How Homophily Affects Learning and Diffusion in Networks

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- “For it often happens that some of us elders of about the same age come together and verify the old saw of like to like." – Cephalus in Plato’s *Republic*, *c*. 380 BC

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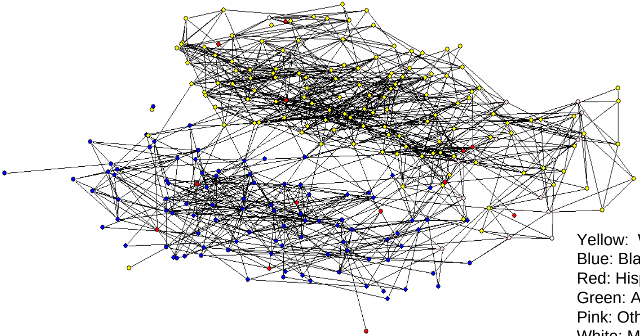
  - About 20% name someone of the opposite sex as their closest friend (Verbrugge 1977).
  - In middle school, less than 10% of “expected” cross-race friendships exist (Shrum et. al. 1988).

Friendships in a High School



Yellow: Whites
Blue: Blacks
Red: Hispanics
Green: Asians
Pink: Other
White: Missing

Currarini, Jackson, and Pin (2009)

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- In this project, we focus on communication and build models of:
  - networks with homophily;
  - diffusion or learning processes happening in them.
- Study how homophily affects the speed of the processes.

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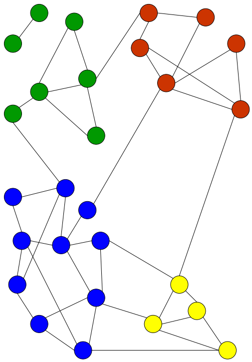
  - convergence to consensus opinions;
  - convergence to equilibrium under myopic updating.

Multi-Type Random Network

- There are n agents, indexed by a set $N = \{1, 2, \ldots, n\}$.



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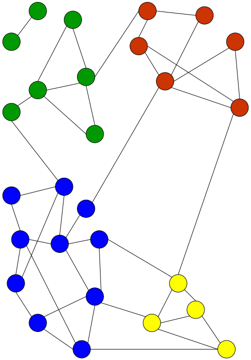

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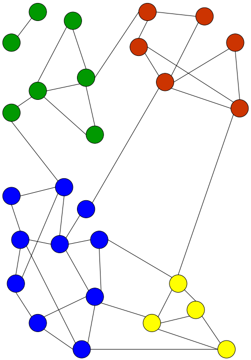

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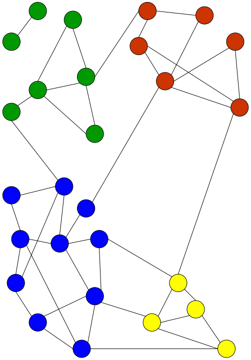
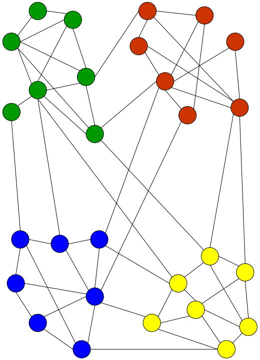
- Links are formed independently.



Islands Model

Special case for this talk:



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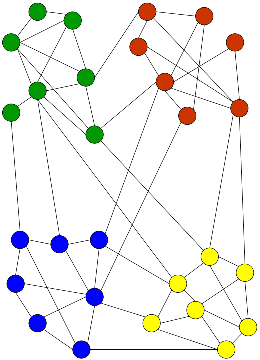

Islands Model

Special case for this talk:

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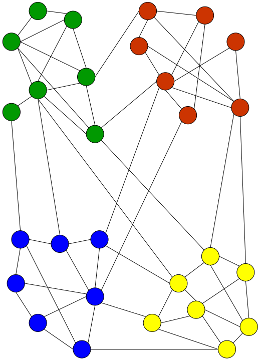
- Only two probabilities:

$$P_{k\ell} = \begin{cases} p_s & \text{if } k = \ell \\ p_d & \text{otherwise.} \end{cases}$$



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  - Broadcasting.

