LEARNING IN SOCIAL NETWORKS

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1. Introduction

Social ties convey information through observations of others' decisions as well as through conversations and the sharing of opinions. The resulting information flows play a role in a range of phenomena, including job search (Montgomery 1991), financial planning (Duflo and Saez 2003), product choice (Trusov et al. 2009), and voting (Beck et al. 2002). Understanding how individuals use information from their social environments, and the aggregate consequences of this learning, is therefore important in many contexts. Is dispersed information aggregated efficiently? Whose opinions or experiences are particularly influential? Can we understand when choices will be diverse, when people will choose to conform, and how the networks in which individuals communicate shape these outcomes?

This chapter surveys several approaches to these questions. We begin with a discussion of sequential social learning models in which each agent makes one decision; these models have admitted a rich analysis within a canonical Bayesian paradigm. We next discuss the DeGroot model of repeated linear updating. This theory employs a simple heuristic learning rule, delivering a fairly complete characterization of learning dynamics as a function of network structure. Finally, we review work that studies repeated Bayesian (or quasi-Bayesian) updating in general networks.

2. The Sequential Social Learning Model

An important early branch of the social learning literature arose to explain widespread conformity within groups, referred to as herd behavior. Banerjee (1992) and Bikhchandani et al. (1992) independently proposed models in which players each take a single action in sequence. Before making a choice, each player observes all previous actions and a private signal. Though individual payoffs are independent of other players' actions, others' choices provide information about their signals, and therefore about what action is best. This information constitutes an externality. Oftentimes, individuals acting later will optimally ignore their private signals and copy the crowd instead. If this happens, the choices of these players cease to reveal new information, and the population can herd on a suboptimal action. Variations on this insight have been used to explain phenomena ranging from fads and fashions to stock bubbles, and there is a rich literature on extensions and applications.[1]

In this section we cover more recent efforts, adapting the classical herding models to a network structure that encodes partial observation of the history. The distinguishing feature in this line of work is the sequential structure, and we collectively refer to these models as the *sequential social learning model* (SSLM). While Bayesian learning is often difficult to analyze in a network, sequential models have proved particularly tractable. Since each player makes only one choice, there is no scope for strategic interactions, and researchers can study the information externality in isolation from other concerns.

Two key principles provide intuition for long-run learning outcomes in the SSLM: the *improvement principle* and the *large-sample principle*. The improvement principle notes that a player always has the option to copy one of her neighbors, so the payoff from imitating provides a lower bound on her expected utility. Since the player additionally has a private signal, she might be able to improve upon imitation. One family of learning results characterizes how well the population learns through this mechanism. The intuition of the improvement principle is generally robust to features of the network, but the extent of learning depends heavily on the private signals because continued improvement on imitation requires the possibility of a strong signal. If players observe many actions, we can additionally employ the large-sample principle: If observations are at least partly independent, then a large sample collectively conveys much information, even if each signal is individually weak. In these networks, the exact distribution of private signals matters less, and learning is far more robust.

2.1 The SSLM with Homogeneous Preferences

Our first pass at the SSLM assumes that players differ only in their initial information, not in their preferences. Our formulation includes as special cases the classical models of Banerjee (1992), Bikhchandani et al. (1992), and Smith and Sørensen (2000), which assume that all players observe the entire history. However, a decision maker's information set is typically far less complete. Authors have taken different approaches to represent limited observation of other players' choices. One approach, adopted by Smith and Sørensen (2008), is for players to take anonymous random samples from the past. We focus more explicitly on the network of connections between players; players know the identities of those they

[1]See Chamley (2004) for an overview. Bose et al. (2008) and Ifrach et al. (2013) offer recent contributions.

observe, and we consider arbitrary sampling processes. Our framework encompasses recent work by Celen and Kariv (2004), Acemoglu et al. (2011), Arieli and Mueller-Frank (2014), and Lobel and Sadler (2015). We can also include the model of Eyster and Rabin (2011) if we relax the assumption that players' signals are identically distributed.

2.1.1 *Information and Strategies*

Each player $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ makes a binary choice of action $x_n \in \{0, 1\}$ in sequence. The state of the world is $\theta \in \{0, 1\}$, and players share a common prior $q_0 = \mathbb{P}(\theta = 1)$. Players prefer to choose the action that matches the state: We assume a common utility function $u(x, \theta)$ with

$$u(1, 1) > u(0, 1) \text{ and } u(0, 0) > u(1, 0).$$

Player n observes a signal s_n taking values in an arbitrary metric space $\mathcal{S}$, and the signals $\{s_n\}_{n \in \mathbb{N}}$ are independent and identically distributed conditional on the underlying state θ. We use $\mathbb{F}_\theta$ to denote the signal distribution conditional on the state. We assume $\mathbb{F}_0$ is not almost everywhere equal to $\mathbb{F}_1$, so some signal realizations provide information.

