. Remember that $ma = mv^2/r$ is *not* a force (see the discussion on page 72), so (d) can't be correct. Choice (e) would be correct if we were working in an accelerating frame and using fictitious forces. But we won't touch accelerating frames until Chapter 12.

4.18. [b] At the bottom of the motion, the upward force from the spring on the mass is maximum (because the spring is stretched maximally there), which means that the downward force from the spring on the box is maximum (because the spring exerts equal and opposite forces at its ends). This in turn means that the upward force from the scale on the box is maximum (because the net force on the box is always zero, because it isn't accelerating). And this force is the reading on the scale.

4.19. [c] The forces on the mass are the spring force, which is $k\,\Delta y = k(3mg/k) = 3mg$ upward, and the gravitational force, which is mg downward. The net force is therefore $2mg$ upward, so the acceleration is $2g$ upward.

4.20. [e] The left mass feels forces from both springs, so it feels equal and opposite forces of $k\ell$. Its acceleration is therefore zero. The right mass feels only the one force of $k\ell$ from the right spring (directed leftward). Its acceleration is therefore $k\ell/m$ leftward. The amount that the left spring is stretched is completely irrelevant as far as the right mass goes. Also, as far as the left mass goes, it is irrelevant that the far ends of the springs are attached to different things (the immovable wall and the movable right mass), at least right at the start.

4.21. [a] Let the stretching, relative to the equilibrium position, of the spring in the original scenario be 2ℓ. Then the given information tells us that half of the spring stretches by ℓ when its tension is $2mg$ (because the whole spring stretches by 2ℓ when its tension is $2mg$, and the tension is the same throughout). Therefore, the top half of the spring in the second scenario is stretched by ℓ, because it is holding up a total mass of $2m$ below it (this mass is split into two pieces, but that is irrelevant as far as the top spring is concerned). But the bottom half of the spring is stretched by only $\ell/2$, because it is holding up only a mass m below it (and $x \propto F$ by Hooke's law). The total stretch is therefore $\ell + \ell/2 = 3\ell/2$. This is less than 2ℓ, so the desired answer is "higher."

REMARK: This answer of "higher" can be made a little more believable by looking at some limiting cases. Consider a more general version of the second scenario, where we still cut the spring into equal pieces, but we now allow for the two masses to be unequal (although they must still add up to the original mass). Let the top and bottom masses be labeled m_t and m_b, and let the original mass be M. In the limit where $m_t = 0$ and $m_b = M$, we simply have the original setup, so the answer is "same height." In the limit where $m_t = M$ and $m_b = 0$, we have a mass M hanging from a shorter spring. But a shorter spring has a larger spring constant (see Problem 4.2), which means that it stretches less. So the answer is "higher." Therefore, since the $m_t = M/2$ and $m_b = M/2$ case presented in the problem lies between the preceding two cases with answers of "same height" and "higher," it is reasonable to expect that the answer to the original problem is "higher." You can also make a similar argument by splitting the original mass into two equal pieces, but now allowing the spring to be cut into unequal pieces.

4.22. [b] $(a_1 + a_2)/2$ is the acceleration of the bottom pulley, because the average height of the bottom two masses always stays the same distance below the bottom pulley. And the acceleration of the bottom pulley equals the acceleration of the middle pulley, which in turn equals $-a_3/2$. This is true because if the middle pulley goes up by d, then $2d$ of string disappears above it, which must therefore appear above m_3; so m_3 goes down by $2d$. Putting all this together yields $a_3 = -(a_1 + a_2)$. See the solution to Problem 4.3 for more discussion of these concepts.

4.23. [b] If the left mass goes up by y_1, there is $2y_1$ less string in the segments above it. Likewise, if the middle mass goes up by y_2, there is $2y_2$ less string in the segments above it. All of this missing string must appear above the right mass, which therefore goes down by $2y_1 + 2y_2$. So $y_3 = -2y_1 - 2y_2$. Taking two time derivatives gives choice (b).

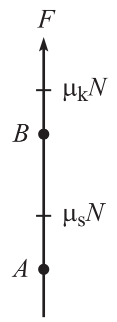
4.24. [d] If the left mass goes up by y_1, then $2y_1$ worth of string disappears from the left region. Similarly, if the right mass goes up by y_3, then $2y_3$ worth of string disappears from the right region. This $2y_1 + 2y_3$ worth of string must appear in the middle region. It gets divided evenly between the two segments there, so the middle mass goes down by $y_1 + y_3$. Hence $y_2 = -(y_1 + y_3)$. Taking two time derivatives gives choice (d).

4.5 Problem solutions

4.1. **Coefficients of friction**

Assume, in search of a contradiction, that μ_k is larger than μ_s. Imagine pushing a block that rests on a surface. If the applied force is smaller than $\mu_s N$ (such as the force indicated by the point A on the scale in Fig. 4.35), then nothing happens. The static friction exactly cancels the applied force, and the block just sits there.

However, if the applied force lies between $\mu_s N$ and $\mu_k N$ (such as the force indicated by the point B), then we have problem. On one hand, the applied force exceeds $\mu_s N$ (which is the maximum static friction force, by definition), so the block should move. But on the other hand, the applied force is smaller than the kinetic friction force, $\mu_k N$, so the block *shouldn't* move. Basically, as soon as the block moves even the slightest infinitesimal amount (at which point the kinetic friction force becomes the relevant force), it will decelerate and stop because the kinetic friction force wins out over the applied force. So it actually never moves at all, even for the applied force represented by point B. This means, by the definition of μ_s, that $\mu_s N$ is actually located higher than point B. The above contradiction (where the block both does move and doesn't move) will arise unless $\mu_s N \geq \mu_k N$. So we conclude that it must be the case that $\mu_s \geq \mu_k$. There then exists no point B below $\mu_k N$ and above $\mu_s N$.



This situation with $\mu_k > \mu_s$ leads to a contradiction

Figure 4.35

In the real world, μ_s is rarely larger than twice μ_k. In some cases the two are essentially equal, with μ_s being a hair larger.

4.2. **Cutting a spring in half**

Take the half-spring and stretch it a distance x. Our goal is to find the force F that it exerts on something attached to an end; the spring constant is then given by $k_{1/2} = F/x$ (ignoring the minus sign in Hooke's law; we'll just deal with the magnitude). Imagine taking two half-springs that are each stretched by x, lying along the same line, and attaching the right end of one to the left end of the other. We are now back to our original spring with spring constant k. And it is stretched by a distance $2x$, so the force it exerts is $k(2x)$. But this is also the force F that each of the half-springs exerts, because we didn't change anything about these springs when we attached them together. So the desired value of $k_{1/2}$ is $k_{1/2} = F/x = k(2x)/x = 2k$.

REMARK: The same type of reasoning shows (as you can verify) that if we have a spring and then cut off a piece with a length that is a factor f times the original (so $f = 1/2$ in the above case), then the spring constant of the new piece is $1/f$ times the spring constant of the original. Actually, the reasoning works only in the case of rational numbers f. (It works with f of the form $f = 1/N$, where N is an integer. And you can show with similar reasoning that it also works with f of the form $f = N$, which corresponds to attaching N springs together in a line. Combining these two results yields all of the rational numbers.) But any real number is arbitrarily close to a rational number, so the result is true for any factor f.

Basically, a shorter spring is a stiffer spring, because for a given total amount of stretching, a centimeter of a shorter spring must stretch more than a centimeter of a longer spring. So the tension in the former centimeter (which is the same as the tension throughout the entire shorter spring) is larger than the tension in the latter centimeter.

4.3. **Useful Atwood's facts**

(a) Assume, in search of a contradiction, that the tension varies throughout the string. Then there exists a segment of the string for which the tension is different at the two ends. This means that there is a nonzero net force on the segment. But the string is massless, so the acceleration of this segment must be infinite. Since this can't be the case, the tension must in fact be the same throughout the string. In short, any massless object must always have zero net force acting on it. (Ignoring photons and such!)

REMARK: If there is friction between a string and a pulley, and if the pulley has a nonzero moment of inertia (a topic covered in Chapter 7), then the tension in the string will vary, even if the string is massless. But the statement, "Any massless object must always have zero net force acting on it," still holds; there is now a nonzero friction force from the pulley acting on a segment of the string touching it. So the net force on the segment is still zero.