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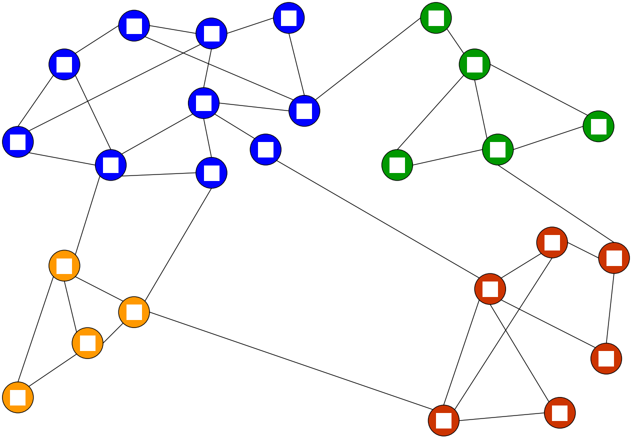

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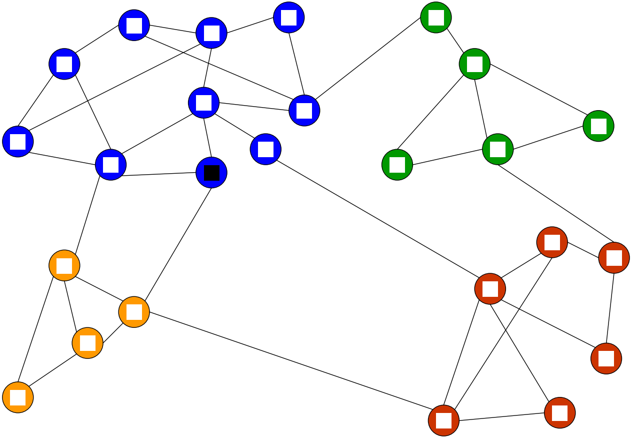

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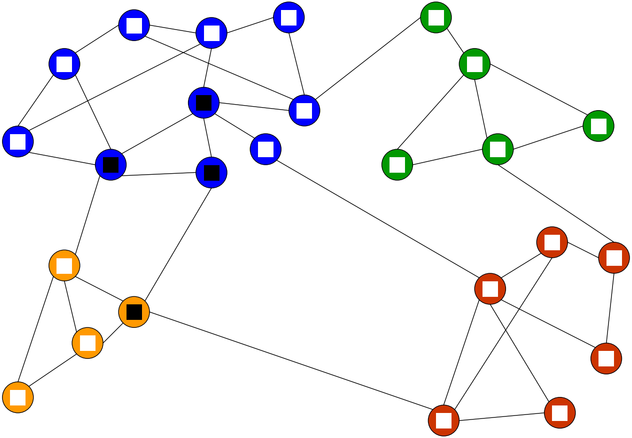

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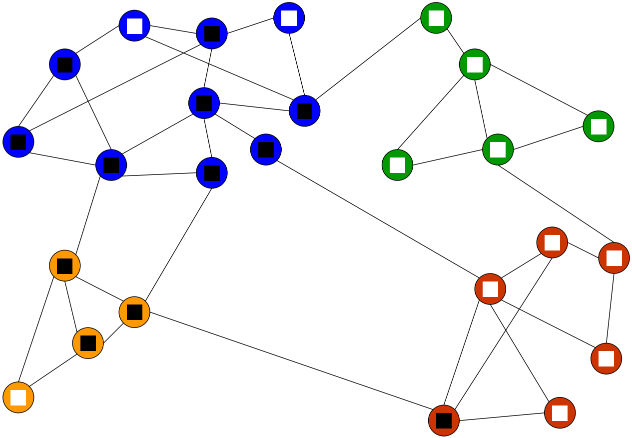

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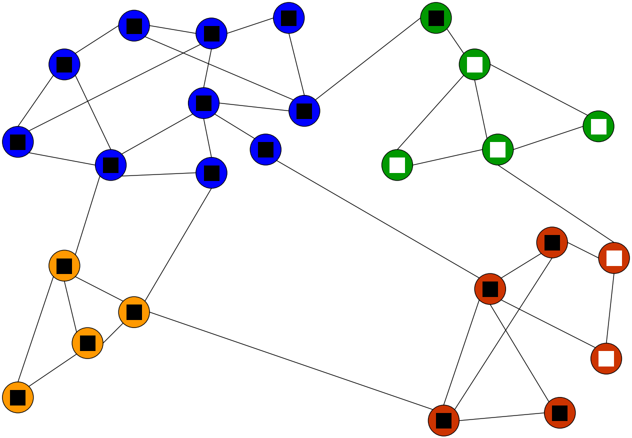

