Chapter 1

Problem-solving strategies

From *Problems and Solutions in Introductory Mechanics* (Draft version, August 2014)
David Morin, morin@physics.harvard.edu

TO THE READER: This book is available as both a paperback and an eBook. I have made a few chapters available on the web, but it is possible (based on past experience) that a pirated version of the complete book will eventually appear on file-sharing sites. In the event that you are reading such a version, I have a request:

If you don't find this book useful (in which case you probably would have returned it, if you had bought it), or if you do find it useful but aren't able to afford it, then no worries; carry on. However, if you do find it useful and are able to afford the Kindle eBook[1] (priced somewhere between $7 and $10), then please consider purchasing it (available on Amazon). I chose to self-publish this book so that I could keep the cost low. The resulting price of around $10, which is very inexpensive for a 350-page physics book, is less than a movie and a bag of popcorn, with the added bonus that the book lasts for more than two hours and has zero calories (if used properly!).

– David Morin

1.1 Basic strategies

In view of the fact that this is a problem book, it makes sense to start off by arming you with some strategies for solving problems. This is the subject of the present chapter. We'll begin with a few strategies that are discussed somewhat in depth, and then we'll provide a long list of 30-ish strategies. You obviously shouldn't try to memorize all of them. Just remember that the list is there, and refer back to it every now and then.

1.1.1 Solving problems symbolically

If you are solving a problem where the given quantities are specified numerically, it is highly advantageous to immediately change the numbers to letters and then solve the problem in terms of the letters. After you obtain a symbolic answer in terms of these letters, you can plug in the actual numerical values to obtain a numerical answer. There are many advantages to using letters:

- IT IS QUICKER. It's much easier to multiply a g by an ℓ by writing them down on a piece of paper next to each other, than it is to multiply their numerical values on a calculator. If solving a problem involves five or ten such operations, the time would add up if you performed all the operations on a calculator.

- YOU ARE LESS LIKELY TO MAKE A MISTAKE. It's very easy to mistype an 8 for a 9 in a calculator, but you're probably not going to miswrite a q for an a on a piece of paper. But even if you

[1] If you don't already have the Kindle reading app for your computer, you can download it free from Amazon.

do, you'll quickly realize that it should be an a. You certainly won't just give up on the problem and deem it unsolvable because no one gave you the value of q!

- You can do the problem once and for all. If someone comes along and says, oops, the value of ℓ is actually 2.4 m instead of 2.3 m, then you won't have to do the whole problem again. You can simply plug the new value of ℓ into your symbolic answer.

- You can see the general dependence of your answer on the various given quantities. For example, you can see that it grows with quantities a and b, decreases with c, and doesn't depend on d. There is *much* more information contained in a symbolic answer than in a numerical one. And besides, symbolic answers nearly always look nice and pretty.

- You can check units and special cases. These checks go hand-in-hand with the previous "general dependence" advantage. We'll discuss these very important checks below.

Two caveats to all this: First, occasionally there are times when things get messy when working with letters. For example, solving a system of three equations in three unknowns might be rather cumbersome unless you plug in the actual numbers. But in the vast majority of problems, it is highly advantageous to work entirely with letters. Second, if you solve a problem that was posed with letters instead of numbers, it's always a good idea to pick some values for the various parameters to see what kinds of numbers pop out, just to get a general sense of the size of things.

1.1.2 Checking units/dimensions

The words *dimensions* and *units* are often used interchangeably, but there is technically a difference: dimensions refer to the general qualities of mass, length, time, etc., whereas units refer to the specific way we quantify these qualities. For example, in the standard meters-kilogram-second (mks) system of units we use in this book, the meter is the unit associated with the dimension of length, the joule is the unit associated with the dimension of energy, and so on. However, we'll often be sloppy and ignore the difference between units and dimensions.

The consideration of units offers two main benefits:

- Considering the units of the relevant quantities before you start solving a problem can tell you roughly what the answer has to look like, up to numerical factors. This practice is called *dimensional analysis*.

- Checking units at the end of a calculation (which is something you should *always* do) can tell you if your answer has a chance at being correct. It won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely *incorrect*. For example, if your goal in a problem is to find a length, and if you end up with a mass, then you know that it's time to look back over your work.

In the mks system of units, the three fundamental mechanical units are the meter (m), kilogram (kg), and second (s). All other units in mechanics, for example the joule (J) or the newton (N), can be built up from these fundamental three. If you want to work with dimensions instead of units, then you can write everything in terms of length (L), mass (M), and time (T). The difference is only cosmetic.

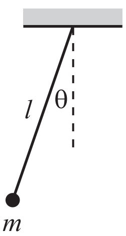

Figure 1.1

As an example of the above two benefits of considering units, consider a pendulum consisting of a mass m hanging from a massless string with length ℓ; see Fig. 1.1. Assume that the pendulum swings back and forth with an angular amplitude θ_0 that is small; that is, the string doesn't deviate far from vertical. What is the period, call it T_0, of this oscillatory motion? (The period is the time of a full back-and-forth cycle.)

With regard to the first of the above benefits, what can we say about the period T_0, by looking only at units and not doing any calculations? Well, we must first make a list of all the quantities the period can possibly depend on. The mass m (with units of kg), the length ℓ (with units of m), and the angular amplitude θ_0 (which is unitless) are given, but additionally there might be dependence on g (the acceleration due to gravity, with units of m/s^2), If you think for a little while, you'll come to the conclusion that there really isn't anything else the period can depend on (assuming that we ignore air resistance).

So the question becomes: How does T_0 depend on m, ℓ, θ_0, and g? Or equivalently: How can we produce a quantity with units of seconds from four quantities with units of kg, m, 1, and m/s^2? (The 1 signifies no units.) We quickly see that the answer can't involve the mass m, because there would be no way to get rid of the units of kg. We then see that if we want to end up with units of seconds, the answer must be proportional to $\sqrt{\ell/g}$, because this gets rid of the meters and leaves one power of seconds in the numerator. Therefore, by looking only at the units involved, we have shown that $T_0 \propto \sqrt{\ell/g}$.[2]

This is all we can say by considering units. For all we know, there is a numerical factor out front, and also an arbitrary function of θ_0 (which won't mess up the units, because θ_0 is unitless). The correct answer happens to be $T_0 = 2\pi\sqrt{\ell/g}$, but there is no way to know this without solving the problem for real.[3] However, even though we haven't produced an exact result, there is still a great deal of information contained in our $T_0 \propto \sqrt{\ell/g}$ statement. For example, we see that the period is independent of m; a small mass and a large mass swing back and forth at the same rate. We also see that if we quadruple the length of the string, then the period gets doubled. And if we place the same pendulum on the moon, where the g factor is $1/6$ of that on the earth, the period increases by a factor of $\sqrt{6} \approx 2.4$; the pendulum swings back and forth more slowly. Not bad for doing nothing other than considering units!

While this is all quite interesting, the second of the above two benefits (checking the units of an answer) is actually the one you will get the most mileage out of when solving problems, mainly because you should make use of it *every* time you solve a problem. It only takes a second. In the present example with the pendulum, let's say that you solved the problem correctly and ended up with $T_0 = 2\pi\sqrt{\ell/g}$. You should then immediately check the units, which do indeed correctly come out to be seconds. If you had made a mistake in your solution, such as flipping the square root upside down (so that you instead had $\sqrt{g/\ell}$), then your units check would yield the incorrect units of s^{-1}. You would then know to go back and check over your work.

Throughout this book, we often won't bother to explicitly write down the units check if the check is a simple one (as with the above pendulum). But you should of course always do the check in your head. In more complicated cases where it actually takes a little algebra to show that the units work out, we'll write things out explicitly.

1.1.3 Checking limiting/special cases

As with units, the consideration of limiting cases (or perhaps we should more generally say special cases) offers two main benefits. First, it can help you get started on a problem. If you are having trouble figuring out how a given system behaves, then you can imagine making, for example, a certain length become very large or very small, and then you can see what happens to the behavior. Having convinced yourself that the length actually affects the system in extreme cases (or perhaps you will discover that the length doesn't affect things at all), it will then be easier to understand how it affects the system in general. This will then make it easier to write down the relevant quantitative equations (conservation laws, $F = ma$ equations, etc.), which will allow you to fully solve the problem. In short, modifying the various parameters and seeing the effects on the system can lead to an enormous amount of information.

Second, as with checking units, checking limiting cases (or special cases) is something you should *always* do at the end of a calculation. As with units, checking limiting cases won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely

[2] In this setup it was easy to determine the correct combination of the given parameters. But in more complicated setups, you might find it simpler to write down a general product of the given dimensionful quantities raised to arbitrary powers, and then solve a system of equations to determine these powers. An example of this method is given in the solution to Problem 1.4.

[3] This $T_0 = 2\pi\sqrt{\ell/g}$ result holds in the approximation where the amplitude θ_0 is small. For a general value of θ_0, the period actually *does* involve a function of θ_0. This function can't be written in closed form, but it starts off as $1 + \theta_0^2/16 + \cdots$. It takes a lot of work to show this, though. See Exercise 4.23 in *Introduction to Classical Mechanics, With Problems and Solutions*, David Morin, Cambridge University Press, 2008; henceforth referred to as "Morin (2008)."

incorrect. Your intuition about limiting cases is generally *much* better than your intuition about generic values of the parameters. You should use this to your advantage.

As an example, consider the trigonometric formula for $\tan(\theta/2)$. The formula can be written in many different ways. Let's say that you're trying to derive it, but you keep making mistakes and getting different answers. However, let's assume that you're pretty sure it takes the form of $\tan(\theta/2) = A(1 \pm \cos\theta)/\sin\theta$, where A is a numerical coefficient. Can you determine the correct form of the answer by checking special cases? Indeed you can, because you know what $\tan(\theta/2)$ equals for a few special values of θ:

- $\theta = 0$: We know that $\tan(0/2) = 0$, so this immediately rules out the $(1 + \cos\theta)/\sin\theta$ form, because this isn't zero when $\theta = 0$; it actually goes to infinity at $\theta = 0$. The answer must therefore take the form of $A(1 - \cos\theta)/\sin\theta$. (This appears to be $0/0$ when $\theta = 0$, but it does indeed go to zero, as you can check by using the Taylor series for $\sin\theta$ and $\cos\theta$; see the subsection on Taylor series below.)
- $\theta = 90^\circ$: We know that $\tan(90^\circ/2) = 1$, which quickly gives $A = 1$. So the correct answer must be $\tan(\theta/2) = (1 - \cos\theta)/\sin\theta$.
- $\theta = 180^\circ$: If you want to feel better about this result, you can note that it gives the correct answer for another special value of θ; it correctly goes to infinity when $\theta = 180^\circ$.