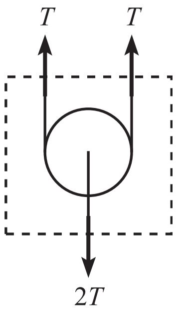


Figure 4.36

(b) The reasoning here is similar to the reasoning in part (a). Since the left pulley is massless, the net force on it must be zero. If we draw a free-body diagram as shown in Fig. 4.36, then we see that two T's protrude from the top of the box. So the downward tension from the bottom string must be $2T$, to make the net force be zero. (By the same reasoning, the tension in the short string above the right pulley is also $2T$.) We'll use this result many times in the Atwood's problems in this chapter.

REMARK: In the above free-body diagram, the top string does indeed get counted twice, as far as the forces go. In a modified scenario where we have two people pulling upward on the ends of the string, each with a force T, as shown in Fig. 4.37, their total upward force is of course $2T$. And the pulley can't tell the difference between this scenario and the original one.

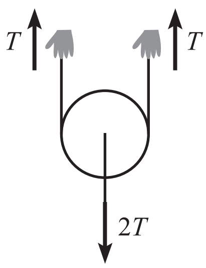


Figure 4.37

(c) Imagine that the left mass (and hence left pulley) goes up by a distance d. Then a length d of string disappears from each of the two segments above the pulley, as shown in Fig. 4.38. So a total length $2d$ of string disappears from these two segments. This string has to go somewhere, so it appears above the right mass. The right mass therefore goes down by $2d$, as shown. So the downward displacement of the right mass is always twice the upward displacement of the left mass. Taking two derivatives of this relation tells us that downward acceleration of the right mass is always twice the upward acceleration of the left mass. (Equivalently, the general relation $d = at^2/2$ holds, so the ratio of the accelerations must be the same as the ratio of the displacements.) This is the so-called "conservation of string" statement for this setup.

The same reasoning applies if the left mass instead goes down. In any case, the sign of the right mass's acceleration is the negative of the sign of the left mass's acceleration.

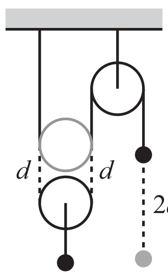


Figure 4.38

REMARKS: When thinking about conservation-of-string statements, it is often helpful to imagine cutting out pieces of string in some parts of the setup and then splicing them into other parts. This isn't what actually happens, of course; the string just slides around like a snake. But if you take a photo at two different times during the motion, the splicing photo will look the same as the actual photo. In short, while it is often hard to visualize what is happening as the string *moves*, it is generally much easier to imagine the string as *having moved.*

Conservation of string by itself doesn't determine the motion of the masses. We still need to apply $F = ma$ to find out how the masses actually move. If you grab the masses and move them around in an arbitrary manner while always making sure that the strings stay taut, then the conservation-of-string condition will be satisfied. But the motion will undoubtedly not be the same as the motion where the masses are acted on by only gravity. There is an infinite number of possible motions consistent with conservation of string, but only one of these motions is also consistent with all of the $F = ma$ equations. And conversely, the $F = ma$ equations alone don't determine the motion; the conservation-of-string relation is required. If the strings aren't present, the motion will certainly be different; all the masses will be in freefall!

(d) The average height of the right two masses always remains a constant distance below the right pulley (because the right string keeps the same length). So $y_p = (y_2 + y_3)/2 + C$. Taking two derivatives of this relation gives $a_p = (a_2 + a_3)/2$. But the downward (or upward) acceleration of the right pulley (which we just showed equals the average of the accelerations of the right two masses) equals the upward (or downward) acceleration of the left mass (because the left string keeps the same length), as desired.

A few of the Atwood's-machine problems in this chapter contain some unusual conservation-of-string relations, but they all involve the types of reasoning in parts (c) and (d) of this problem.

4.4. **Block and tackle**

(a) Let T be the tension in the long rope (the tension is the same throughout the rope, from the reasoning in Problem 4.3(a)). Since the bottom pulley is massless, the reasoning in Problem 4.3(b) tells us that the tension in the short rope attached to the block is $4T$ (there would now be four T's protruding from the top of the dashed box in Fig. 4.36). We want this $4T$ tension to balance the Mg weight of the block, so the person must pull on the rope with a force $T = Mg/4$. (The angle of the rope to the person doesn't matter.)

(b) The free-body diagram for the bottom part of the setup is shown in Fig. 4.39. Five tensions protrude from the top of the dashed box, and the two weights Mg and mg protrude from the bottom. The net force must be zero if the setup is at rest (or moving with constant speed), so the tension must equal $T = (M + m)g/5$. This is the desired downward force exerted by the person on the rope.

LIMITS: In the case where $M = 0$ (so the person is standing on a massless platform), he must pull down on the rope with a force equal to one fifth his weight, if he is to hoist himself up. In this case, his mg weight is balanced by an upward $mg/5$ tension force from the rope he is holding, plus an upward $4mg/5$ normal force from the platform (which basically comes from the $4mg/5$ tension in the short rope attached to the platform).

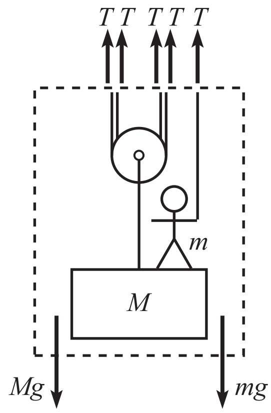


Figure 4.39

4.5. **Equivalent mass**

With the various parameters defined as in Fig. 4.40, the two $F = ma$ equations are (with upward taken as positive for the left mass, and downward positive for the right)

$$T - m_1 g = m_1 a \qquad \text{and} \qquad m_2 g - T = m_2 a. \tag{4.6}$$

We have used the fact that since your hand (and hence the pulley) is held at constant height, the accelerations of the masses are equal in magnitude and opposite in direction. We can solve for T by multiplying the first equation by m_2, the second by m_1, and then subtracting them. This eliminates the acceleration a, and we obtain $T = 2m_1 m_2 g/(m_1 + m_2)$. The upward force you apply equals the tension $2T$ in the upper string; see Problem 4.3(b) for the explanation of the $2T$. This force of $2T$ equals the weight Mg of a single mass M if

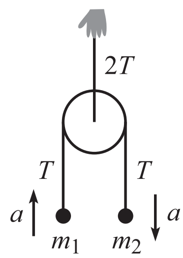


Figure 4.40

$$M = \frac{4m_1 m_2}{m_1 + m_2}. \tag{4.7}$$

This result is *not* equal to the sum of the masses, $m_1 + m_2$. In the special case where the masses are equal ($m_1 = m_2 \equiv m$), the equivalent mass does simply equal $2m$. But in general, M isn't equal to the sum.

LIMITS: In the limit where m_1 is very small, we can ignore the m_1 in the denominator of Eq. (4.7) (but not in the numerator; see the discussion in Section 1.1.3), which yields $M \approx 4m_1$. So the combination of a marble and a bowling ball looks basically like four marbles (this limit makes it clear that M can't be equal to $m_1 + m_2$ in general). This can be seen fairly intuitively: the bowling ball is essentially in freefall downward, so the marble accelerates upward at g. The tension T must therefore be equal to $2m_1 g$ to make the net upward force on the marble be $m_1 g$. The tension $2T$ in the upper string is then $2(2m_1 g)$, which yields $M = 4m_1$.

If we want to solve for a in Eq. (4.6), we can simply add the equations. The result is

$$a = g\frac{m_2 - m_1}{m_2 + m_1}. \tag{4.8}$$

This acceleration makes sense, because a net force of $m_2 g - m_1 g$ pulls down on the right side, and this force accelerates the total mass of $m_1 + m_2$. Various limits check correctly: if $m_2 = m_1$ then $a = 0$; if $m_2 \gg m_1$ then $a \approx g$; and if $m_2 \ll m_1$ then $a \approx -g$ (in Fig. 4.40 we defined positive a to be upward for m_1).

4.6. **Atwood's 1**

If m_1 is at rest, the tension in the string supporting it must be $m_1 g$. From the reasoning in Problem 4.3(b), the tension in the lower string is then $m_1 g/2$. The $F = ma$ equations for the two lower masses are therefore (with upward taken to be positive for m_2 and downward positive for M)

$$\begin{aligned} m_2: \quad & \frac{m_1 g}{2} - m_2 g = m_2 a, \\ M: \quad & Mg - \frac{m_1 g}{2} = Ma. \end{aligned} \tag{4.9}$$

We have used the fact that the accelerations of m_2 and M have the same magnitude, because the bottom pulley doesn't move if m_1 is at rest. Equating the two resulting expressions for a from the above two equations gives

$$\begin{aligned} \frac{m_1 g}{2m_2} - g = g - \frac{m_1 g}{2M} \quad & \Longrightarrow \quad \frac{m_1}{m_2} - 4 = -\frac{m_1}{M} \\ & \Longrightarrow \quad M = \frac{m_1 m_2}{4m_2 - m_1}. \end{aligned} \tag{4.10}$$

In order for a physical M to exist, we need the denominator of M to be positive. So we need $m_2 > m_1/4$.