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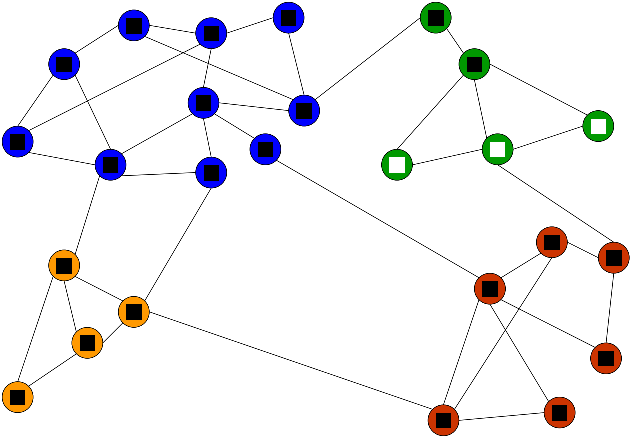

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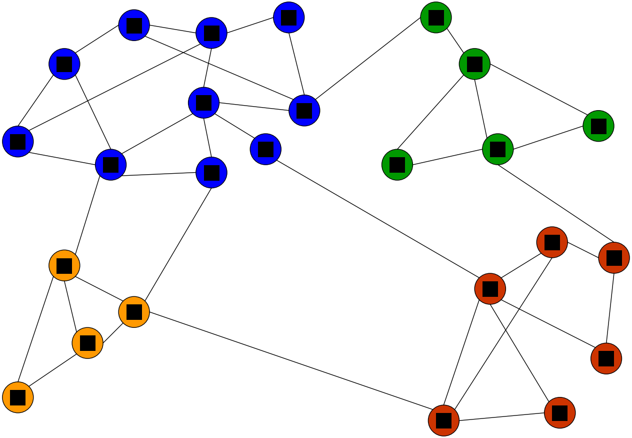

Measuring Speed with Shortest Path Communication

A sufficient statistic for time to communicate (in a *given, fixed* network) in this case is just the expected distance between two randomly chosen nodes.

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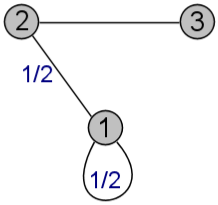

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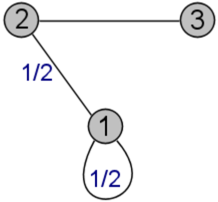

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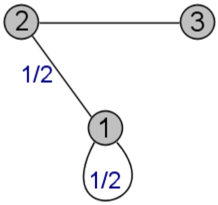

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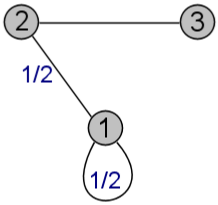

$$b_1(t+1) = \frac{1}{2} b_1(t) + \frac{1}{2} b_2(t)$$

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- This gives the linear updating process.

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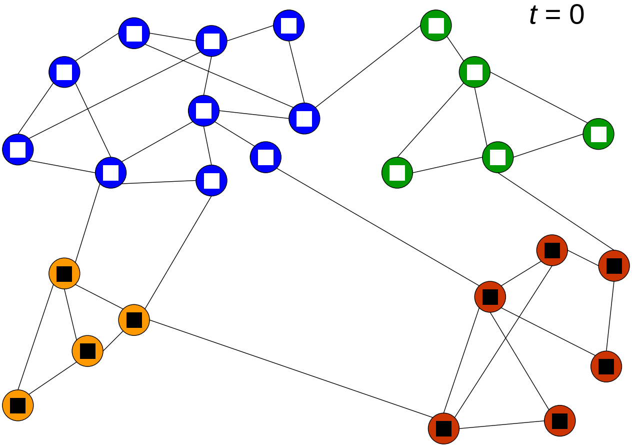

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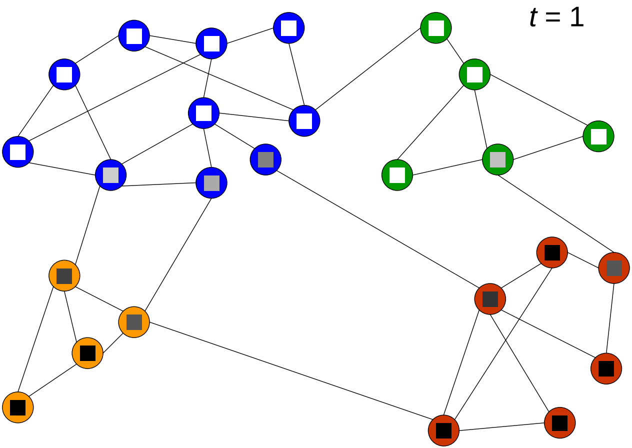