Of course, none of what we've done here demonstrates that $(1 - \cos\theta)/\sin\theta$ is actually the correct answer. But checking the above special cases does two things: it rules out some incorrect answers, and it makes us feel better about the correct answer.

A type of approximation that often comes up involves expressions of the form $ab/(a + b)$, that is, a product over a sum. For example, the equivalent mass in Problem 4.5 turns out to be

$$M = \frac{4m_1 m_2}{m_1 + m_2}\,. \tag{1.1}$$

What does M look like in the limit where m_1 is much smaller than m_2? In this limit we can ignore the m_1 in the denominator, but we *can't* ignore it in the numerator. So we obtain $M \approx 4m_1m_2/(0 + m_2) = 4m_1$. Why can we can ignore one of the m_1's but not the other? We can ignore the m_1 in the denominator because it appears there as an *additive* term. If m_1 is small, then erasing it essentially doesn't change the value of the denominator. However, in the numerator m_1 appears as a *multiplicative* term. Even if m_1 is small, its value certainly affects the value of the numerator. Decreasing m_1 by a factor of 10 would decrease the numerator by the same factor of 10. So we certainly can't just erase it. (That would change the units of M anyway.)

Alternatively, you can obtain the $M \approx 4m_1$ result in the limit of small m_1 by applying a Taylor series (discussed below) to M. But this would be overkill. It's much easier to just erase the m_1 in the denominator. In any case, if you're ever unsure about which terms you should keep and which terms you can ignore, just plug some very small numbers (or very large numbers, depending on what limit you're dealing with) into a calculator to see how the expression depends on the various parameters.

It should be noted that there is no need to wait until the end of a solution to check limiting cases (or units, too). Whenever you arrive at an intermediate result that lends itself to checking limiting cases, you should check them. If you find that something is amiss, this will prevent you from wasting time carrying onward with incorrect results.

1.1.4 Taylor series

A tool that often comes up when checking limiting cases is the Taylor series. A Taylor series expresses a function $f(x)$ as a series expansion in x (that is, a sum of terms involving different powers of x). Perhaps the most well-known Taylor series is the one for the function $f(x) = e^x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\,. \tag{1.2}$$

A number of other Taylor series are listed near the beginning of Appendix B (Section 13.2). The rest of Appendix B contains a discussion of Taylor series and various issues that arise when using them. If you've never seen Taylor series before, you should take a moment and read the appendix. For the present purposes, we'll just take the above expression for e^x as given and see where it leads us.[4]

As an example of the utility of Taylor series, consider a beach ball that is dropped from rest. It can be shown that if air drag is taken into account, and if the drag force is proportional to the velocity (so that it takes the form $F_d = -bv$, where b is the drag coefficient), then the ball's velocity (with upward taken as positive) as a function of time equals

$$v(t) = -\frac{mg}{b}\left(1 - e^{-bt/m}\right). \tag{1.3}$$

This is a somewhat complicated expression, so you might be a little doubtful of its validity. Let's therefore look at some limiting cases. If these limiting cases yield expected results, then we can feel more confident that the expression is actually correct.

If t is very small (more precisely, if $bt/m \ll 1$; see the discussion in Section 13.2.3), then we can use the Taylor series in Eq. (1.2) to make an approximation to $v(t)$, to leading order in t. (The leading-order term is the smallest power of t with a nonzero coefficient.) To first order in x, Eq. (1.2) gives $e^x \approx 1 + x$. If we let x be $-bt/m$, then we see that Eq. (1.3) can be written as

$$\begin{aligned} v(t) &\approx -\frac{mg}{b}\left(1 - \left(1 - \frac{bt}{m}\right)\right) \\ &\approx -gt. \end{aligned} \tag{1.4}$$

This answer makes sense, because the drag force is negligible at the start (because v, and hence bv, is very small), so we essentially have a freely falling body with acceleration g downward. And $v(t) = -gt$ is the standard expression in that case (see the introduction to Chapter 2). This successful check of a limiting case makes us have a little more faith that Eq. (1.3) is actually correct.

If we mistakenly had, say, $-2mg/b$ as the coefficient in Eq. (1.3), then we would have obtained $v(t) \approx -2gt$ in the small-t limit, which is incorrect. So we would know that we needed to go back and check over our work. Although it isn't obvious that an extra factor of 2 in Eq. (1.3) is incorrect, it *is* obvious that it is incorrect in the limiting $v(t) \approx -2gt$ result. As mentioned above, your intuition about limiting cases is generally much better than your intuition about generic values of the parameters.

We can also consider the limit of large t (or rather, large bt/m). In this limit, $e^{-bt/m}$ is essentially zero, so the $v(t)$ in Eq. (1.3) becomes (there's no need for a Taylor series in this case)

$$v(t) \approx -\frac{mg}{b}. \tag{1.5}$$

This is the "terminal velocity" that the ball approaches as time goes on. Its value makes sense, because it is the velocity for which the total force (gravitational plus air drag), $-mg - bv$, equals zero. And zero force means constant velocity. Mathematically, the velocity never quite reaches the value of $-mg/b$, but it gets extremely close as t becomes large.

Whenever you derive approximate answers as we just did, you gain something and you lose something. You lose some truth, of course, because your new answer is an approximation and therefore technically not correct (although the error becomes arbitrarily small in the appropriate limit). But you gain some aesthetics. Your new answer is invariably much cleaner (often involving only one term), and that makes it a lot easier to see what's going on.

In the above beach-ball example, we checked limiting cases of an answer that was correct. This whole process is more useful (and a bit more fun) when you check limiting cases of an answer that is *incorrect* (as in the case of the erroneous coefficient of $-2mg/b$ we mentioned

[4]Calculus is required if you want to *derive* a Taylor series. However, if you just want to *use* a Taylor series (which is what we will do in this book), then algebra is all you need. So although some Taylor-series manipulations might look a bit scary, there's nothing more than algebra involved.

above). When this happens, you gain the unequivocal information that your answer is wrong (assuming that your incorrect answer doesn't just happen to give the correct result in a certain limit, by pure luck). However, rather than leading you into despair, this information is something you should be quite happy about, considering that the alternative is to carry on in a state of blissful ignorance. Once you know that your answer is wrong, you can go back through your work and figure out where the error is (perhaps by checking limiting cases at various intermediate stages to narrow down where the error could be). Personally, if there's any way I'd like to discover that my answer is garbage, this is it. So you shouldn't check limiting cases (and units) because you're being told to, but rather because you *want* to.

1.2 List of strategies

This section contains a list of all the problem-solving strategies I can think of. *The list is long, so there is certainly no need to memorize it.* It would be a step backward if you spent your time worrying about covering all of the strategies, when you should instead be thinking about actually solving a problem. The best way to use this list is to read through it now, and then occasionally refer back to it, especially if you get stuck.

You will inevitably apply many of the strategies without even trying to. But others in the list might seem like meaningless gibberish for now; we're not applying them to any problems here, so there isn't much context. However, if you refer back to the list when solving problems, a given strategy will mean much more if it helps you solve a problem.

Different people think differently, of course. Some strategies might work for you, while others might not. In the end, there's no overall magic bullet for solving all problems. It just comes down to practice and doing lots of problems. But the strategies listed below should help make the practice more efficient. We've divided them into five categories: (1) Getting started, (2) Solving the problem, (3) Troubleshooting, (4) Finishing up, and (5) Looking ahead.

1.2.1 Getting started

The following nine strategies will help you get started on a problem. They don't require too much thinking; they're standard mechanical things that you can do on auto-pilot.

1. Read the problem slowly and carefully

There's no better way to waste time than to read a problem quickly in an effort to save time. If you miss a piece of the given information, you'll end up just spinning your wheels, trying to solve an unsolvable problem.

There is famous statement about the existence of known knowns (things that we know we know), known unknowns (things that we know we don't know), and unknown unknowns (things that we don't know we don't know). Leaving firmly aside who made the statement and why, you might wonder about the fourth permutation: the unknown knowns. What might those be? Well, one thing that certainly falls into this category is the information you miss when you read a problem too quickly. The information is certainly known, but you just don't know that you know it!

2. Identify the things you know, and the things you are trying to find

Identifying the known quantities enables you to see what you have to work with. And identifying the unknown quantities enables you to see what you're aiming for, which gives you some guidance in thinking about what physical principles you should consider (Strategy 10 below). Of course, as mentioned in Strategy 1, identifying the things you know requires reading the problem carefully!

The "knowns and unknowns" reference in Strategy 1 is relevant here too. We mentioned there that you want to avoid unknown knowns. You also want to avoid unknown unknowns.

These are things you're going to need to find, but you don't know yet that you need to look for them. It's much easier, of course, to reach a destination if you know what that destination is. So do your best to make sure you know what all of the unknowns are. Basically, try to make sure everything is known – even if it's (an) unknown!

3. Draw a picture

Draw a nice *big* picture, one where you can label everything clearly. Make a note of which quantities you know, and which quantities you're trying to find. Although a small set of mechanics problems involve doing only some math, the vast majority involve a setup that you really need to visualize in order to get anywhere. A picture makes things much more concrete.

4. Draw free-body diagrams

A special kind of picture is a free-body diagram. This is a picture where you draw all of the external forces acting on a given object. As with a general picture, make a note of the known and unknown quantities. Free-body diagrams are absolutely critical when solving problems involving forces. More precisely, they are necessary, and nearly sufficient. That is, many problems are impossible if you don't draw the free-body diagrams, and trivial if you do. Often the only thing that remains to be done after drawing the diagrams is to solve some $F = ma$ equations by doing some math. The physics is all contained in the diagrams. See the introduction to Chapter 4 for further discussion of free-body diagrams.