Alternatively, you can solve this problem by solving for m_2 in Eq. (4.7) in the solution to Problem 4.5 and then relabeling the masses appropriately.

Remark: If m_2 is smaller than $m_1/4$, then even an infinitely large M won't keep m_1 from falling. The reason for this is that the best-case scenario is where M is so large that it is essentially in freefall. So m_2 gets yanked upward with acceleration g, which means that the tension in the string pulling on it is $2m_2 g$ (so that the net upward force is $2m_2 g - m_2 g = m_2 g$). The tension in the upper string is then $4m_2 g$. (We've basically just repeated the reasoning for the small-m_1 limit discussed in the solution to Problem 4.5.) This is the largest the tension can be, so if $4m_2 g$ is smaller than $m_1 g$ (that is, if m_2 is smaller than $m_1/4$), then m_1 will fall downward. Note that if $m_2 \to \infty$ (with m_2 finite), then Eq. (4.10) says that $M = m_1/4$, which is consistent with the preceding reasoning. You can show, as you might intuitively expect, that the smallest sum of m_2 and M that supports a given m_1 is achieved when m_2 and M are both equal to $m_1/2$.

4.7. **Atwood's 2**

From the reasoning in Problem 4.3(b), the tensions in the two strings are T and $2T$, as shown in Fig. 4.41. The $F = ma$ equations are therefore

$$\begin{aligned} T &= ma_1, \\ T &= (2m)a_2, \\ (2m)g - 2T &= (2m)a_3. \end{aligned} \tag{4.11}$$

We have three equations but four unknowns here: a_1, a_2, a_3, and T. So we need one more equation – the conservation-of-string relation. The average position of the left two masses remains the same distance behind the left pulley, which moves the same distance as the right pulley, and hence right mass. So the conservation-of-string relation is $a_3 = (a_1 + a_2)/2$. This is the same reasoning as in Problem 4.3(d).

The first two of the above $F = ma$ equations quickly give $a_1 = 2a_2$. Plugging this into the conservation-of-string relation gives $a_3 = 3a_2/2$. The second two $F = ma$ equations are

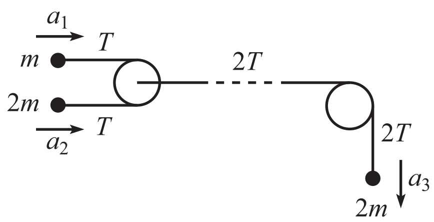


Figure 4.41

then

$$T = (2m)a_2,$$
$$(2m)g - 2T = (2m)(3a_2/2). \tag{4.12}$$

The second equation plus twice the first gives $2mg = 7ma_2 \implies a_2 = 2g/7$. We then have $a_1 = 2a_2 = 4g/7$, and $a_3 = 3a_2/2 = 3g/7$. As a check, a_3 is indeed the average of a_1 and a_2. Additionally, the tension is $T = ma_1 = 4mg/7$.

Remark: When solving any problem, especially an Atwood's problem, it is important to (1) identify all of the unknowns and (2) make sure that you have as many equations as unknowns, as we did above.

4.8. **Atwood's 3**

If T is the tension in the lowest string, then from Problem 4.3(b) the tensions in the other strings are shown in Fig. 4.42. Let all of the accelerations be defined with upward being positive. Then the three $F = ma$ equations are

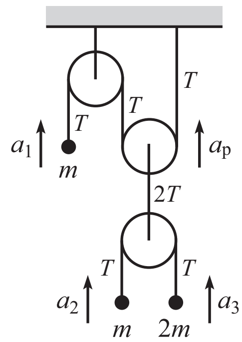


Figure 4.42

$$T - mg = ma_1,$$
$$T - mg = ma_2,$$
$$T - (2m)g = (2m)a_3. \tag{4.13}$$

The first two of these equations quickly give $a_1 = a_2$.

Now for the conservation-of-string statement. Let a_p be the acceleration of the upper right pulley (which is the same as the acceleration of the lower right pulley). The average height of the two right masses always remains the same distance below this pulley. Therefore $a_p = (a_2 + a_3)/2$. But we also have $a_1 = -2a_p$, because if the pulley goes up a distance d, then a length d of string disappears from both segments above the pulley, so $2d$ of string appears above the left mass. This means that it goes down by $2d$; hence $a_1 = -2a_p$. (We've just redone Problem 4.3(c) and (d) here.) Combining this with the $a_p = (a_2 + a_3)/2$ relation gives

$$a_1 = -2\left(\frac{a_2 + a_3}{2}\right) \implies a_1 + a_2 + a_3 = 0. \tag{4.14}$$

Since we know from above that $a_1 = a_2$, we obtain $a_3 = -2a_2$. The last two of the above $F = ma$ equations are then

$$T - mg = ma_2,$$
$$T - (2m)g = (2m)(-2a_2). \tag{4.15}$$

Taking the difference of these equations yields $mg = 5ma_2$, so $a_2 = g/5$. (And this is also a_1.) The desired acceleration of the mass $2m$ is then $a_3 = -2a_2 = -2g/5$. This is negative, so the mass $2m$ goes downward, which makes sense. (If all three masses are equal to m, you can quickly show that $T = mg$ and all three accelerations are zero. Increasing the right mass to $2m$ therefore makes it go downward.)

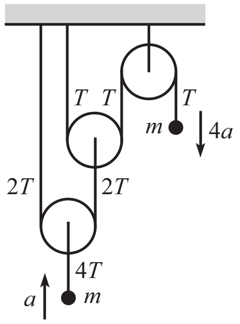


Figure 4.43

4.9. **Atwood's 4**

Let T be the tension in the string connected to the right mass. Then from the reasoning in Problem 4.3(b), the tensions in the other strings are $2T$ and $4T$, as shown in Fig. 4.43.

The conservation-of-string relation tells us that the accelerations are a and $4a$, as shown. This is true because if the bottom pulley goes up by d, then the middle pulley goes up by $2d$, from the reasoning in Problem 4.3(c). From the same reasoning, the right mass then goes down by $4d$. So the ratio of the distances moved is 4. Taking two time derivatives of this relation tells us that the (magnitudes of the) accelerations are in the same ratio. The $F = ma$ equations are therefore

$$\begin{aligned} 4T - mg &= ma, \\ mg - T &= m(4a). \end{aligned} \tag{4.16}$$

The first equation plus four times the second gives $3mg = 17ma \implies a = 3g/17$. This is the upward acceleration of the left mass. The downward acceleration of the right mass is then $4a = 12g/17$.

REMARK: Consider the more general case where we have a "tower" of n movable pulleys extending down to the left (so the given problem has $n = 2$). If we define $N \equiv 2^n$, then as an exercise you can show that the upward acceleration of the left mass and the downward acceleration of the right mass are

$$a_{\text{left}} = g\frac{N-1}{N^2+1} \qquad \text{and} \qquad a_{\text{right}} = g\frac{N^2-N}{N^2+1}. \tag{4.17}$$

These expressions correctly reproduce the above results when $n = 2 \implies N = 4$. If $N \to \infty$, we have $a_{\text{left}} \to 0$ and $a_{\text{right}} \to g$. You can think physically about what is going on here; in some sense the system behaves like a lever, where forces and distances are magnified. The above expressions for the a's are also valid when $n = 0 \implies N = 1$, in which case both accelerations are zero; we just have two masses hanging over the fixed pulley connected to the ceiling.

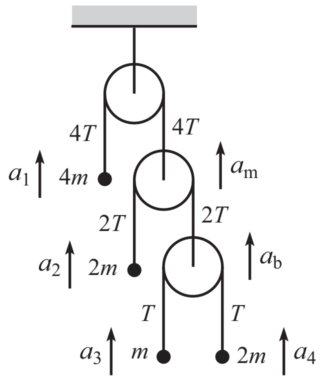


Figure 4.44

4.10. **Atwood's 5**

Let T be the tension in the bottom string. Then from the reasoning in Problem 4.3(b), the tensions in the other strings are $2T$ and $4T$, as shown in Fig. 4.44. If all of the accelerations are defined with upward being positive, the four $F = ma$ equations are

$$\begin{aligned} 4T - 4mg &= 4ma_1, \\ 2T - 2mg &= 2ma_2, \\ T - mg &= ma_3, \\ T - 2mg &= 2ma_4. \end{aligned} \tag{4.18}$$

The first three of these equations quickly give $a_1 = a_2 = a_3$.