For concreteness, consider some examples of signal structures. We could have binary signals, with each s_n taking values in $\mathcal{S} = \{0, 1\}$ and where

$$\mathbb{P}(s_n = 0 \,|\, \theta = 0) = \mathbb{P}(s_n = 1 \,|\, \theta = 1) = g > \frac{1}{2}.$$

The realization $s_n = 0$ provides evidence in favor of $\theta = 0$, while $s_n = 1$ provides evidence in favor of $\theta = 1$. This is the signal structure studied by Banerjee (1992) and Bikhchandani et al. (1992). We could also consider real-valued signals with $\mathbb{F}_0$ and $\mathbb{F}_1$ being probability measures on $\mathbb{R}$. For instance, suppose $\mathbb{F}_0$ and $\mathbb{F}_1$ are both supported on $\mathcal{S} = [0, 1]$, where $\mathbb{F}_0$ has density $2 - 2s$ and $\mathbb{F}_1$ has density $2s$. In this case, lower signal realizations provide stronger evidence in favor of $\theta = 0$: the signal $s \in [0, 1]$ induces the likelihood ratio $\frac{s}{1-s}$.

In addition to the signal s_n, player n observes the choices of those in her neighborhood $B(n) \subseteq \{1, 2, \ldots, n-1\}$. That is, player n observes the value x_m for each $m \in B(n)$. This neighborhood is a subset of players who have already acted before player n, and the sequence of neighborhoods $\{B(n)\}_{n \in \mathbb{N}}$ constitutes the observational network. Each neighborhood is randomly drawn according to a distribution, and we use $\mathbb{Q}$ to denote the joint probability distribution of the sequence of neighborhoods. We assume the distribution $\mathbb{Q}$, which we call the network, is common knowledge among the players.

The following are examples of networks:

- For each $n \in \mathbb{N}$, $B(n) = \{1, 2, \ldots, n-1\}$ with probability 1. This is the complete network of Banerjee (1992), Bikhchandani et al. (1992), and Smith and Sørensen (2000), in which each player observe all of her predecessors.
- For each $n \in \mathbb{N}$, $B(n) = \{n-1\}$ with probability 1. This is the network of Celen and Kariv (2004) in which each player observes only her immediate predecessor.

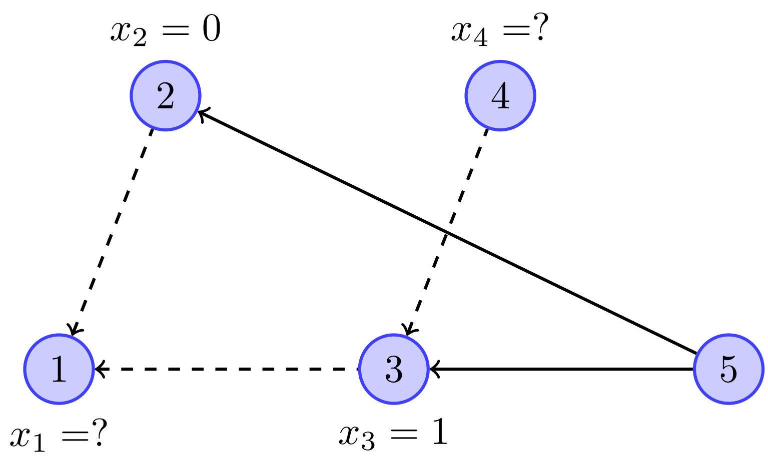


Figure 1: A realized network. Player 5 observes players 2 and 3, but is uncertain about other decisions and links.

- For each $n \in \mathbb{N}$, $B(n)$ contains one element drawn uniformly at random from $\{1, 2, \ldots, n-1\}$, and all these draws are independent. This example of observing a random predecessor demonstrates how neighborhood realizations can be stochastic.

- With equal probability, $B(2) = \{1\}$ or $B(2) = \emptyset$. If $B(2) = \emptyset$, then $B(n) = \emptyset$ for all n; otherwise $B(n) = \{n-1\}$ for all n. Here, every agent observes only her immediate predecessor or every agent observes nobody. So neighborhood realizations are again random and, in contrast to the previous example, correlated.

The information player n observes is then $I_n = \{s_n,\, x_m,\, m \in B(n)\}$: her private signal and the actions of all the predecessors in her realized neighborhood. This is a natural generalization of the early sequential learning literature, which implicitly assumed a complete network.