5. Strip the problem down to its basics

Some problems are posed as idealized "toy model" problems, for example a point mass colliding with a uniform stick with negligible thickness. Other problems deal with more realistic setups that you might encounter in the real world, for example two skaters colliding and grabbing on to each other. When dealing with the latter type, the first thing you should do is strip the problem down to its basics. If possible, reduce the problem to point masses, sticks, massless strings, etc. Many real-life problems that look different at first glance end up being the same when reduced to the underlying toy model. So when you solve the toy-model version, you're actually solving a more general problem, which is a good thing.

Of course, you need to be careful that your toy model mimics the original setup correctly. For example, simplifying an object to a point mass works fine if you're using forces, but not necessarily if you're using torques. Your goal is to simplify the setup as much as possible without changing the physics. It takes some thought not to go too far, but this thought process is helpful in solving the problem. It helps you decide which aspects of the problem are important, and which aspects are irrelevant. This in turn helps you decide which physical principles you need to use (Strategy 10 below). Along these lines, if the original real-life setup is one for which you have some physical intuition, remember to use it when you start dealing with the toy model!

6. Choose wisely your coordinate system or reference frame

There are always only a couple of reasonable coordinate systems and reference frames to choose from, but a particular choice may greatly simplify things. For example, when dealing with an inclined plane, choosing tilted axes (parallel and perpendicular to the plane) is often helpful. And when dealing with circular motion, it is of course usually best to work with polar coordinates. And when two (or more) objects are moving with respect to each other, it is often helpful to analyze the setup in a new reference frame (the CM frame, or perhaps a frame moving along with one of the objects).

Part of choosing a coordinate system involves choosing the positive directions for the coordinate axes. This is completely your choice, but you must remember that once you pick a convention, you must stick with it. It's fine to let downward be positive for a falling object; just don't forget later on in your solution that you've made that choice.

7. Identify the initial and final states of the system

This is especially important in problems involving conservation principles (conservation of energy, momentum, angular momentum). In many cases, you can ignore the specifics of what happens during a process and simply equate the initial and final values of a particular quantity.

Another class of problems is "initial condition" problems. If you've calculated a general expression for, say, an object's position involving some unknown parameters, you can determine the values of these parameters by invoking the initial conditions (usually the initial position and velocity).

8. Identify the constraints

Is an object constrained to lie on a plane? Or travel in a circle? Or move with constant velocity? Is the system static? In the end, a constraint means that you have one fewer unknown than you otherwise might have thought. For example, if an object lies on a plane inclined at angle θ, then its coordinates are related by $y = x \tan\theta$. So if you choose x as your unknown, then y is determined.

9. Convert numbers to letters, so that you can solve things symbolically

This strategy is extremely helpful and very simple to apply. It is discussed in depth in Section 1.1.1 above, where its many benefits are noted.

1.2.2 Solving the problem

Having taken the above mechanical steps, it's now time to start thinking. There's no sure-fire way to guarantee that you'll solve every problem you encounter, but the following five strategies will certainly help.

10. Identify the physical principles involved

Think about what physical principle(s) will allow you to solve the problem. The most fundamental principles in mechanics are $F = ma$ and $\tau = I\alpha$ (or more accurately $F = dp/dt$ and $\tau = dL/dt$), and conservation of E, p, and L. A given problem can invariably be solved in multiple ways. For example, since conservation of energy can be derived from $F = ma$, any problem that can be solved with conservation of energy can also be solved with $F = ma$, although the latter solution may be more cumbersome.

In addition to the overarching fundamental principles listed above, there are many other physical principles/facts that you may need to use. For example, the radial acceleration is v^2/r (Eq. (3.7)); $v_y = 0$ at the highest point in projectile motion; the energy of an object that is both translating and rotating consists of two terms (Eq. (7.8)); Hooke's law for a spring is $F = -kx$ (Eq. (10.1)); Newton's law of gravitation is an inverse-square law (Eq. (11.1)); and so on.

11. Convert physical statements into mathematical equations

Having identified the relevant physical principles, you must now convert them into mathematical equations. For example, having noted that the horizontal speed in projectile motion is constant, you need to write down $x = (v_0 \cos\theta)t$, or something equivalent. Or having decided that you will use $F = ma$ to solve a problem, you need to explicitly write down the F_x (and maybe F_y and F_z) equations, which may involve breaking vectors into their components. Or having decided to use conservation of energy, you need to determine what kinds of energy are involved, and then equate the initial total energy with the final total energy.

12. Think initially in terms of physical statements, rather than equations

It is important to *first* think about the physical principles, and *then* think about how you can express them with equations. Don't just write down a bunch of equations and look for ways to plug things into them. *The initial goal when attacking a problem isn't to write down the correct equation; rather, it's to say the correct thing in words.* If you proceed by blindly writing down all the equations you can think of that seem somewhat relevant, you might end up just going around in circles. You wouldn't try to get to a certain destination by randomly walking around with the hope that you'll eventually stumble upon it. And that strategy doesn't work any better in problem solving!

13. Make sure you have as many facts/equations as unknowns

If you are trying to solve for three unknowns and you have only two equations/facts, then there's no way you're going to be successful. Along the same lines, if you've identified an unknown but haven't incorporated it into any of your equations/facts, then there's no way you're going to be able to solve for it. If you can't think of which additional physical principle to apply to generate the necessary equation, it's helpful to run through all of the given information and think about the implications of each bit.

14. Be organized

Sometimes you can see right away exactly how to solve a problem, in which case you can fly right through it, without much need for organization. But unless you're positive that the solution will be quick, it is critical to be organized about the other strategies in this list, by explicitly writing things out. For example, you should write out the knowns and unknowns, as opposed to just thinking them. And likewise for the physical principles involved, etc. There's no need to write a book, but some brief notes will do wonders in organizing your thoughts.

1.2.3 Troubleshooting

In many cases the preceding strategies are sufficient for solving a problem. But if you get stuck, the following thirteen strategies should be helpful.

The following three strategies are bread-and-butter ones.

15. Reduce the problem to an intermediate one

Equivalently, work backwards. Say to yourself, "I'd be able to get the answer to the problem if I somehow knew the quantity A. And I'd be able to get A if I somehow knew B." And so on. Eventually you'll hit a quantity that you can figure out from the given information. For example, you can find the distance x traveled by a projectile if you somehow know the time t (because $x = (v_0 \cos\theta)t$). So the problem reduces to finding t. And you can find t by (among other ways) noting that at the top of the projectile motion (after time $t/2$), the y component of the velocity is zero, so $v_0 \sin\theta - g(t/2) = 0$.

As an analogy, if you can't remember how to get to a certain destination, you're still in luck if you remember that it's just north of a park, which you remember is a few blocks down a certain street from a statue, which you remember is around the corner from a school, which you remember how to get to.

16. Exaggerate/change the parameters to understand their influence

This is basically the same as checking limiting cases of your final answer (Strategy 28 below, discussed in detail in Section 1.1.3). However, there is no need to wait until you obtain your final answer (or even an intermediate result) to take advantage of this extremely useful strategy. Your intuition about extreme cases is much better than your intuition about normal scenarios, so you should use it. Once you see that a certain parameter influences the result, you can hone in on how exactly this influence comes about. This can then lead you to the relevant physical principle (Strategy 10).

17. Think about how the various quantities (known or unknown) are related

The task of Strategy 10 is to identify the relevant physical principles. This will yield relations among the various quantities. If you've missed some of the principles, it might be possible to figure out what they are by thinking about how the various quantities relate. For example, consider a mass on the end of a spring, and let's say you pull the mass a distance d away from its equilibrium position and then let go. It is intuitively clear that the larger d is, the larger the mass's speed v will be when it passes through the equilibrium position during the resulting oscillatory motion. If your goal is to find v, the preceding qualitative statement might help lead you to the useful physical principle of energy conservation, which will then allow you to write down a quantitative mathematical equation.

The following three strategies are quick checks.

18. Check that you have incorporated all of the given information

Part of the task of Strategy 2 is to identify everything that you know. When immersed in a problem, it's easy to forget some of this information, and this will likely make the problem unsolvable. So double check that for every given piece of information, you've either incorporated it or declared it to be irrelevant.

19. Check your math

Check over your algebra, of course. It's good to do at least a cursory check after each step. If you eventually hit a roadblock, go back and do a more careful check through all the steps.

20. Check the signs in all equations

In some sense this is just a subcase of the preceding strategy of checking your math. But often when people check through algebra, they fixate on the numerical values of the various terms and neglect the signs. So if you're stuck, just do a quick check where you ignore the numerical values and look only at the signs, just to make sure that at least those are correct. This check should be very quick. Pay special attention to the initial equation that you wrote down. A common mistake is to have an incorrect sign right from the start (for example, having the wrong sign in a vector component), which won't show up as an algebra mistake.

The following three strategies involve building on other knowledge.

21. Think of similar problems you know how to do

Try to reduce the problem (all, or part of it) to a previously solved problem. There are only so many types of problems in introductory mechanics, so odds are that if you've done a good number of problems, they should start looking familiar. How is the present problem similar to an old one, and how is it different?

You might wonder whether someone becomes an expert problem solver by being brilliant, or by solving a zillion problems, which has the effect of making any new problem look vaguely familiar. Elite athletes, chess players, debaters, comedians, etc., rely on recognizing familiar situations that they know how to react to. You can argue about what percentage of their strategy/action is based on this reaction. But you can't argue with the fact that a huge arsenal of familiar situations, built up from endless hours of practice, is a necessary condition for elite status in pretty much anything.

22. Think of a real-life setup that behaves the same way

This strategy in is the same spirit as the preceding one. We continually gather physical intuition from everyday life, so in a sense we're always practicing physics without even knowing it. For example, if you need to push a large object, say, a car, then you know that you should lean forward with your feet behind you, as opposed to pushing with your body upright. We have much more physical intuition about Newtonian mechanics than we do about other subfields of physics (electricity and magnetism, relativity, quantum mechanics), so you should use it to your advantage when solving mechanics problems!

Of course, there are times when your intuition might lead you astray, sometimes due to the fact that certain observations dominate others. For example, based on observations of skidding and general braking in a car or on a bike, you might think that friction always slows things down. It's easy to forget that when you accelerate from rest (often a more gentle acceleration than braking), friction is what speeds you up. You won't go anywhere if you're on ice!