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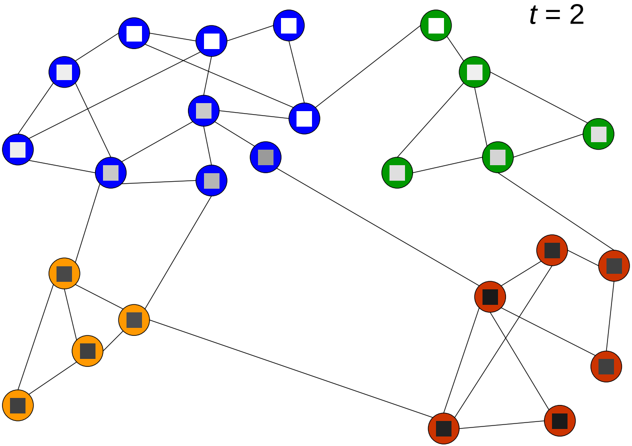

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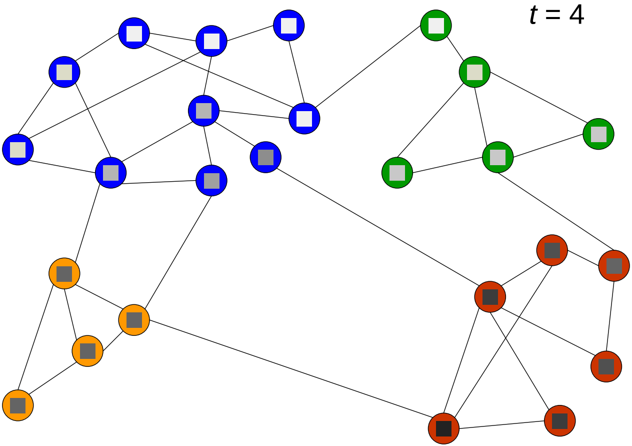

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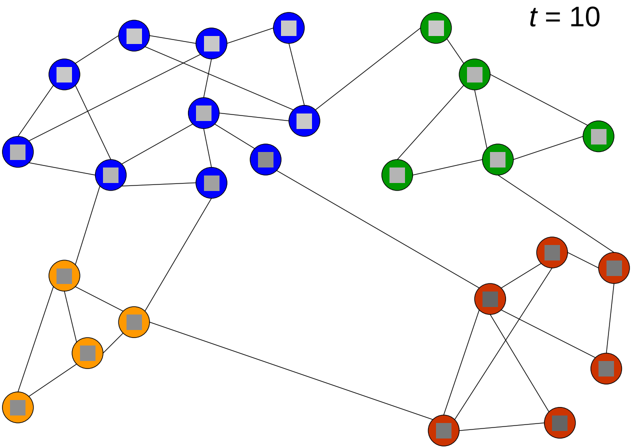

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    (Essentially root-mean-squared distance from consensus.)

Measuring Speed with Linear Updating

Definition

The *consensus time* $\mathrm{CT}(\epsilon; \mathbf{A})$ is the time it takes in network $\mathbf{A}$ until the distance from consensus remains below ϵ, in the worst case, assuming beliefs start in $[0, 1]$.

The Big Picture: How Communication Speed Depends on Density and Homophily

		Independent Variable	
		Density	Homophily
Process	Shortest Path	↑	0
	Linear Updating	0	↓

Arrows indicate how communication speed is affected when the independent variable is increased.

An Approximation Notion

Definition

$$f(n) \approx g(n)$$

means that for any $\delta > 0$,

$$\mathbb{P}\left[\frac{f(n)}{g(n)} \in (1/2 - \delta, 2 + \delta)\right] \xrightarrow{n \to \infty} 1.$$

How Homophily Affects Shortest Path Based Communication: Assumptions

- $d(n) := np(n) \geq (1 + \varepsilon) \log n$ for some $\varepsilon > 0$

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- $h(n) \leq \bar{h} \qquad$ for some $\bar{h} < 1$

  (islands are not completely introspective)

Density, not Homophily, Matters for Shortest Path Communication

Theorem (Jackson 2008)

Under the assumptions just stated,

$$\text{average distance} \approx \frac{\log n}{\log d(n)}$$

and, asymptotically, does not depend at all on homophily.

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- Homophily doesn't matter.
- Only density matters (more = faster).