We must now determine the conservation-of-string relation. Let a_{m} and a_{b} be the accelerations of the middle and bottom pulleys (with upward being positive). The average position of the bottom two masses stays the same distance below the bottom pulley, so $a_{\text{b}} = (a_3 + a_4)/2$. Similarly, $a_{\text{m}} = (a_2 + a_{\text{b}})/2$. Substituting the value of a_{b} from the first of these relations into the second, and using the fact that $a_1 = -a_{\text{m}}$, we have

$$a_1 = -\left(\frac{a_2 + (a_3 + a_4)/2}{2}\right). \tag{4.19}$$

Using $a_1 = a_2 = a_3$, this becomes

$$a_3 = -\left(\frac{a_3 + (a_3 + a_4)/2}{2}\right) \implies a_4 = -7a_3. \tag{4.20}$$

The last two of the $F = ma$ equations in Eq. (4.18) are then

$$\begin{aligned} T - ma &= ma_3, \\ T - 2mg &= 2m(-7a_3). \end{aligned} \tag{4.21}$$

Subtracting these equations eliminates T, and we find $a_3 = g/15$. The acceleration of the rightmost mass is then $a_4 = -7a_3 = -7g/15$. This is negative, so this mass goes down. The other three masses all move upward with acceleration $g/15$.

4.11. **Atwood's 6**

The important thing to note in this problem is that the tension in the long string is *different* above and below the middle mass. The standard fact that we ordinarily use (that the tension is the same everywhere throughout a massless string; see Problem 4.3(a)) holds only if the string is *massless*. If there is a mass attached to the string, then the tensions on either side of the mass will be different (unless the mass is in freefall, as we see from the second $F = ma$ equation below). Let these tensions be T_1 and T_2, as shown in Fig. 4.45. If we label the accelerations as a_1, a_2, and a_3 from left to right (with upward defined to be positive for all), then the $F = ma$ equations are

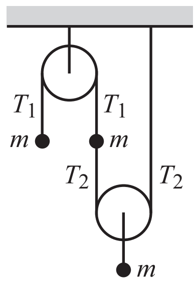


Figure 4.45

$$\begin{aligned} T_1 - mg &= ma_1, \\ T_1 - T_2 - mg &= ma_2, \\ 2T_2 - mg &= ma_3. \end{aligned} \tag{4.22}$$

Conservation of string quickly gives $a_2 = -a_1$. And it also gives $a_3 = -a_1/2$, from the reasoning in Problem 4.3(c). The above $F = ma$ equations then become

$$\begin{aligned} T_1 - mg &= ma_1, \\ T_1 - T_2 - mg &= m(-a_1), \\ 2T_2 - mg &= m(-a_1/2). \end{aligned} \tag{4.23}$$

We now have three equations in three unknowns (T_1, T_2, and a_1). Solving these by your method of choice gives $a_1 = 2g/9$. Hence $a_2 = -2g/9$ and $a_3 = -g/9$. The tensions turn out to be $T_1 = 11mg/9$ and $T_2 = 4mg/9$.

4.12. **Atwood's 7**

Let the tension in the upper left part of the string in Fig. 4.46 be T. Then the two other upper parts also have tension T, as shown. Because the left pulley is massless, the net force on it must be zero, so the string below it must have tension $2T$. This then implies the other $2T$ shown. Finally, the tension in the bottom string is $4T$ because the net force on the bottom pulley must be zero. Note that the tension in the long string is allowed to change from T to $2T$ at the mass, because the "massless string" reasoning in Problem 4.3(a) doesn't hold here.

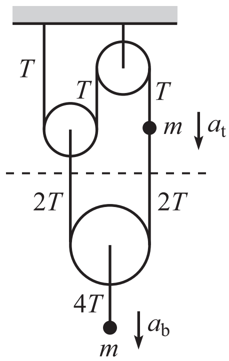


Figure 4.46

With the accelerations defined as shown in the figure, the $F = ma$ equations for the bottom and top masses are

$$\begin{aligned} mg - 4T &= ma_b, \\ mg + 2T - T &= ma_t. \end{aligned} \tag{4.24}$$

We must now determine the conservation-of-string relation. From the reasoning in Problem 4.3(c), the acceleration of the top mass is twice the acceleration of the left pulley (with the opposite sign).

Additionally, if the left pulley goes up by d, then an extra length d of string appears below the dotted line in Fig. 4.46. This is true because d disappears from each of the two parts of the string above the left pulley, but d must also be inserted right below the pulley. So a net length d is left over, and this appears below the dotted line. It gets divided evenly between the two parts of the string touching the bottom pulley, so the bottom pulley (and hence the bottom mass) goes down by $d/2$. So the acceleration of the bottom mass is half the acceleration of the left pulley (with the opposite sign). But from the previous paragraph, the acceleration of the left pulley is half the acceleration of the top mass (with the opposite sign). Putting this all together gives $a_b = -(-a_t/2)/2$, or $a_t = 4a_b$. The above $F = ma$ equations therefore become

$$\begin{aligned} mg - 4T &= ma_b, \\ mg + 2T - T &= m(4a_b). \end{aligned} \tag{4.25}$$

The first equation plus four times the second gives $5mg = 17ma_b \Longrightarrow a_b = 5g/17$. And then $a_t = 4a_b = 20g/17$. It is perfectly fine that $a_t > g$. In the limit where the top mass is zero, the bottom mass is in freefall, so we have $a_b = g$ and $a_t = 4g$.

4.13. **Atwood's 8**

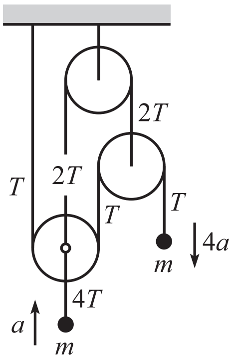


Figure 4.47

Let T be the tension in the string connected to the right mass. Then from Problem 4.3(b), zero net force on the right pulley tells us that the tension in the other long string is $2T$, as shown in Fig. 4.47. And then again from Problem 4.3(b), zero net force on the left pulley tells us that the tension in the bottom short string is $4T$.

Let a be the upward acceleration of the left mass. We claim that the conservation-of-string relation says that the downward acceleration of the right mass is $4a$. This is true for the following reason. If the left pulley (and hence left mass) moves up by a distance d, then a length d of string disappears from each of the two segments of string above it that touch it tangentially. Likewise, if the right pulley moves down by a distance d (and it does indeed move the same distance as the left pulley, because the centers of the two pulleys are connected by a string), then a length d of string disappears from each of the two segments of string below it (assuming for a moment that the right mass doesn't move). A total of $4d$ of string has therefore disappeared (temporarily) from the long string touching the right mass. This $4d$ must end up being inserted above the right mass. So this mass goes down by $4d$, which is four times the distance the left mass goes up, hence the accelerations of a and $4a$ shown in the figure. The $F = ma$ equations for the left and right masses are therefore

$$\begin{aligned} 4T - mg &= ma, \\ mg - T &= m(4a). \end{aligned} \tag{4.26}$$

The first equation plus four times the second gives $3mg = 17mg \Longrightarrow a = 3g/17$. This is the upward acceleration of the left mass. The downward acceleration of the right mass is then $4a = 12g/17$. These accelerations happen to be the same as the ones in Problem 4.9.

4.14. **No relative motion**

First note that by the following continuity argument, there must exist a value of a for which there is no relative motion among the masses. If a is zero, then m_2 falls and m_1 moves to the right. On the other hand, if a is very large, then intuitively m_1 will drift backward with respect to M, and m_2 will rise (remember that there is no friction anywhere). So for some intermediate value of a, there will be no relative motion.

Now let's find the desired value of a. If there is no relative motion among the masses, then m_2 stays at the same height, which means that the net vertical force on it must be zero. The tension in the string connecting m_1 to m_2 must therefore be $T = m_2 g$. The horizontal $F = ma$ equation on m_1 is then $T = m_1 a \Longrightarrow m_2 g = m_1 a \Longrightarrow a = (m_2/m_1)g$. This is the acceleration of the system, and hence the desired acceleration of M.

Limits: a is small if m_2 is small (more precisely, if $m_2 \ll m_1$) and large if m_2 is large ($m_2 \gg m_1$). These results make intuitive sense.