23. Solve a simpler problem

If you can't get anywhere, if never hurts to solve a simpler version of the problem, to get a feel for what's going on. A particular example of this (which appears more often in math than in physics) is a problem that involves a large number or a general number N. In such problems, you definitely want to try solving things for the case of $N = 1, 2, 3$. Once you see what's going on for small numbers, it's much easier to generalize to an arbitrary number N.

The following four strategies involve helping your brain get going.

24. Explain (or imagine explaining) the solution to someone else

This strategy might seem a little silly, but it's really just a way of forcing yourself to organize your thoughts and proceed slowly. I assume I'm not the only person who has checked over an incorrect solution multiple times in my head, only to make the same mistake each time. It's very easy to repeatedly slide over a mistake or faulty assumption in the confines of one's own head. This can often be remedied by explaining your solution to someone else. And in most cases, you will see the mistake even before the other person says anything. But in the event that there's no one else around to lend an ear, a little talking to yourself never hurt.

25. Imagine your teacher explaining the solution

This strategy might also seem a little silly, but it does help sometimes, due to the fact that you're a human and not a machine. When floundering with a problem, it's easy to lose confidence and give up, even if you don't consciously know that you're giving up. If you imagine your teacher (or another student you look up to) explaining the solution, then since you expect them to be able to solve the problem with confidence, you just might end up solving it yourself. Occasionally it's better to ditch the "I think I can" mantra for "I *know* someone *else* can." Hey, if it gets the job done…

26. When you can't think of anything to do, just do something!

If all else fails, just start trying some random things. It can't hurt. You might hit something that gets you on track. A reasonable analogy is the unrealistic scenario in which you're lost in the woods with an infinite supply of food, and with zero chance that anyone else is going to help you. It doesn't do any good to just sit there. If you can't think of a reason to head in any particular direction, you should just head in *some* direction. Maybe you'll hit a stream that heads somewhere. Of course, after the fact, you'll probably see why you should have known the stream was there in the first place (a gulley between two hills, the dripping wet moose that walked past you, etc.). But that's knowledge you can use the next time you get stuck.

27. Set the problem aside and go for a walk

If nothing is working and you're truly at an impasse, take a break from the problem. It's easy to get stuck in a rut in your thinking, where you keep making the same mistakes and your mind keeps heading from a correct thought to an incorrect or unhelpful one. If you take a break and let your mind wander a little, it might semi-randomly jump to a helpful thought.

1.2.4 Finishing up

Assuming you've produced an answer to a problem, the following three strategies are ones you should always apply, as double checks on your answer.

28. Check limits and special cases

This strategy is immensely helpful. It is discussed in depth in Section 1.1.3, where its many benefits are explained.

29. Check units

This strategy is also very helpful and is discussed in depth in Section 1.1.2.

30. Check the rough size of numerical values

If your final answer involves actual numbers, be sure to check that they make sense. You can do this by checking that the rough "order of magnitude" (the nearest power of 10) is plausible. It's quite possible that you dropped a factor of 10 somewhere in the calculation. Or maybe you dropped one of the parameters when going from one line of the math to the next (although checking units is often a safeguard against this, too).

1.2.5 Looking ahead

After solving a problem, you can use the following four strategies to build upon what you've done and better prepare yourself for future problems.

31. Review the solution

After you're finished with a problem, go back and carefully review the entire solution. First think about the big-picture idea(s). Then think not only about what each step was, but also *why* you performed it. That is, how did each step logically follow from the previous one, as opposed to just being a random step? This review is extremely helpful in making the solution sink in, and it usually takes only a minute or two. You get a lot of bang for your buck, timewise.

Even if the solution *did* sink in first time around, there is another benefit to a careful review. It's often not enough just to know something; you also need to *know that you know it.* That way, you can confidently apply it to a future problem. If a certain tool is in your arsenal but you don't know that it's there, it doesn't help you much. (Think of all the memories that reside in your brain but that you'll never ever think of again, because you don't know that you have them.) A careful review of something will let you know that you know it.

32. Analyze where/why you went wrong

This strategy is actually relevant *while* you are solving the problem. As mentioned in the preface, it is critical that you never just read a solution straight through, unless you've already solved the problem. Just read enough to get a hint to get started. If you *do* need to read a little bit to get a hint, then before you use the hint to move on with the problem, think about your previous train of thought. Where exactly did you get stuck? What would you have needed to realize to get unstuck? How will you modify your thinking in the future so that you don't hit the same roadblock?

In short, it doesn't hurt to obsess a little about where you went wrong and what you can do better. If you get schooled on the basketball court with a crossover dribble that leads to the game-winning layup, you're probably going to obsess for a while about what to do differently next time. That's a good thing. And it works the same way with problem solving. And for that matter, anything else you're trying to get good at.

33. Think of another approach

Just because you solved a problem once, that doesn't mean you can't solve it again. There are often many ways to solve a given problem, and you can learn a lot by working through another solution. Furthermore, most of what you learn actually comes from the things you do wrong, so if you happened to have breezed through your first solution, you might not have learned much. But if you work through a second solution and have to struggle here and there, you'll significantly increase what you take away.

34. Think of a variation or extension

Try to make up a similar problem. You could vary the parameters, of course. But if you've solved the problem symbolically instead of with numbers (as you should always do!), then you've actually already solved the problem for any values of the parameters. So there's nothing new there. So what can you change? Perhaps add new forces and/or objects, or change the shape of an object, or allow something that was fixed to be moveable, or change the direction of the motion, or relax some of the given information and see what the most general motion is, or generalize from 1-D to 2-D, etc. Thinking about variations will not only solidify the problem you just solved, but also make you much more prepared for new problems that come your way.

As mentioned at the start of this section, the above list of strategies is long, so you certainly shouldn't try to memorize it. Just refer back to it now and then.

1.3 How to use this book

The preface contained some advice on the proper use of the solutions in this book. We'll repeat some of that advice here and also give a few more pointers for using this book.

- Read the introduction to each chapter, to become familiar with the material.
- Solve the foundational problems in a given chapter first, to make sure that you have all the tools you will need. These problems are the first few in each chapter.
- Solve lots of problems.
- Don't look at the solution to a problem too soon. If you do need to look at it, don't just read it straight through. Read it line by line until you get a hint to get going again. (Have a piece of paper handy, to cover up the rest of the solution.) Then set it aside and solve the problem on your own. Repeat as necessary.
- When solving the multiple-choice questions, be sure to *fully commit* to an answer before checking to see if it's correct. Don't just make a reasonable guess and then cross your fingers. Think hard until you're sure of your answer. If it turns out to be wrong, then solve the question again, without looking at the explanation in the solution.

The most important piece of advice is the fourth bullet point above, which also appeared in the boxed paragraph in the preface. But it's so important that one more appearance, now in bold, can't hurt:

Unless you have already completely solved a given problem,

Don't just read through the solution!

If you read through a solution without first solving the problem, you will gain essentially nothing from it.

1.4 Multiple-choice questions

As mentioned above: In the multiple-choice questions, be sure to fully commit to an answer before checking to see if it is correct.

1.1. If the task of a given problem is to find a certain length, which of the following quantities could be the answer? (The ℓ, v, a, t, and m in this and the following two questions are given quantities with the dimensions of length, velocity, acceleration, time, and mass.)

(a) at (b) mvt (c) $\sqrt{a\ell}$ (d) v/t (e) v^2/a

1.2. If the task of a given problem is to find a certain time, which of the following quantities could be the answer?

(a) a/t (b) mv/ℓ (c) v^2/a (d) $\sqrt{\ell/a}$ (e) $\sqrt{v/a}$

1.3. If the task of a given problem is to find a certain force (with units kg m/s^2), which of the following quantities could be the answer?

(a) mv^2 (b) mat (c) mv/t (d) mv/ℓ (e) v^2/ℓ

1.4. One mile per hour equals how many meters per second? (There are 1609 meters in a mile.)

(a) 0.04 (b) 0.45 (c) 1 (d) 2.2 (e) 27

In the following seven questions, don't solve things from scratch. Just use dimensional analysis.

1.5. A block rests on an inclined plane with coefficient of friction μ (which is dimensionless). Let θ_{max} be the largest angle of inclination for which the block doesn't slide down. Which of the following is true?

(a) θ_{max} is larger on the moon than on the earth.
(b) θ_{max} is larger on the earth than on the moon.
(c) θ_{max} is the same on the earth and the moon.

1.6. A mass m oscillates back and forth on a spring with spring constant k (with units kg/s^2). If the amplitude (the maximum displacement) is A, which of the following quantities is the maximum speed the mass achieves as it passes through the equilibrium point?

(a) $\dfrac{kA}{m}$ (b) $\dfrac{kA^2}{m}$ (c) $\sqrt{\dfrac{kA}{m}}$ (d) $\sqrt{\dfrac{kA^2}{m}}$ (e) $\sqrt{mkA^2}$

1.7. A bucket of water with mass density ρ (with units kg/m^3) has a small hole in it, at a depth h below the surface. Assuming that the viscosity of the water is negligible, which of the following quantities is the speed of the water as it exits the hole?

(a) $\sqrt{2gh}$ (b) $\sqrt{2\rho gh}$ (c) $\sqrt{2g/h}$ (d) $\sqrt{2h/g}$ (e) $\sqrt{2gh/\rho}$

1.8. The increase in pressure ΔP (force per area) as you descend in a lake depends on your depth h, the density of water ρ, and g. Which of the following quantities is ΔP?

(a) $\rho g/h$ (b) ρgh (c) $\rho^2 gh$ (d) $\rho g^2 h$ (e) ρgh^3

1.9. The drag force F_d on a sphere moving slowly through a viscous fluid depends on the viscosity of the fluid η (with units kg/(m s)), the radius R, and the speed v. Which of the following quantities is F_d?

(a) $6\pi\eta R/v$ (b) $6\pi\eta/Rv$ (c) $6\pi\eta Rv$ (d) $6\pi\eta R^2 v$ (e) $6\pi\eta R^2 v^2$

1.10. The drag force F_d on a sphere moving quickly through a nonviscous fluid depends on the density of the fluid ρ, the radius R, and the speed v. Which of the following quantities is F_d proportional to?

(a) ρv (b) $\rho R v$ (c) $\rho R v^2$ (d) $\rho R^2 v$ (e) $\rho R^2 v^2$

1.11. The Schwarzschild radius R_S of a black hole depends on its mass m, the speed of light c, and the gravitation constant G (with units $m^3/(kg\, s^2)$). Which of the following quantities is R_S?