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- Thus, the average agent can still reach the same number people after t steps, with or without homophily.
  - Homophily does change who is close and who is far; the first hearers of the news are predominantly of the originator's type.
  - But order does not matter – only the overall speed at which the news spreads.

Homophily, not Density, Matters for Linear Updating

Theorem

If $d(n)/\log^2 n \to \infty$ and $m \to \infty$

$$\mathrm{CT}\left(\gamma/n; \mathbf{A}(n)\right) \approx \frac{\log n}{\log(h^{-1})}$$

where the network $\mathbf{A}(n)$ is the islands network with

- n nodes
- m islands
- homophily h.

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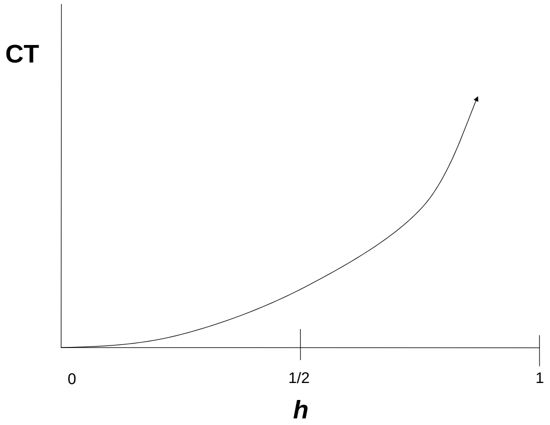

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- Homophily matters (more = slower).

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- Beyond a low threshold, density doesn't matter.

Homophily, not Density, Matters for Linear Updating

Basic intuition: each island reaches its own internal consensus, and if islands put low weight outside themselves, then it will take a long time for the differences to erode.

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Steps of proof:

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  can be written as

  $$\mathbf{b}(t) = \mathbf{T}^t \mathbf{b}(0).$$

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- $$b_i(t+1) = \sum_j \frac{A_{ij}}{d_i} b_j(t)$$

  can be written as

  $$\mathbf{b}(t) = \mathbf{T}^t \mathbf{b}(0).$$

- Convergence of this process to steady state is controlled by second largest eigenvalue in magnitude of **T**.

Homophily, not Density, Matters for Linear Updating

Steps of proof (continued):

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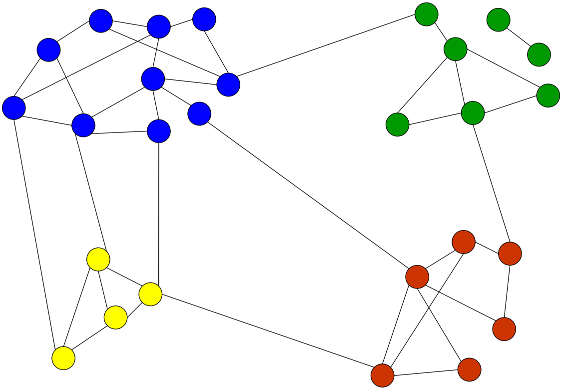
Homophily, not Density, Matters for Linear Updating

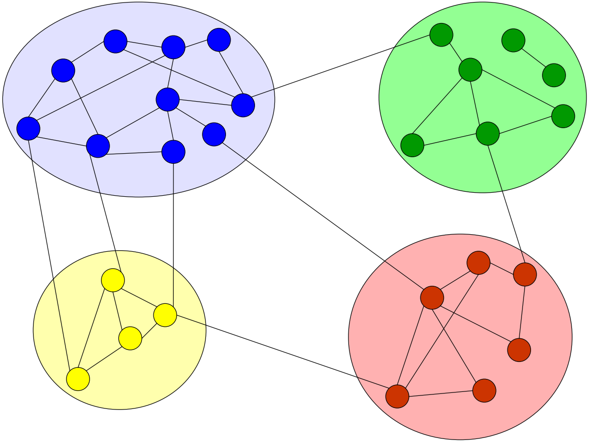
Steps of proof (continued):

- For a multi-type random network, we can look at a *representative agent matrix* with
  - one agent for each type;
  - realized links replaced by expected link densities.
- Theorem: the second eigenvalue of the big random matrix is well-approximated by the second eigenvalue of the small deterministic matrix.