4.15. **Slipping blocks**

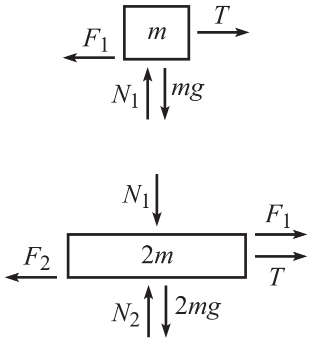


Figure 4.48

(a) The free-body diagrams are shown in Fig. 4.48. The normal forces N_1 (between the blocks) and N_2 (between the bottom block and the table) are quickly found to be $N_1 = mg$ and $N_2 = 3mg$. And the tension in the string is $T = 3mg$, because twice this must equal the applied $6mg$ force (because the net force on the massless pulley must be zero). The friction forces F_1 (between the blocks) and F_2 (between the bottom block and the table) are as yet unknown.

Assume that the bottom block *doesn't* slip with respect to the table. The maximum possible leftward static friction force from the table acting on the bottom block is $\mu_s N_2 = 3mg$. This can cancel out the rightward tension $T = 3mg$ acting on the block. However, there also the friction force F_1 between the blocks. (This is the kinetic friction force $\mu_k N_1 = mg$, because you can quickly show that if the bottom block is at rest, then the top block must slip with respect to the bottom block.) This friction force acts rightward on the bottom block, which means that the net rightward force on the bottom block is nonzero. It will therefore slip with respect to the table. Hence our initial non-slipping assumption was incorrect.

(b) Assume that the top block *doesn't* slip with respect to the bottom block (which we know must be moving, from part (a)). Then the two blocks can be treated like a single block with mass $3m$. The leftward kinetic friction force from the table is $F_2 = \mu_k N_2 = 3mg$. The net force on the effective $3m$ block is therefore $6mg - 3mg = 3mg$ rightward, so the acceleration is $a = g$ rightward. The horizontal $F = ma$ equation for the top block is then

$$T - F_1 = ma \implies 3mg - F_1 = mg \implies F_1 = 2mg. \tag{4.27}$$

But this friction force isn't possible, because it exceeds the maximum possible static friction force between the blocks, which is $\mu_s N_1 = mg$. This contradiction implies that our initial non-slipping assumption must have been incorrect. The blocks therefore slip with respect to each other.

REMARK: If the coefficient of static friction between the blocks were instead made sufficiently large ($\mu_s \geq 2$), then the blocks would in fact move as one effective mass $3m$. The static friction force $2mg$ between the blocks would act backward on m and forward on $2m$, and both blocks would have acceleration g. If the friction force were then suddenly decreased to the kinetic value of $\mu_k N_1 = mg$ relevant to the stated problem, m would accelerate faster than g (because there isn't as much friction holding it back), and $2m$ would accelerate slower than g (because there isn't as much friction pushing it forward). This is consistent with the accelerations we will find below in part (c).

(c) Since we know that all surfaces slip with respect to each other, the friction forces are all kinetic friction forces. Their values are therefore $F_1 = \mu_k N_1 = mg$ and $F_2 = \mu_k N_2 = 3mg$. The $F = ma$ equations for the two blocks are then

$$\begin{aligned} m: \quad & T - F_1 = ma_1 \implies 3mg - mg = ma_1 \\ & \implies a_1 = 2g, \\ 2m: \quad & T + F_1 - F_2 = (2m)a_2 \implies 3mg + mg - 3mg = 2ma_2 \\ & \implies a_2 = g/2. \end{aligned} \tag{4.28}$$

By conservation of string, the average position of the two blocks stays the same distance behind the pulley, and hence also behind your hand. So

$$a_{\text{hand}} = \frac{a_1 + a_2}{2} = \frac{2g + g/2}{2} = \frac{5g}{4}. \tag{4.29}$$

4.16. **Block and wedge**

Let N be the normal force between the block and the wedge. Then the vertical $F = ma$ equation for the block (with downward taken to be positive) is

$$Mg - N\cos\theta = Ma_M. \tag{4.30}$$

And the horizontal $F = ma$ equation for the wedge (with rightward taken to be positive) is

$$N\sin\theta = ma_m. \tag{4.31}$$

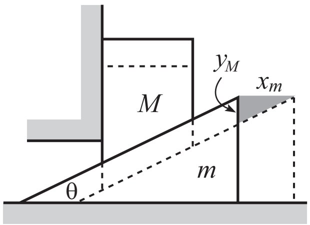


Figure 4.49

We have two equations but three unknowns (N, a_M, a_m), so we need one more equation. This equation is the constraint that the block remains on the wedge at all times. At a later time, let the positions of the block and wedge be indicated by the dotted lines in Fig. 4.49. The block has moved downward a distance y_M, and the wedge has moved rightward a distance x_m. From the triangle that these lengths form (the shaded triangle in the figure), we see that $y_M = x_m \tan\theta$. Taking two time derivatives of this relation gives the desired third equation,

$$a_M = a_m \tan\theta. \tag{4.32}$$

There are various ways to solve the preceding three equations. If we multiply the first by $\sin\theta$ and the second by $\cos\theta$, and then add them, the N terms cancel. If we then use $a_M = a_m \tan\theta$ to eliminate a_M, we obtain

$$\begin{aligned} Mg\sin\theta &= Ma_M \sin\theta + ma_m\cos\theta \\ &= M(a_m\tan\theta)\sin\theta + ma_m\cos\theta. \end{aligned} \tag{4.33}$$

Solving for a_m gives

$$a_m = \frac{Mg\sin\theta\cos\theta}{M\sin^2\theta + m\cos^2\theta}. \tag{4.34}$$

Limits: There are many limits we can check. You should verify that all of the following results make sense.

- If $\theta \to 0$ or $\theta \to 90^\circ$, then $a_m \to 0$.
- If $M \ll m$ then $a_m \approx 0$.
- If $M \gg m$ then $a_m \approx g/\tan\theta \implies g \approx a_m \tan\theta$. This is simply Eq. (4.32) with $a_M = g$, because the block is essentially in freefall.
- If $M = m$, then $a_m = g\sin\theta\cos\theta$, which achieves a maximum when $\theta = 45^\circ$. And since the constraining force on the left side of the block equals $N\sin\theta$, which in turn equals ma_m from Eq. (4.31), we see that $\theta = 45^\circ$ necessitates the maximum force by the constraining wall (for the special case where $M = m$).

You can check that various limits for a_M (given in Eq. (4.32)) also work out correctly.

4.17. **Up and down a plane**

(a) When the block stops (at least instantaneously) at its highest point, the forces along the plane are the $mg\sin\theta$ gravity component downward, and the static friction force F_f upward. We know that $F_f \le \mu_s N = 1 \cdot mg\cos\theta$. The block will accelerate downward if the gravitational force $mg\sin\theta$ is larger than the maximum possible friction force $mg\cos\theta$. So the block will slide back down if

$$mg\sin\theta > mg\cos\theta \implies \tan\theta > 1 \implies \theta > 45^\circ. \tag{4.35}$$

(b) On the way up the plane, both the gravity component and friction point down the plane, so the force along the plane is $mg\sin\theta + \mu_k mg\cos\theta$ downward. Therefore, since $\mu_k = 1$, the acceleration during the upward motion points down the plane and has magnitude

$$a_u = g(\sin\theta + \cos\theta). \tag{4.36}$$

The total time for the upward motion is then

$$t_{\rm u} = \frac{v_0}{a_{\rm u}} = \frac{v_0}{g(\sin\theta + \cos\theta)}\,. \tag{4.37}$$

To find the time $t_{\rm d}$ for the downward motion, we need to find the maximum distance d up the plane that the block reaches. From standard kinematics we have $d = v_0^2/2a_{\rm u}$. (This can be obtained from either $v_{\rm f}^2 - v_{\rm i}^2 = 2ad$, or $d = v_0 t - at^2/2$ with $t = v_0/a$.) During the downward motion, the friction force points up the plane, so the net acceleration points down the plane and has magnitude

$$a_{\rm d} = g(\sin\theta - \cos\theta). \tag{4.38}$$

Using the value of d we just found, the relation $d = a_{\rm d}t_{\rm d}^2/2$ for the downward motion gives

$$\frac{v_0^2}{2a_{\rm u}} = \frac{a_{\rm d}t_{\rm d}^2}{2} \implies t_{\rm d} = \frac{v_0}{\sqrt{a_{\rm u}a_{\rm d}}} = \frac{v_0}{g\sqrt{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)}}\,. \tag{4.39}$$