(a) $\dfrac{2G}{mc^2}$ (b) $\dfrac{2Gm}{c^2}$ (c) $\dfrac{2Gm}{c^3}$ (d) $\dfrac{2c^2}{Gm}$ (e) $\dfrac{2c^3}{Gm}$

In the remaining questions, don't solve things from scratch. Just check special cases.

1.12. The plane in Fig. 1.2 is inclined at an angle θ, and two vectors are drawn. One vector is perpendicular to the plane, and its horizontal and vertical components are shown. The other vector is horizontal, and its components parallel and perpendicular to the plane are shown. Which of the following angles equal(s) θ? (Circle all that apply.)

(a) A (b) B (c) C (d) D

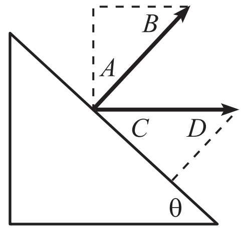

Figure 1.2

1.13. Two massless strings support a mass m as shown in Fig. 1.3. Which of the following quantities is the tension (that is, force) T in each string?

(a) $\dfrac{mg}{2}$ (b) $\dfrac{mg\sin\theta}{2}$ (c) $\dfrac{mg\cos\theta}{2}$ (d) $\dfrac{mg}{2\sin\theta}$ (e) $\dfrac{mg}{2\cos\theta}$

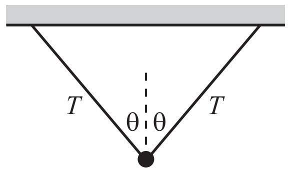

Figure 1.3

1.14. A block slides down a plane inclined at angle θ. What should the coefficient of kinetic friction μ be so that the block slides with constant velocity?

(a) 1 (b) $\sin\theta$ (c) $\cos\theta$ (d) $\tan\theta$ (e) $\cot\theta$

1.15. Consider the "endcap" of the sphere shown in Fig. 1.4, obtained by slicing the sphere with a vertical plane perpendicular to the plane of the paper. Which of the following expressions is the volume of the cap?

(a) $\pi R^3\big(4/3 - (2/3)\sin\theta\big)$

(b) $\pi R^3\big((2/3)\sin\theta\big)$

(c) $\pi R^3\big(2/3 - (2/3)\cos\theta + \sin\theta\big)$

(d) $\pi R^3\big(2/3 + (1/3)\cos^3\theta - \cos\theta\big)$

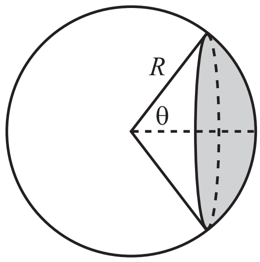

Figure 1.4

1.16. Consider the line described by $ax + by + c = 0$. Which of the following expressions is the distance from this line to the point (x_0, y_0)?

(a) $\dfrac{bx_0 + ay_0 + c}{\sqrt{a^2+b^2}}$ (b) $\dfrac{ax_0 + by_0 + c}{\sqrt{a^2+b^2}}$ (c) $\dfrac{ax_0 + by_0}{\sqrt{a^2+b^2}}$ (d) $\dfrac{ax_0 + by_0 + c}{\sqrt{a^2+b^2+c^2}}$

1.17. A person throws a ball with a given speed v (at the optimal angle for the following task) toward a wall of height h. Which of the following quantities is the maximum distance the person can stand from the wall and still be able to throw the ball over the wall?

(a) $\dfrac{gh^2}{v^2}$ (b) $\dfrac{v^2}{g}$ (c) $\dfrac{v^4}{g^2 h}$ (d) $\sqrt{\dfrac{v^2 h}{g}}$ (e) $\dfrac{v^2}{g}\sqrt{1-\dfrac{2gh}{v^2}}$ (f) $\dfrac{v^2/g}{1+2gh/v^2}$

1.5 Problems

1.1. **Furlongs per fortnight squared**

Convert $g = 9.8\,\text{m/s}^2$ into furlongs/fortnight^2. A fortnight is two weeks, a furlong is 220 yards, and there are 1.09 yards in a meter.

1.2. **Miles per gallon**

The efficiency of a car is commonly rated in miles per gallon, the dimensions of which are length per volume, or equivalently inverse area. What is the value of this area for a car that gets 30 miles per gallon? What exactly is the physical interpretation of this area? *Note*: There are 3785 milliliters (cubic centimeters) in a gallon, and 1609 meters in a mile.

1.3. **Painting a funnel**

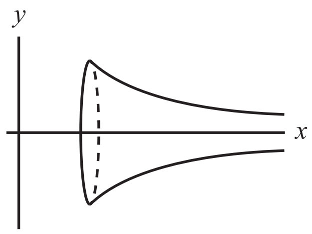

Figure 1.5

Consider the curve $y = 1/x$, from $x = 1$ to $x = \infty$. Rotate this curve around the x axis to create a funnel-like surface of revolution, as shown in Fig. 1.5. By slicing up the funnel into disks with radii $r = 1/x$ and thickness dx (and hence volume $(\pi r^2)\,dx$) stacked side by side, we see that the volume of the funnel is

$$V = \int_1^\infty \frac{\pi}{x^2}\,dx = -\frac{\pi}{x}\bigg|_1^\infty = \pi, \tag{1.6}$$

which is finite. The surface area, however, involves the circumferential area of the disks, which is $(2\pi r)\,dx$ multiplied by a $\sqrt{1+y'^2}$ factor accounting for the tilt of the area. The surface area of the funnel is therefore

$$A = \int_1^\infty \frac{2\pi\sqrt{1+y'^2}}{x}\,dx \;>\; \int_1^\infty \frac{2\pi}{x}\,dx, \tag{1.7}$$

which is infinite. (The square root factor is irrelevant for the present purposes.) Since the volume is finite but the area is infinite, it appears that you can fill up the funnel with paint but you can't paint it. However, we then have a problem, because filling up the funnel with paint implies that you can certainly paint the *inside* surface. But the inside surface is the same as the outside surface, because the funnel has no thickness. So we should be able to paint the outside surface too. What's going on here? Can you paint the funnel or not?

1.4. **Planck scales**

Three fundamental physical constants are Planck's constant, $\hbar = 1.05 \cdot 10^{-34}\,\text{kg m}^2/\text{s}$; the gravitational constant, $G = 6.67\cdot 10^{-11}\,\text{m}^3/(\text{kg s}^2)$; and the speed of light, $c = 3.0\cdot 10^8\,\text{m/s}$. These constants can be combined to yield quantities with dimensions of length, time, and mass (known as the *Planck length*, etc.). Find these three combinations and the associated numerical values.

1.5. **Capillary rise**

If the bottom end of a narrow tube is placed in a cup of water, the surface tension of the water causes the water to rise up in the tube. The height h of the column of water depends on the surface tension γ (with dimensions of force per length), the radius of the tube r, the mass density of the water ρ, and g. Is it possible to determine from dimensional analysis alone how h depends on these four quantities? Is it possible if we invoke the fact that h is proportional to γ? (This is believable; doubling the surface tension γ should double the height, because the surface tension is what is holding up the column of water.)

1.6. **Fluid flow**

Poiseuille's equation gives the flow rate Q (volume per time) of a fluid in a pipe, in the case where viscous drag is important. How does Q depend on the following four quantities: the pressure difference ΔP (force per area) between the ends of the pipe, the radius R and length L of the pipe, and the viscosity η (with units of kg/(m s)) of the fluid? To answer this, you will need to invoke the fact that Q is inversely proportional to L. (This is believable; doubling the length of the pipe will double the effect of friction between the fluid and the walls, and thereby halve the flow rate.)

1.7. **1-D collision**

If a mass M moving with velocity V collides head-on elastically with a mass m that is initially at rest, it can be shown (see Problem 6.3) that the final velocities are given by

$$V_M = \frac{(M-m)V}{M+m} \qquad \text{and} \qquad v_m = \frac{2MV}{M+m}. \tag{1.8}$$

Check the $M = m$, $M \ll m$, and $M \gg m$ limits of these expressions.

1.8. **Atwood's machine**

Consider the Atwood's machine in Fig. 1.6, consisting of three masses and two frictionless pulleys. It can be shown that the acceleration of m_2, with upward taken to be positive, is given by (just accept this)

$$a_2 = -g\frac{4m_2m_3 + m_1(m_2 - 3m_3)}{4m_2m_3 + m_1(m_2 + m_3)}. \tag{1.9}$$

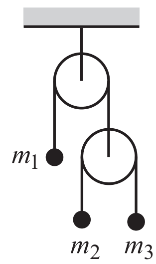

Figure 1.6

Find a_2 for the following special cases:

(a) $m_1 = 2m_2 = 2m_3$

(b) m_2 much larger than both m_1 and m_3

(c) m_2 much smaller than both m_1 and m_3

(d) $m_1 \gg m_2 = m_3$

(e) $m_1 = m_2 = m_3$

1.9. **Dropped ball**

In Section 1.1.4, we looked at limiting cases of the velocity, given in Eq. (1.3), of a beach ball dropped from rest. Let's now look at the height of the ball. If the ball is dropped from rest at height h, and if the drag force from the air takes the form $F_d = -bv$, then it can be shown that the ball's height as a function of time equals

$$y(t) = h - \frac{mg}{b}\left(t - \frac{m}{b}\left(1 - e^{-bt/m}\right)\right). \tag{1.10}$$

Find an approximate expression for $y(t)$ in the limit where t is very small (or more precisely, in the limit where $bt/m \ll 1$).

1.6 Multiple-choice answers

1.1. [e] This is the only choice with dimensions of length.

1.2. [d] This is the only choice with dimensions of time.

1.3. [c] This is the only choice with units of kg m/s^2.

1.4. [b] There are $60 \cdot 60 = 3600$ seconds in an hour, so one mile per hour equals

$$1\,\frac{\text{mile}}{\text{hour}} = \frac{1609\text{ meters}}{3600\text{ seconds}} = 0.447\text{ m/s}. \tag{1.11}$$

A common automobile speed of, say, 60 mph is therefore about 27 m/s. The inverse relation, going from m/s to mph, is 1 m/s = 2.24 mph.