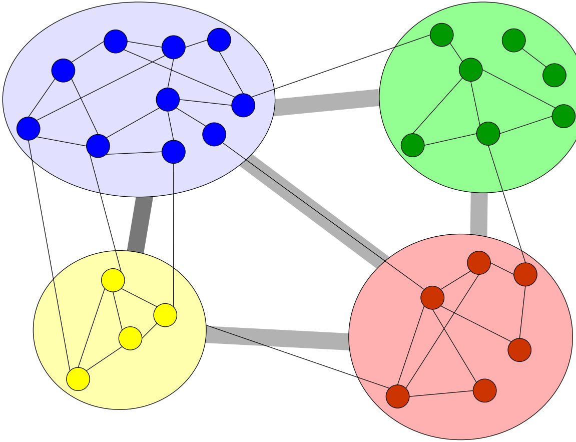
Representative Agent Matrix



Representative Agent Matrix



Representative Agent Matrix



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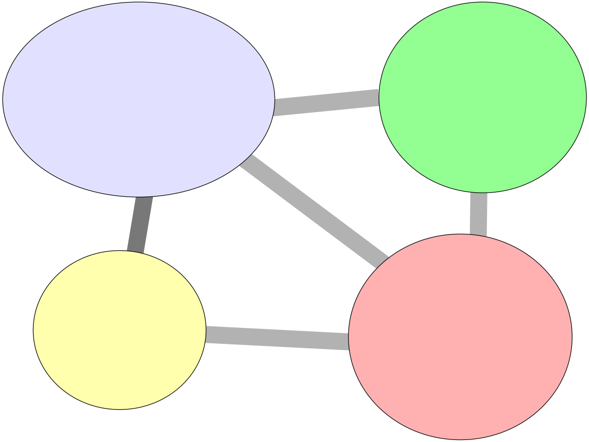


The Data

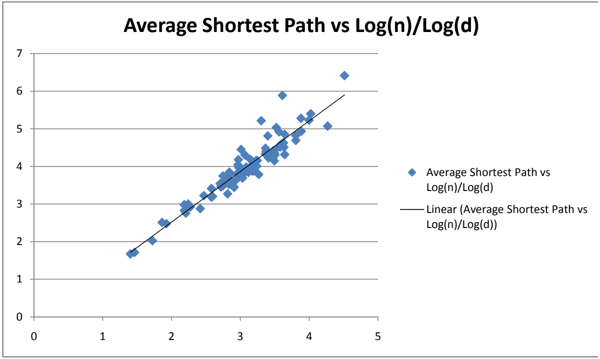
- Adolescent Health data set.
- 84 schools (2 outliers removed).
- For each student:
    - grade in school (6–12);
    - gender;
    - race.
- Friendships.

Testing the Shortest Path Theorem

Recall that the theorem predicts

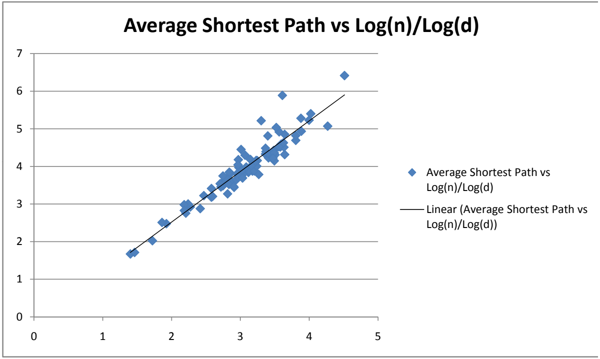
$$\text{average distance} \approx \frac{\log n}{\log d(n)}.$$

Testing the Shortest Path Theorem



without homophily: $R^2 = 0.93$

Testing the Shortest Path Theorem



without homophily: $R^2 = 0.93$ with homophily: $R^2 = 0.94$

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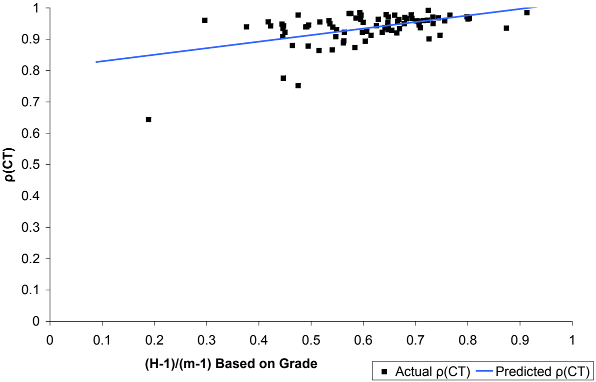
- Can manipulate this around and find a function ρ so that

$$\rho(\text{CT}) - c \propto \frac{H-1}{m-1}.$$

Testing the Consensus Time Theorem



$R^2 = 0.231$