(c) The total time with friction is

$$T_{\rm F} = t_{\rm u} + t_{\rm d} = \frac{v_0}{a_{\rm u}} + \frac{v_0}{\sqrt{a_{\rm u}a_{\rm d}}}\,, \tag{4.40}$$

where $a_{\rm u}$ and $a_{\rm d}$ are given in Eqs. (4.36) and (4.38). Without friction, the acceleration along the plane is simply $g\sin\theta$ downward for both directions of motion, so the total time with no friction is

$$T_{\rm no\,F} = \frac{2v_0}{g\sin\theta}\,. \tag{4.41}$$

(Equivalently, just erase the $\cos\theta$ terms in $T_{\rm F}$, since those terms came from the friction.) If we let $s \equiv \sin\theta$, $c \equiv \cos\theta$, and $x \equiv \cot\theta \equiv c/s$, then $T_{\rm F}$ will be larger than $T_{\rm no\,F}$ if

$$\begin{aligned}
\frac{1}{s+c} + \frac{1}{\sqrt{(s+c)(s-c)}} > \frac{2}{s} \quad &\implies \quad \frac{1}{1+x} + \frac{1}{\sqrt{1-x^2}} > 2 \\
\implies \quad \frac{1}{\sqrt{1-x^2}} > 2 - \frac{1}{1+x} \quad &\implies \quad \frac{1}{1-x^2} > \frac{(1+2x)^2}{(1+x)^2} \\
&\implies \quad \frac{1}{1-x} > \frac{(1+2x)^2}{(1+x)}\,.
\end{aligned} \tag{4.42}$$

Cross multiplying and simplifying yields

$$4x^3 - 2x > 0 \implies x > \frac{1}{\sqrt{2}}\,. \tag{4.43}$$

(There is technically also a range of negative solutions to this equation, but x is defined to be a positive number.) However, we also need $x < 1$ for Eq. (4.42) to hold. (If $x > 1$ then our cross multiplication switches the order of the inequality.) So $T_{\rm F} > T_{\rm no\,F}$ if $1 > \cot\theta > 1/\sqrt{2}$, or equivalently if $1 < \tan\theta < \sqrt{2} \implies 45^\circ < \theta < 54.7^\circ$. (Of course, we already knew that $\theta > 45^\circ$ from part (a).) To summarize:

- If $45^\circ < \theta < 54.7^\circ$, then $T_{\rm F} > T_{\rm no\,F}$. That is, the process takes longer with friction.
- If $54.7^\circ \le \theta$, then $T_{\rm no\,F} \ge T_{\rm F}$. That is, the process takes longer without friction.

The first of these results is clear in the limiting case where θ is only slightly larger than 45°, because the block will take a very long time to slide back down the plane, since $a_{\rm d} \approx 0$. In the other extreme where $\theta \to 90^\circ$, we have $T_{\rm F} = T_{\rm no\,F}$ because the friction force vanishes on the vertical plane.

Plots of T_F and $T_{\text{no F}}$ for θ values between 45° and 90° are shown in Fig. 4.50. Note that T_F achieves a local minimum. You can show numerically that this minimum occurs at $\theta = 1.34$ radians, which is about 77°. It isn't obvious that there should exist a local minimum. But what happens is that below 77°, T_F is larger than the minimum value because the slowness of the downward motion dominates other competing effects. And above 77°, T_F is larger because the larger distance up the plane dominates other competing effects (this isn't terribly obvious).

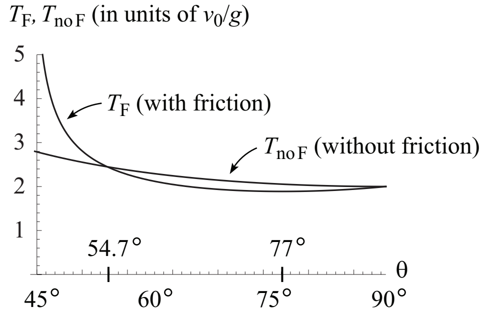


Figure 4.50

4.18. **Rope in a tube**

(a) First solution: Each little piece of the rope with tiny mass m has a radially inward acceleration v^2/R, so it feels a radially inward force of mv^2/R (applied by the outer surface of the tube). The sideways components of these forces cancel in pairs in Fig. 4.28(a), so we are left with a net downward force (in the plane of the page).

Second solution: The center of mass (a topic in Chapter 6) of the rope is located somewhere on the y axis at the given instant. It happens to be at radius $2R/\pi$, but that isn't important here. The relevant fact is that the CM travels in a circle. And since $\mathbf{F}_{\text{net}} = m\mathbf{a}_{\text{CM}}$ (see Chapter 6), we can simply imagine a point mass moving around in a circle. So the acceleration is radially inward (that is, downward in Fig. 4.28(a)) at the given instant.

(b) First solution: Each little piece of the rope has a tangential acceleration a (but no radial acceleration since v is instantaneously zero), so it feels a tangential force of ma. The vertical (in the plane of the page) components of these forces cancel in pairs, so we are left with a net rightward force in Fig. 4.28(b).

Remark: Since the rope is being pulled downward, where does this rightward force come from? It comes from the inner surface of the tube, with the important fact being that the force from the right part of the inner circle in Fig. 4.28(b) is larger than the force from the left part. This can be traced to the fact that the tension in the rope is larger closer to the right end that is being pulled; the tension approaches zero at the left end.

Second solution: The CM is initially located on the y axis, and it is accelerated to the right at the given instant (because it will end up moving in a circle once the rope gains some speed). Since $\mathbf{F}_{\text{net}} = m\mathbf{a}_{\text{CM}}$, the net force is therefore rightward.

If you pull on the rope with a nonzero force while it has a nonzero v, then the total force vector will have both downward and rightward components.

4.19. **Circling bucket**

Consider a little volume of the water, with mass m. Assuming that the water stays inside the bucket, then at the top of the motion the forces on the mass m are both the downward gravitational force mg and the downward normal force N from the other water in the bucket. So the radial $F = ma$ equation is

$$mg + N = \frac{mv^2}{R}. \tag{4.44}$$

(If you want, you can alternatively consider a little rock at the bottom of an empty bucket; that's effectively the same setup.) If v is large, then N is large. The cutoff case where the water barely stays in the bucket occurs when $N = 0$. The minimum v is therefore given by

$$mg = \frac{mv^2}{R} \implies v_{\text{min}} = \sqrt{gR}. \tag{4.45}$$

If we take R to be, say, 1 m, then this gives $v_{\text{min}} \approx 3\,\text{m/s}$. The time for each revolution is then $2\pi R/v_{\text{min}} \approx 2\,\text{s}$. If you swing your arm around with this period of revolution, you'll probably discover that the minimum speed is a lot slower that you would have guessed.

Remark: The main idea behind this problem is that although at the top of the motion the water is certainly accelerating downward under the influence of gravity, if you accelerate the bucket downward fast enough, then the bucket will maintain contact with the water. So the requirement is that at the top of the motion,[2] you must give the bucket a centripetal (downward) acceleration of at least g. That is, $v^2/R \ge g$, in agreement with Eq. (4.45). If the centripetal acceleration of the bucket is larger than g, then the bucket will need to push downward on the water to keep the water moving along with the bucket. That is, the normal force N will be positive. If the centripetal acceleration of the bucket is smaller than g, then the water will accelerate downward faster than the bucket. That is, the water will leave the bucket.

It isn't necessary to use circular motion to keep the water in the bucket; linear motion works too. If you turn a glass of water upside down and immediately accelerate it straight downward with an acceleration greater than or equal to g, the water will stay in the glass. Of course, you will soon run out of room and smash the glass into the floor! The nice thing about circular motion is that it can go on indefinitely. That's why centrifuges involve circular motion and not linear motion.

4.20. **Banking an airplane**

First solution: If you *do* feel like you are getting flung to the side in your seat, then you will need to counter this tendency with some kind of force parallel to the seat, perhaps friction from the seat or a normal force by pushing on the wall, etc. If you *don't* feel like you are getting flung to the side in your seat, then you could just as well be asleep on a frictionless seat, and you would remain at rest on the seat. So the goal of this problem is to determine the banking angle that is consistent with the only forces acting on you being the gravitational force and the normal force from the seat (that is, no friction), as shown in Fig. 4.51.