REMARK: We see that a meter per second is larger than a mile per hour, by a factor of slightly more than 2. This factor of about 2.2 is easy to remember, because it happens to be essentially the same as the conversion factor between kilograms and pounds: 1 kg weighs 2.2 pounds. (Note that since a kilogram is a unit of mass, and a pound is a unit of weight, we used the word "weighs" here, instead of "equals.")

1.5. c The only things that θ_{max} can possibly depend on are the mass m of the block, the relevant acceleration due to gravity g, and the coefficient of friction μ. But θ_{max} is dimensionless, and there is no way to form a dimensionless quantity involving g or m. So θ_{max} can depend only on μ (which is dimensionless). Therefore, since θ_{max} can't depend on g, it is the same on the earth and the moon.

1.6. d This is the only choice with units of m/s. The answer must involve the ratio k/m, to get rid of the kg units. And it must involve $\sqrt{k}$ in the numerator to produce the desired single power of seconds in the denominator. Choice (d) additionally produces the desired single power of meters in the numerator. If you want to solve the problem for real, you can quickly use conservation of energy (discussed in Chapter 5) to say that $mv^2/2 = kA^2/2$.

1.7. a This is the only choice with units of m/s. Note that the answer can't involve ρ, because there would then be no way to get rid of the units of kg.

REMARKS: This speed of $\sqrt{2gh}$ is the same as the speed of a dropped ball after it has fallen a height h, assuming that air drag can be neglected. (This speed can't depend on the mass m of the ball for the same dimensional reason.) If air drag *is* included, then this introduces a new parameter which involves units of kg, so now the speed of the ball can (and does) depend on m. A metal ball falls faster than a styrofoam ball.

There is one issue we've glossed over. The size of the hole introduces another length scale ℓ (which you can take to be the radius or diameter or whatever). This means that there are now an infinite number of possible answers. Any expression of the form $\sqrt{2gh^{n+1}/\ell^n}$ has the correct units. However, assuming that the viscosity is negligible, it can be shown that the size of the hole doesn't matter. That is, $n = 0$. At any rate, choice (a) is certainly the only correct choice among the given options.

1.8. b Let's be a little more systematic here than in the previous few questions. Since the units of force are kg m/s^2, the units of pressure (force per area) are kg/(m s^2). So the units of the various quantities are:

$$\Delta P : \frac{\text{kg}}{\text{m s}^2}, \qquad h : \text{m}, \qquad \rho : \frac{\text{kg}}{\text{m}^3}, \qquad g : \frac{\text{m}}{\text{s}^2}. \tag{1.12}$$

Our goal is to create ΔP from the other three quantities. By looking at the powers of kg and s in ΔP, we quickly see that the answer must be proportional to ρg. This then means that we need one power of h, to produce the correct power of m.

1.9. c The units of the various quantities are:

$$F_{\text{d}} : \frac{\text{kg m}}{\text{s}^2}, \qquad \eta : \frac{\text{kg}}{\text{m s}}, \qquad R : \text{m}, \qquad v : \frac{\text{m}}{\text{s}}. \tag{1.13}$$

Our goal is to create F_{d} from the other three quantities, and we quickly see that the simple product $\eta R v$ gets the job done. A detailed calculation is required to generate the numerical coefficient of 6π.

1.10. e The units of the various quantities are:

$$F_{\text{d}} : \frac{\text{kg m}}{\text{s}^2}, \qquad \rho : \frac{\text{kg}}{\text{m}^3}, \qquad R : \text{m}, \qquad v : \frac{\text{m}}{\text{s}}. \tag{1.14}$$

Our goal is to create F_{d} from the other three quantities. By looking at the powers of kg and s in F_{d}, we see that we need one power of ρ and two powers of v. This then implies that we need two powers of R, to produce the correct power of m. The actual numerical coefficient in F_{d} depends on the specifics of the surface of the sphere (how rough it is).

1.11. b The units of the various quantities are:

$$R_S : \text{m}, \qquad m : \text{kg}, \qquad c : \frac{\text{m}}{\text{s}}, \qquad G : \frac{\text{m}^3}{\text{kg}\,\text{s}^2}. \tag{1.15}$$

Our goal is to create the Schwarzschild radius R_S from the other three quantities. Since R_S doesn't involve units of kg, we see that the answer must involve the product Gm (raised to some power). And since R_S also doesn't involve units of seconds, the answer must involve the quotient G/c^2 (raised to some power). We quickly see that Gm/c^2 makes the units of meters work out correctly. The factor of 2 requires a detailed calculation.

REMARK: The Schwarzschild radius of an object is the radius with the property that if you shrink the object down to that radius (keeping the mass the same), it will be a black hole. That is, not even light can escape from it. The R_S for the sun is about 3 km, and the R_S for the earth is about 1 cm. These objects are (of course) not black holes, because R_S is smaller than the radius of the object. However, note that R_S is proportional to m, which in turn is proportional to the cube of the radius r of the object, for a given density ρ. This means that for any given ρ, if r is increased, m grows faster than r. So eventually R_S will become as large as r, and the object will be a black hole. Since $m = (4\pi r^3/3)\rho$, you can quickly show that the critical radius at which this occurs is $r_{\text{crit}} = \sqrt{3c^2/8\pi G\rho}$. If $\rho = 1000\,\text{kg/m}^3$ (the density of water), then $r_{\text{crit}} \approx 4 \cdot 10^{11}$ m, which is about three times the radius of the earth's orbit.

1.12. A,C You can figure out the angles by doing some geometry, but it's much easier to just check the special case where θ is very small or very close to 90°. Fig. 1.7 shows the case where θ is small. In setups like this, every angle is either θ or $90° - \theta$ (or 90°). So all of the small angles in the figure must be θ, as shown.

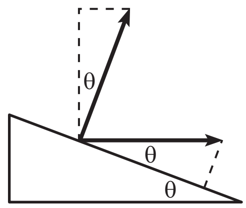

Figure 1.7

REMARK: Even if you want to work out the geometry to determine the angles, it would be silly to pass up the quick and easy double check of small or large θ. Of course, once you get used to doing this quick check, you'll realize that there's not much need to work through the geometry in the first place. A corollary of this is that you should never draw anything close to a 45-45-90 triangle (as we purposely did in the statement of this question), because in that case you can't tell which are the big angles and which are the small angles!

1.13. e Intuitively, the tension must go to infinity when $\theta \to 90°$, that is, when the strings approach being horizontal. Imagine pulling the top ends of the strings outward, to try to make the strings horizontal. This will take a large (infinite) force, because the mass will always sag a little in the middle. If the strings were exactly horizontal, then they would have no vertical force component to balance the mg weight. Choice (e) is the only one that goes to infinity in the $\theta \to 90°$ limit.

Some additional reasoning: (a) is incorrect because T should depend on θ, (b) is incorrect because T shouldn't be zero when $\theta = 0$, (c) is incorrect because T shouldn't be zero when $\theta = 90°$, and (d) is incorrect because T shouldn't be infinite when $\theta = 0$.

1.14. d The coefficient of friction must be very small in the $\theta \to 0$ limit, otherwise the block won't move. And it must be very large in the $\theta \to \pi/2$ limit, otherwise the block will keep accelerating. The $\mu = \tan\theta$ choice is the only one that satisfies these conditions.

1.15. d The volume must be zero when $\theta = 0$; this rules out (a). And it must be $(2/3)\pi R^3$ when $\theta = \pi/2$ (half of the whole sphere); this rules out (c). And it must be $(4/3)\pi R^3$ when $\theta = \pi$ (the whole sphere); this rules out (b). So the answer must be (d).

1.16. b For a horizontal line (with $a = 0$), the distance can't depend on x_0; this rules out (a). The answer must depend on c, because c affects the position (the height) of the line; this rules out (c). Furthermore, in the $c \to \infty$ limit, the distance should go to infinity. But choice (d) approaches 1. So the answer must be (b).

Another bit of reasoning: if the point (x_0, y_0) lies on the line, that is, if $ax_0 + by_0 + c = 0$, then the distance is zero. This implies that the correct answer must be (b) or (d).

Alternatively, you can eliminate all of the wrong answers by explicitly using the fact that in the case of a horizontal line (with $a = 0$, so the line is given by $y = -c/b$), the distance is $y_0 - (-c/b)$.

1.17. $\boxed{\text{e}}$ All of the possible answers have the correct units, so we'll have to figure things out by looking at special cases. Let's look at each choice in turn:

Choice (a) is incorrect, because the answer shouldn't be zero for $h = 0$. Also, it shouldn't grow with g. And furthermore it shouldn't be infinite for $v \to 0$.

Choice (b) is incorrect, because the answer should depend on h.

Choice (c) is incorrect, because the answer shouldn't be infinite for $h = 0$.

Choice (d) is incorrect, because the answer shouldn't be zero for $h = 0$.

Choice (e) can't be ruled out, and it happens to be the correct answer.

Choice (f) is incorrect, because there should be no possible distance when $h \to \infty$ (you can't throw the ball over an infinitely high wall). But this answer gives zero in this limit. This reasoning actually gets rid of all the answers except the correct one in one fell swoop. There are certainly cases for which there is no distance from which it is possible to throw the ball over the wall (for example, if h or g is very large, or v is very small). So any expression that gives a real result for all values of the parameters cannot be correct. Choice (e) is the only one that correctly gives an imaginary (and hence nonphysical) result in certain cases.

1.7 Problem solutions

1.1. **Furlongs per fortnight squared**

The systematic way of doing the conversion is to trade certain units for certain other units, by multiplying by 1 in the appropriate form. We want to trade meters for furlongs, and seconds for fortnights. This can be done as follows:

$$\begin{aligned} g &= 9.8\,\frac{\text{m}}{\text{s}^2}\left(\frac{1.09\text{ yard}}{1\text{ m}}\cdot\frac{1\text{ furlong}}{220\text{ yard}}\right)\left(\frac{60\text{ s}}{1\text{ min}}\cdot\frac{60\text{ min}}{1\text{ hr}}\cdot\frac{24\text{ hr}}{1\text{ day}}\cdot\frac{14\text{ day}}{1\text{ fortnight}}\right)^2 \\ &= 7.1\cdot 10^{10}\,\frac{\text{furlongs}}{\text{fortnight}^2}\,. \end{aligned} \tag{1.16}$$

All we've done here is multiply by 1 six times, so we haven't changed the value. The right-hand side still equals g; it's just that it's now expressed in different units. You can see that the m, yard, s, min, hr, and day units all cancel (don't forget that the second set of fractions is squared), so we're left with only furlongs and fortnights (squared), as desired.

REMARKS: This numerical result of $7.1 \cdot 10^{10}$ is very large. The reason for this is the following. The $g = 9.8\,\text{m/s}^2$ expression tells us that after one second, a falling body will be traveling at a speed of 9.8 m/s. So the question we want to ask is "If a body has been falling for one fortnight, what will its speed be, as expressed in furlongs per fortnight?" (We'll ignore the fact that a body certainly can't freefall for two weeks on the earth!) The numerical answer to this question is large for *two* reasons, consistent with the fact that the fortnight is *squared* in the expression for g. First, a fortnight is a long time, so the body will be moving very fast after falling for all this time. Second, because the body is moving so fast, it will (if it were to continue to travel with that speed) travel a very large distance after another lengthy time of one fortnight. And this distance is the numerical value of the speed when expressed in furlongs per fortnight. A competing effect is that since a furlong is larger than a meter by a factor of 220, the numerical result in Eq. (1.16) is decreased by this factor. But this effect is washed out by the squared larger affect of the lengthy fortnight.

There are many conversions that you can just do in your head. If someone asks you to convert 1 minute into seconds, you know that the answer is simply 60 seconds. There is no need to multiply 1 min by $(60\,\text{s})/(1\,\text{min})$ to cancel the minutes and be left with only seconds. But for more complicated conversions, you can systematically multiply by 1 in the appropriate form.

Note the word “appropriate” in the previous sentence. You can’t just blindly multiply by 1; you need to think about which units you’re trying to cancel out. If you unwisely multiply 1 min by (1 min)/(60 s), which still equals 1, then you will end up with $(1\,\text{min})^2/(60\,\text{s})$. This does indeed equal 1 minute, but it isn’t very informative. No one will know what you’re talking about if you say to take a $(5\,\text{min}^2)/(60\,\text{s})$ break!

1.2. **Miles per gallon**

30 miles per gallon equals

$$\frac{30\text{ miles}}{1\text{ gallon}} = \frac{30\cdot 1609\text{ m}}{3785\text{ cm}^3} = \frac{30\cdot 1609\text{ m}}{3785\,(10^{-2}\text{ m})^3} = \frac{1}{7.84\cdot 10^{-8}\text{ m}^2}\,. \tag{1.17}$$

This tiny area of $7.84\cdot 10^{-8}\text{ m}^2$ corresponds to a square with side length $2.8\cdot 10^{-4}$ m, or about 0.3 millimeters. For comparison, a common diameter for the pencil lead in a mechanical pencil is 0.5 or 0.7 millimeters.

What does this area actually have to do with a car that gets 30 miles per gallon? Note that Eq. (1.17) can alternatively be written as

$$(7.84\cdot 10^{-8}\text{ m}^2)(30\text{ miles}) = 1\text{ gallon}. \tag{1.18}$$

What this says is that if we have a narrow tube of gasoline running parallel to the road, with a cross-sectional area of $7.84\cdot 10^{-8}\text{ m}^2$, and if our car gobbles up the gasoline in the tube as it travels along, then it will gobble up one gallon every 30 miles, which is exactly what it needs to operate. This interpretation of the area gives you an intuitive sense of how much gasoline you’re using; think of a long pencil lead running parallel to the road. If you instead want to think in terms of a small unit of volume, you can work with drops. In medicine, the unit of one “drop” equals 1/20 of a milliliter. You can show that you burn about one drop of gasoline for every two feet you travel (assuming 30 miles per gallon).

1.3. **Painting a funnel**

It is true that the volume of the funnel is finite, and that you can fill it up with paint. It is also true that the surface area is infinite, but you actually *can* paint it.

The apparent paradox arises from essentially comparing apples and oranges. In our case we are comparing *volumes* (which are three dimensional) with *areas* (which are two dimensional). When someone says that the funnel can’t be painted, he is saying that it would take an infinite *volume* of paint to cover it. But the fact that the surface *area* is infinite does *not* imply that it takes an infinite *volume* of paint to cover it. To be sure, if we try to paint the funnel with a given fixed thickness of paint, then we would indeed need an infinite volume of paint. But in this case, if we look at very large values of x where the funnel has negligible thickness, we would essentially have a tube of paint with a fixed radius, extending to $x = \infty$, with the funnel taking up a negligible volume at the center of the tube. This tube certainly has an infinite volume.

But what if we paint the funnel with a decreasing thickness of paint, as x gets larger? For example, if we make the thickness be proportional to $1/x$, then the volume of paint is proportional to $\int_1^\infty (1/x)(1/x)\,dx$, which is finite. (The first $1/x$ factor here comes from the $2\pi r$ factor in the area, and the second $1/x$ factor comes from the thickness of the paint. We have ignored the $\sqrt{1+y'^2}$ factor, which goes to 1 for large x.) In this manner, we can indeed paint the funnel. To sum up, you buy paint by the gallon, not by the square meter. And a gallon of paint can cover an infinite area, as long as you make the thickness go to zero fast enough. The moral of this problem, therefore, is to not mix up things with different units!

1.4. **Planck scales**

First solution: Since only G and $\hbar$ involve units of kilograms (one in the numerator and one in the denominator), it’s fairly easy to see what the three desired combinations are. For example, the Planck length and time must involve the product $\hbar G$, because this eliminates the kilograms. You should check that the expressions below in Eq. (1.20) all have the correct units (the subscript P is for Planck). Using the numerical values,

$$\begin{aligned} \hbar &= 1.05 \cdot 10^{-34}\, \frac{\text{kg m}^2}{\text{s}}\,, \\ G &= 6.67 \cdot 10^{-11}\, \frac{\text{m}^3}{\text{kg s}^2}\,, \\ c &= 3.0 \cdot 10^{8}\, \frac{\text{m}}{\text{s}}\,, \end{aligned} \tag{1.19}$$

we obtain the three Planck scales:

$$\begin{aligned} \text{Length}: &\qquad \ell_{\text{P}} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \cdot 10^{-35}\,\text{m}, \\ \text{Time}: &\qquad t_{\text{P}} = \sqrt{\frac{\hbar G}{c^5}} = 5.4 \cdot 10^{-44}\,\text{s}, \\ \text{Mass}: &\qquad m_{\text{P}} = \sqrt{\frac{\hbar c}{G}} = 2.2 \cdot 10^{-8}\,\text{kg}. \end{aligned} \tag{1.20}$$

Another such quantity is the Planck energy:

$$\text{Energy}: \qquad E_{\text{P}} = \sqrt{\frac{\hbar c^5}{G}} = 2.0 \cdot 10^{9}\,\text{J}. \tag{1.21}$$

Second solution: If you want to be more systematic, you can write down a general expression of the form $\hbar^\alpha G^\beta c^\gamma$ and then solve for the values of the three exponents that make the overall units be correct. For example, if we want to find the Planck length, then we need the overall units to be meters, so we want

$$\left(\frac{\text{kg m}^2}{\text{s}}\right)^\alpha \left(\frac{\text{m}^3}{\text{kg s}^2}\right)^\beta \left(\frac{\text{m}}{\text{s}}\right)^\gamma = \text{m}. \tag{1.22}$$

Equating separately the powers of kg, m, and s on the two sides of the equation gives a system of three equations in three unknowns:

$$\begin{aligned} \text{kg}: &\qquad \alpha - \beta = 0, \\ \text{m}: &\qquad 2\alpha + 3\beta + \gamma = 1, \\ \text{s}: &\qquad -\alpha - 2\beta - \gamma = 0. \end{aligned} \tag{1.23}$$

The first equation gives $\alpha = \beta$. The third equation then gives $\gamma = -3\alpha$. And the second equation then gives $\alpha = 1/2$. So $\beta = 1/2$ and $\gamma = -3/2$. This agrees with the result for the Planck length in Eq. (1.20). The Plank time and mass proceed similarly. However, even though this method will always get the job done, it was certainly quicker in this problem to just fiddle around with the units by noting, for example, that the Planck length and time must involve the product $\hbar G$.

Remark: The physical significance of all these scales is that they are the scales at which the quantum effects of gravity can no longer be ignored. Said in another way, they are the scales at which the four known forces in nature (gravitational, electromagnetic, weak, strong) all become roughly equal. This equality should be contrasted with the fact that at, say, atomic length scales, the electrical force between two protons is vastly larger (about 10^{36} times larger) than the gravitational force.

To get a sense of how small the planck length is, the size of an atomic nucleus is on the order of a femtometer (10^{-15} m). The Planck length is therefore 10^{20} times smaller. So the Planck length is to the nuclear size as the nuclear size is to 100 km. And to get a sense of how large the Planck energy E_P is, the energies presently probed at the CERN collider are only on the order of $10^{-15}E_{\text{P}}$.

1.5. **Capillary rise**

Since the units of force are kg m/s^2, the units of surface tension (force per length) are kg/s^2. So the units of the various quantities are:

$$h: \text{m}, \qquad \gamma: \frac{\text{kg}}{\text{s}^2}, \qquad r: \text{m}, \qquad \rho: \frac{\text{kg}}{\text{m}^3}, \qquad g: \frac{\text{m}}{\text{s}^2}. \tag{1.24}$$

Our goal is to create h from the other four quantities. We need to get rid of the units of kg and s, and we quickly see that the combination $\gamma/\rho g$ accomplishes this. It leaves us with m^2 in the numerator. But now we have a problem, because although simply dividing by r will give us the desired units of meters, there are many other possibilities that work too. In fact, any expression of the form $(\gamma/\rho g)^n/r^{2n-1}$ has units of meters. The issue here is that we have four unknowns (the powers of each of γ, r, ρ, and g), but only three equations (the facts that the overall powers of kg, m, and s must be 0, 1, and 0, respectively). So the system is under-determined; we can't uniquely solve for the four unknowns.