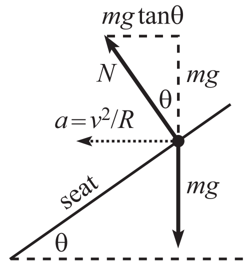


Figure 4.51

Let the banking angle be θ. The vertical component of the normal force must be $N_y = mg$, to make the net vertical force be zero. This implies that the horizontal component is $N_x = mg\tan\theta$. The horizontal $F = ma$ equation for the circular motion of radius R is then

$$N_x = \frac{mv^2}{R} \implies mg\tan\theta = \frac{mv^2}{R} \implies \tan\theta = \frac{v^2}{gR}. \tag{4.46}$$

Your apparent weight is simply the normal force, because this is the force with which a scale on the seat would have to push up on you. So your weight is

$$N = \sqrt{N_x^2 + N_y^2} = m\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}. \tag{4.47}$$

Limits: If R is very large or v is very small (more precisely, if $v^2 \ll gR$), then $\theta \approx 0$ and $N \approx mg$. (Of course, v can't be too small, or the plane won't stay up!) These limits make sense, because

[2]As an exercise, you can show that if the water stays inside the bucket at the highest point, then it will stay inside at all other points too, as you would intuitively expect.

for all you know, you are essentially moving in a straight line. If R is very small or v is very large (more precisely, if $v^2 \gg gR$), then $\theta \approx 90°$ and $N \approx mv^2/R$, which makes sense because gravity is inconsequential in this case.

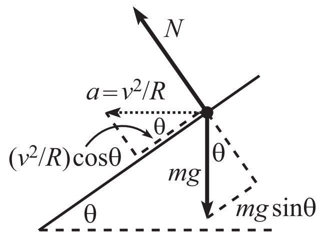


Figure 4.52

SECOND SOLUTION: Let's use tilted axes parallel and perpendicular to the seat. If we break the forces and the acceleration into components along these tilted axis, then we see that the force component parallel to the seat is $mg\sin\theta$, and the acceleration component parallel to the seat is $(v^2/R)\cos\theta$, as shown in Fig. 4.52. So the $F = ma$ equation for the motion parallel to the seat is

$$F = ma \implies mg\sin\theta = \frac{mv^2}{R}\cos\theta \implies \tan\theta = \frac{v^2}{gR}, \tag{4.48}$$

in agreement with the result in Eq. (4.46). The $F = ma$ equation for the motion perpendicular to the seat gives us the normal force N:

$$N - mg\cos\theta = \frac{mv^2}{R}\sin\theta \implies N = mg\cos\theta + \frac{mv^2}{R}\sin\theta. \tag{4.49}$$

You can verify that the $\sin\theta$ and $\cos\theta$ values implied by the $\tan\theta = v^2/gR$ relation in Eq. (4.48) make this expression for N reduce to the one we found above in Eq. (4.47).

4.21. **Breaking and turning**

If you brake on a straight road, your acceleration vector points along the road. The friction force satisfies $F_f = ma$. (We'll just deal with magnitudes here, so both F_f and a are positive quantities.) But $F_f \le \mu N = \mu(mg)$. Therefore,

$$ma = F_f \le \mu mg \implies a \le \mu g. \tag{4.50}$$

So your maximum possible deceleration is μg. It makes sense that this should be zero if μ is zero.

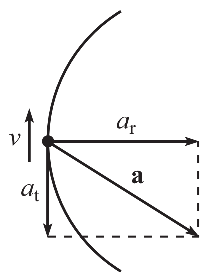


Figure 4.53

If you brake while traveling around a bend, your acceleration vector does *not* point along the road. It has a tangential component a_t pointing along the road (this is the deceleration we are concerned with) and also a radial component $a_r = v^2/R$ pointing perpendicular to the road. These components are shown in Fig. 4.53. The total acceleration vector (and hence also the total friction force vector) points backward and radially inward. Since $\mathbf{F}_f = m\mathbf{a}$, we still have $F_f = ma$, where a is the magnitude of the $\mathbf{a}$ vector, which is now $a = \sqrt{a_t^2 + a_r^2}$. So the $F_f \le \mu N$ restriction on F_f takes the form

$$m\sqrt{a_t^2 + a_r^2} = F_f \le \mu mg \implies \sqrt{a_t^2 + (v^2/R)^2} \le \mu g$$
$$\implies a_t \le \sqrt{(\mu g)^2 - (v^2/R)^2}. \tag{4.51}$$

LIMITS: This result correctly reduces to $a_t \le \mu g$ when $R = \infty$, that is, when the road is straight. It also reduces to $a_t \le \mu g$ when v is very small (more precisely, $v \ll \sqrt{\mu g R}$), because in this case the radial acceleration is negligible; you are effectively traveling on a straight road. If $v = \sqrt{\mu g R}$, then Eq. (4.51) tells us that a_t must be zero. In this case the maximum friction force $\mu N = \mu mg$ is barely large enough to provide the $mv^2/R = m(\mu g R)/R = \mu mg$ force required to keep you going in a circle. Any additional acceleration caused by braking will necessitate a friction force larger than the μN limit.

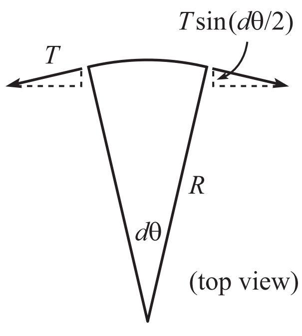


Figure 4.54

4.22. **Circle of rope**

The forces on a small piece of rope subtending an angle $d\theta$ are the tensions at its ends. In Fig. 4.54 these tensions point slightly downward; they make an angle of $d\theta/2$ with respect to the horizontal. (This is true because each of the long radial sides of the pie piece in the figure makes an angle of $d\theta/2$ with respect to the vertical, and the tensions are perpendicular to these sides.) If T is the tension throughout the rope, then from the figure the net force on the small piece is $F = 2 \cdot T \sin(d\theta/2)$. This force points downward (radially inward). Using the approximation $\sin x \approx x$ for small x, we see that the net force is $F \approx T\,d\theta$. This force is what causes the centripetal acceleration of the small piece. The length of the piece is $R\,d\theta$, so its mass is $\lambda R\,d\theta$. The radial $F = ma$ equation is therefore

$$F = \frac{mv^2}{R} \implies T\,d\theta = \frac{(\lambda R\,d\theta)v^2}{R} \implies T = \lambda v^2. \tag{4.52}$$

Remarks: Note that this result of λv^2 is independent of R. This means that if we have an arbitrarily shaped rope (that is, the local radius of curvature may vary), and if the rope is moving along itself (so if the rope were featureless, you couldn't tell that it was actually moving) at constant speed, then the tension equals λv^2 everywhere.

If the rope is stretchable, it will stretch under the tension as it spins around. You should convince yourself that the tension's independence of R implies that if two circles (made of the same material) with different radii have the same v, then the stretching will cause their radii to increase by the same factor.

This λv^2 result comes up often in physics. In addition to being the answer to the present problem, λv^2 is the tension in a rope with density λ if the speed of a traveling wave is v (this is a standard result that can be found in a textbook on waves). And λv^2 is also the tension needed if you have a heap of rope and you grab one end and pull with speed v to straighten it out (see Multiple-Choice Question 6.18). All three of these results appear in a gloriously unified way in the intriguing phenomenon of the "chain fountain."[3]

4.23. **Cutting the string**

We must first find the speed of the circular motion. The free-body diagram is shown in Fig. 4.55, where we have included the acceleration for convenience. The vertical $F = ma$ equation tells us that $T_y = mg$, because the net vertical force must be zero. The horizontal component of the tension is therefore $T_x = T_y \tan\theta = mg\tan\theta$. So the horizontal $F = ma$ equation is (using the fact that the radius of the circular motion is $r = \ell\sin\theta$)

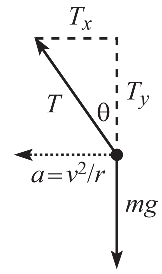


Figure 4.55

$$T_x = \frac{mv^2}{r} \implies mg\tan\theta = \frac{mv^2}{\ell\sin\theta} \implies v = \sqrt{g\ell\sin\theta\tan\theta}. \tag{4.53}$$

After the string is cut, we simply have a projectile problem in which the initial velocity is horizontal (tangential to the circle when the string is cut). The distance down to the floor is $d = \ell - \ell\cos\theta$, so the time to fall to the floor is given by

$$\frac{1}{2}gt^2 = \ell - \ell\cos\theta \implies t = \sqrt{\frac{2\ell(1-\cos\theta)}{g}}. \tag{4.54}$$

The horizontal distance traveled is therefore

$$\begin{aligned} x = vt &= \sqrt{g\ell\sin\theta\tan\theta}\sqrt{\frac{2\ell(1-\cos\theta)}{g}} \\ &= \ell\sqrt{2\sin\theta\tan\theta(1-\cos\theta)}. \end{aligned} \tag{4.55}$$

Limits: If $\theta \approx 0$ then $x \approx 0$ (because both $v \to 0$ and $t \to 0$). And if $\theta \to 90^\circ$ then $x \to \infty$ (because $v \to \infty$). These limits make intuitive sense.