However, if we use the additional fourth piece of information that the power of γ is 1, then this tells us that the value of n in the above general expression is 1. So we can now say that

$$h \propto \frac{\gamma}{\rho g r}. \tag{1.25}$$

The actual result has a numerical factor of 2 in the numerator, but it takes a little more effort to show that. Additionally, h depends on the contact angle θ between the water and the tube, at the top of the meniscus; this brings in a factor of $\cos\theta$ (so the correct result is $h = 2\gamma\cos\theta/\rho g r$). But θ is a dimensionless quantity, so we can't say anything about it by using dimensional analysis.

1.6. **Fluid flow**

Since the units of force are kg m/s^2, the units of pressure (force per area) are $\text{kg/(m s}^2)$. So the units of the various quantities are:

$$Q: \frac{\text{m}^3}{\text{s}}, \qquad \Delta P: \frac{\text{kg}}{\text{m s}^2}, \qquad R: \text{m}, \qquad L: \text{m}, \qquad \eta: \frac{\text{kg}}{\text{m s}}. \tag{1.26}$$

Our goal is to create Q from the other four quantities, with the condition that there is one power of L in the denominator. As in Problem 1.5, we can't solve for four unknowns (the powers of ΔP, R, L, and η) with only three pieces of information (the required powers of kg, m, and s). This fact about L is the necessary fourth piece of information.

To produce the units of Q, we need to get rid of the kg and have one power of s in the denominator. We quickly see that the quotient $\Delta P/\eta$ accomplishes this. This expression has no m's, but we need an m^3 in the numerator of Q. Since we are told that there is one power of L in the denominator, we must have four powers of R in the numerator. So the desired expression is

$$Q \propto \frac{\Delta P R^4}{\eta L}. \tag{1.27}$$

The actual result has a numerical factor of $\pi/8$ out front, but it requires a detailed calculation to show that.

Remark: The R^4 dependence in Q is stronger than the R^2 dependence you might expect by simply considering the fact that the cross-sectional area of the pipe is proportional to R^2. What happens is that the wider the pipe, the faster the average speed of the fluid (for a given ΔP). So we have a faster fluid flowing through a wider pipe. The fourth power of R isn't obvious, though.

This fourth power implies that if the pipe's radius is decreased to, say, 0.8 of what it was, then the flow rate (with ΔP held constant) is decreased to $(0.8)^4 \approx 0.41$ of what it was. In other words, a 20% reduction in radius produces a nearly 60% reduction in flow rate, which is more than you might naively expect. This fact is highly relevant, for example, when dealing with plaque buildup in arteries.

1.7. **1-D collision**

The given expressions are

$$V_M = \frac{(M-m)V}{M+m} \qquad \text{and} \qquad v_m = \frac{2MV}{M+m}\,. \tag{1.28}$$

If $M = m$ (for example, two identical billiard balls) then we have

$$V_M = 0 \qquad \text{and} \qquad v_m = V. \tag{1.29}$$

The mass that was initially moving ends up at rest, and the the mass that was initially at rest ends up moving with whatever velocity the other mass had. This result is familiar to pool players; barring any effects of spin, the cue ball ends up at rest in a head-on collision.

If $M \ll m$ (for example, a marble bouncing off a bowling ball) then we can ignore the M's in the expressions in Eq. (1.28). (More precisely, we can ignore the additive M's; see the discussion following Eq. (1.1).) This yields

$$V_M \approx \frac{0-m}{0+m}V = -V \qquad \text{and} \qquad v_m \approx \frac{2M}{0+m}V \approx 0. \tag{1.30}$$

In this case, M is basically a ball bouncing backward off a unmoveable brick wall.

If $M \gg m$ (for example, a bowling ball colliding with a marble) then we can ignore the m's in the expressions in Eq. (1.28). This yields

$$V_M \approx \frac{M-0}{M+0}V = V \qquad \text{and} \qquad v_m \approx \frac{2M}{M+0}V = 2V. \tag{1.31}$$

In this case the bowling ball M plows forward with the same velocity V, as expected. But interestingly the marble m picks up *twice* this velocity.

Remark: This result of $2V$ for the speed of m in the $M \gg m$ limit isn't so obvious in the given lab frame, but it's fairly easy to understand if you imagine riding along with M. In the reference frame where M is at rest, m comes flying in with speed V and then bounces off with essentially the same speed V (because M is a brick wall in the $M \gg m$ limit). So the final relative speed of M and m is V. But we must now shift back to the lab frame and remember that M is still plowing forward with speed V (its speed hardly changes during the collision). This means that m is moving forward with speed V *relative* to M, which itself is moving forward with speed V relative to the lab frame. So m is moving with speed $V + V = 2V$ relative to the lab frame, as desired. In terms of energy (the subject of Chapter 5), the transfer of energy from the large object to the small object occurs via the energy stored in the elasticity of the balls as they deform during the collision.

The fact that you can use a large object moving with speed V to make a small object move faster than V is the basic principle behind a whip. Initially the thick heavy part of the whip is moving with a given speed, and eventually the thin light part at the end moves with a much larger speed. A whip is a continuous object, whereas the above problem involved two discrete balls, but see Problem 6.14 for a discussion of how to transition from the discrete case to the continuous case.

Whip-like motions are ubiquitous in sports. Examples include throwing a baseball, throwing a frisbee™, shooting a hockey puck, and kicking a football. In all cases, you start by moving a large object, and you end up with a small object that is moving much faster. There's a reason for the "Put your body into it" mantra. A baseball pitch starts with a relatively slow motion of the body/torso/shoulder and ends with a much faster motion of the forearm/hand/fingers. The transfer of energy occurs via the energy stored in the elasticity of tendons and ligaments. Incidentally, the fastest you can make any part of your body move (relative to your center of mass) is your fingers when throwing something.

1.8. **Atwood's machine**

The given expression for a_2 is

$$a_2 = -g\frac{4m_2m_3 + m_1(m_2 - 3m_3)}{4m_2m_3 + m_1(m_2 + m_3)}\,. \tag{1.32}$$

(a) If $m_1 = 2m_2 = 2m_3 \equiv 2m$, then a_2 becomes

$$a_2 = -g\frac{4m^2 + (2m)(m - 3m)}{4m^2 + (2m)(m + m)} = 0. \tag{1.33}$$

In this case, m_2 and m_3 balance m_1.

(b) If m_2 is very large, then we can ignore the $m_1 m_3$ terms in Eq. (1.32). This gives

$$a_2 \approx -g\frac{4m_2m_3 + m_1(m_2 - 0)}{4m_2m_3 + m_1(m_2 - 0)} = -g. \tag{1.34}$$

In this case, m_2 is simply in freefall.

(c) If m_2 is very small, then we can ignore the m_2 terms in Eq. (1.32). This gives

$$a_2 \approx -g\frac{0 + m_1(0 - 3m_3)}{0 + m_1(0 + m_3)} = 3g. \tag{1.35}$$

In this case, m_2 accelerates upward at $3g$. To understand this factor of 3, you can convince yourself that if m_1 and m_3 both freefall a distance ℓ, then m_2 must rise up a distance 3ℓ. This then implies that the acceleration of m_2 is 3 times the freefall acceleration g of m_1 and m_3. You may want to wait to do this until we discuss Atwood's machines (in particular, the topic of conservation of string) in Chapter 4.

(d) If $m_1 \gg m_2 = m_3 \equiv m$, then we can ignore the $m_2 m_3$ terms in Eq. (1.32). This gives

$$a_2 \approx -g\frac{0 + m_1(m - 3m)}{0 + m_1(m + m)} = g. \tag{1.36}$$

In this case, m_1 is in freefall, so m_2 and m_3 accelerate upward at g.

(e) If $m_1 = m_2 = m_3 \equiv m$, then Eq. (1.32) gives

$$a_2 = -g\frac{4m^2 + m(m - 3m)}{4m^2 + m(m + m)} = -\frac{g}{3}\,. \tag{1.37}$$

In this case, a_2 is correctly negative, but the factor of 1/3 isn't obvious; we would have to solve the problem for real to derive that.

1.9. **Dropped ball**

As with the beach ball's velocity $v(t)$ given in Eq. (1.3), the position $y(t)$ in Eq. (1.10) is a somewhat complicated expression, so it's hard to feel too confident about its validity by just looking at it. But if we can verify that it gives the correct answer in a particular limit, then we'll feel much better about it.

In the limit of small bt/m, we can use the Taylor series $e^{-x} \approx 1 - x + x^2/2$ to produce an approximate expression for $y(t)$, to leading order in t. We obtain (as you can verify, being careful with all the minus signs!)

$$\begin{aligned} y(t) &= h - \frac{mg}{b}\left[t - \frac{m}{b}\left(1 - \left(1 - \frac{bt}{m} + \frac{1}{2}\left(\frac{bt}{m}\right)^2 - \cdots\right)\right)\right] \\ &\approx h - \frac{gt^2}{2}\,. \end{aligned} \tag{1.38}$$

This answer is expected, because we essentially have a freely falling body at the start (v is small, so there is hardly any drag force), which implies that the distance fallen is the standard $gt^2/2$. (This is covered in Chapter 2, so just take it on faith for now. But the $gt^2/2$ term probably looks familiar to you anyway.) Note that in obtaining the leading-order term (the smallest power of t with a nonzero coefficient) in $y(t)$, we needed to go to second order in the Taylor series for e^{-x}, whereas we needed to go only to first order in obtaining the expression for $v(t)$ in Eq. (1.4).

Remark: We can also look at the limit of large bt/m. In this case, $e^{-bt/m}$ is essentially zero, so the $y(t)$ in Eq. (1.10) becomes (there's no need for a Taylor series in this case)

$$y(t) \approx h - \frac{mgt}{b} + \frac{m^2 g}{b^2}\,. \tag{1.39}$$

Apparently, after a long time, $m^2 g/b^2$ is the distance that our ball lags behind another ball that started out already at the terminal velocity $-mg/b$, because that ball has $y(t) = h - (mg/b)t$. (The terminal velocity is in fact $-mg/b$; see the discussion in Section 1.1.4.) You can verify that the quantity $m^2 g/b^2$ does indeed have dimensions of length, using the fact that the original expression for the drag force, $F_d = -bv$, tells us that b has units of N/(m/s), or equivalently kg/s. This $m^2 g/b^2$ result is by no means obvious, so our check of the large bt/m limit doesn't do anything to make us feel better about the original expression in Eq. (1.10). But that's fine; when checking limiting cases, the result is either that we feel better about our answer, or we get an interesting result that we didn't expect.