Units: Note that x doesn't depend on g. Intuitively, if g is large, then for a given θ the speed v is large (it is proportional to $\sqrt{g}$). But the falling time t is short (it is proportional to $1/\sqrt{g}$). These two effects exactly cancel. The independence of g also follows from dimensional analysis. The distance x must be some function of ℓ, θ, g, and m. But there can't be any dependence on g, which has units of m/s^2, because there would be no way to eliminate the seconds from the units to obtain a pure length. (Likewise for the kilograms in the mass m.)

[3]See "Understanding the chain fountain" (by The Royal Society) at http://www.youtube.com/watch?v=-eEi7fO0_O0.

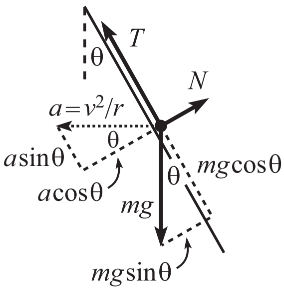


Figure 4.56

4.24. **Circling around a cone**

(a) The free-body diagram is shown in Fig. 4.56. The net force produces the horizontal centripetal acceleration of $a = v^2/r$, where $r = \ell \sin\theta$. Let's work with axes parallel and perpendicular to the cone. Horizontal and vertical axes would work fine too, but things would be a little messier because the tension T and the normal force N would each appear in both of the $F = ma$ equations, so we would have to solve a system of equations.
The $F = ma$ equation along the cone is (using $a = v^2/\ell\sin\theta$)

$$T - mg\cos\theta = m\left(\frac{v^2}{\ell\sin\theta}\right)\sin\theta \implies T = mg\cos\theta + \frac{mv^2}{\ell}. \quad (4.56)$$

Limits: If $\theta \to 0$ then $T \to mg + mv^2/\ell$. We will find below that if contact with the cone is to be maintained, then v must be essentially zero in this case. So we simply have $T \to mg$, which makes sense because the mass is hanging straight down. If $\theta \to \pi/2$ then $T \to mv^2/\ell$, which makes sense because the mass is moving in a circle on a horizontal table.

(b) The $F = ma$ equation perpendicular to the cone is

$$mg\sin\theta - N = m\left(\frac{v^2}{\ell\sin\theta}\right)\cos\theta \implies N = mg\sin\theta - \frac{mv^2}{\ell\tan\theta}. \quad (4.57)$$

Limits: If $\theta \to 0$ then $N \to 0 - mv^2/\ell\tan\theta$. Again, we will find below that v must be essentially zero in this case, so we have $N \to 0$, which makes sense because the cone is vertical. If $\theta \to \pi/2$ then $N \to mg$, which makes sense because, as above, the mass is moving in a circle on a horizontal table.

(c) The mass stays in contact with the cone if $N \ge 0$. Using Eq. (4.57), this implies that

$$mg\sin\theta \ge \frac{mv^2}{\ell\tan\theta} \implies v \le \sqrt{g\ell\sin\theta\tan\theta} \equiv v_{\text{max}}. \quad (4.58)$$

If v equals v_{max} then the mass is barely in contact with the cone. If the cone were removed in this case, the mass would maintain the same circular motion.

Limits: If $\theta \to 0$ then $v_{\text{max}} \to 0$. And if $\theta \to \pi/2$ then $v_{\text{max}} \to \infty$. These limits make intuitive sense.

4.25. **Penny in a dryer**

The penny loses contact with the dryer when the normal force N becomes zero. If θ is measured with respect to the vertical, then the radially inward component of the gravitational force is $mg\cos\theta$. The radial $F = ma$ equation is therefore $N + mg\cos\theta = mv^2/R$, which implies that the $N = 0$ condition is

$$mg\cos\theta = \frac{mv^2}{R} \implies v^2 = gR\cos\theta. \quad (4.59)$$

This equation tells us how v and θ are related at the point where the penny loses contact. (If $v > \sqrt{gR}$, then there is no solution for θ, so the penny never loses contact.) To solve for θ and v (and hence $\omega = v/R$) individually, we must produce a second equation that relates v and θ. This equation comes from the projectile motion and the condition that the landing point is diametrically opposite.

When the penny loses contact with the dryer, the initial position (relative to the center of the dryer) for the projectile motion is $(R\sin\theta, R\cos\theta)$; remember that θ is measured with respect to the vertical. The initial angle of the velocity is θ upward to the left, as you can check. Therefore, since the initial speed is v, the initial velocity components are $x_x = -v\cos\theta$ and $v_y = v\sin\theta$. The coordinates of the projectile motion are then

$$\begin{aligned} x(t) &= R\sin\theta - (v\cos\theta)t, \\ y(t) &= R\cos\theta + (v\sin\theta)t - gt^2/2. \end{aligned} \quad (4.60)$$

If the penny lands at the diametrically opposite point, then the final position is the negative of the initial position. So the final time must satisfy

$$\begin{aligned} R\sin\theta - (v\cos\theta)t = -R\sin\theta &\implies (v\cos\theta)t = 2R\sin\theta, \qquad (4.61)\\ R\cos\theta + (v\sin\theta)t - gt^2/2 = -R\cos\theta &\implies gt^2/2 - (v\sin\theta)t = 2R\cos\theta. \end{aligned}$$

Solving for t in the first of these equations and plugging the result into the second gives

$$\begin{aligned} \frac{g}{2}\left(\frac{2R\sin\theta}{v\cos\theta}\right)^2 - v\sin\theta\left(\frac{2R\sin\theta}{v\cos\theta}\right) &= 2R\cos\theta \\ \implies \frac{gR\sin^2\theta}{v^2\cos^2\theta} &= \frac{\sin^2\theta}{\cos\theta} + \cos\theta \\ \implies \frac{gR\sin^2\theta}{v^2\cos^2\theta} &= \frac{1}{\cos\theta} \\ \implies v^2 &= \frac{gR\sin^2\theta}{\cos\theta}. \qquad (4.62) \end{aligned}$$

This is the desired second equation that relates v and θ. Equating this expression for v^2 (which guarantees a diametrically opposite landing point) with the one in Eq. (4.59) (which gives the point where the penny loses contact) yields

$$\frac{gR\sin^2\theta}{\cos\theta} = gR\cos\theta \implies \tan^2\theta = 1 \implies \theta = 45^\circ. \qquad (4.63)$$

(And $\theta = -45^\circ$, works too, if the dryer is spinning the other way. Equivalently, the corresponding value of t is negative.) There must be a reason why the angle comes out so nice, but it eludes me.

With this value of θ, Eq. (4.59) gives

$$v = \sqrt{\frac{gR}{\sqrt{2}}} \implies \omega = \frac{v}{R} = \sqrt{\frac{g}{\sqrt{2}R}}. \qquad (4.64)$$

LIMITS: ω grows with g, as expected. The decrease with R isn't as obvious, but it does follow from dimensional analysis.

REMARKS: The relation in Eq. (4.62) describes many different types of trajectories. If v is large, then the corresponding θ is close to 90°. In this case we have a very tall projectile path that generally lies outside the dryer, so it isn't physical. If v is small, then θ is close to 0. In this case the penny just drops from the top of the dryer down to the bottom; the trajectory lies completely inside the dryer. The cutoff between these two cases (that is, the case where the penny barely stays inside), turns out to be the $\theta = 45^\circ$ case that solves the problem (by also satisfying Eq. (4.59)). You are encouraged to think about why this is true.

As an exercise, you can also produce the second equation relating v and θ (Eq. (4.62), derived from the projectile motion) by using axes that are tilted along the diameter and perpendicular to it. The magnitudes of the accelerations in these directions are $g\cos\theta$ and $g\sin\theta$, respectively, and the penny hits the diameter a distance $2R$ down along it. The math is fairly clean due to the fact that the initial velocity is perpendicular to the diameter. So the solution turns out to be a bit quicker than the one we used above. Problem 3.18(b) involved similar reasoning with the axis perpendicular to the diameter/plane.