We often reference player n's *private belief* $p_n = \mathbb{P}(\theta = 1 \,|\, s_n)$ separately from player n's *social belief* $q_n = \mathbb{P}(\theta = 1 \,|\, x_m,\, m \in B(n))$, and we use $\mathbb{G}_\theta$ to denote the distribution function of p_n conditional on the state. A strategy σ_n for player n maps each possible realization of her information I_n to an action $x_n \in \{0, 1\}$. A strategy profile $\sigma \equiv \{\sigma_n\}_{n\in\mathbb{N}}$ induces a probability distribution $\mathbb{P}_\sigma$ over the sequence of actions. The profile σ is a *perfect Bayesian equilibrium* if each player maximizes her expected utility, given the strategies of the other players:

$$\mathbb{E}_\sigma\left[u(\sigma_n, \theta) \,\middle|\, I_n\right] \geq \mathbb{E}_\sigma\left[u(\sigma'_n, \theta) \,\middle|\, I_n\right]$$

for any strategy σ'_n. Since each player acts once in sequence, an inductive argument establishes the existence of an equilibrium, though in general this is non-unique since some players may be indifferent between the two actions.

2.1.2 Long-Run Learning Metrics

Our study of equilibrium behavior centers on asymptotic outcomes. In particular, we consider two metrics—diffusion and aggregation—based on players' expected utility as the index n approaches infinity. *Aggregation* occurs if players' utility approaches what they would obtain with perfect information:

$$\lim_{n\to\infty} \mathbb{E}_\sigma\left[u\left(x_n,\theta\right)\right] = q_0 u(1,1) + (1-q_0)u(0,0).$$

This represents the best asymptotic outcome we can hope to achieve: For later players, it is as though the private information of those that came before them is aggregated into a single, arbitrarily precise signal.

The definition of *diffusion* depends on the signal distribution, and the support of private beliefs in particular. The support of the private beliefs is the region $[\underline{\beta},\overline{\beta}]$, where

$$\underline{\beta} = \inf\{r\in[0,1]\,|\,\mathbb{P}(p_1\le r)>0\} \text{ and } \overline{\beta} = \sup\{r\in[0,1]\,|\,\mathbb{P}(p_1\le r)<1\}.$$

It can be shown that there is a unique binary signal $\tilde{s}\in\{0,1\}$, a random variable such that

$$\mathbb{P}(\theta=1\,|\,\tilde{s}=0)=\underline{\beta} \text{ and } \mathbb{P}(\theta=1\,|\,\tilde{s}=1)=\overline{\beta}.$$

We shall call $\tilde{s}$ the *expert signal*. Diffusion occurs if we have

$$\liminf_{n\to\infty} \mathbb{E}_\sigma\left[u\left(x_n,\theta\right)\right] \ge \mathbb{E}\left[u(\tilde{s},\theta)\right] \equiv u^*.$$

Intuitively, we have diffusion if players perform as though they were guaranteed to receive one of the strongest possible signals.

In any particular network $\mathbb{Q}$, diffusion or aggregation might occur or not, depending on the signal structure or the equilibrium. To focus our attention on the network's role, we say that a network diffuses or aggregates information only if this occurs for *any* signal structure and *any* equilibrium.

Definition 1. We say that a network $\mathbb{Q}$ **aggregates (diffuses) information** if aggregation (diffusion) occurs for every signal distribution and every equilibrium strategy profile.

Note that aggregation is generally a stronger criterion than diffusion; if $1-\underline{\beta}=\overline{\beta}=1$, the two metrics coincide. In this case, we say that *private beliefs are unbounded*, whereas private beliefs are bounded if $\underline{\beta}>0$ and $\overline{\beta}<1$. Much work on the SSLM studies conditions under which aggregation occurs. In their seminal paper, Smith and Sørensen (2000) demonstrate that aggregation occurs in a complete network—a network in which all players observe the entire history—if and only if private beliefs are unbounded. This means there are informative signal structures for which the complete network fails to aggregate information, so the complete network does not aggregate according to Definition 1. Acemoglu et al. (2011) find this characterization holds much more generally when the neighborhoods $\{B(n)\}_{n\in\mathbb{N}}$ are mutually independent. Our metrics provide an alternative perspective that emphasizes the *network's* role in social learning. Often, when aggregation turns on whether beliefs are bounded or unbounded, the network diffuses information according to Definition 1, but it does not aggregate information.

2.1.3 Necessary Conditions for Learning

Connectedness is the most basic requirement for information to spread in a network. In the SSLM, this corresponds to each player having at least indirect access to a large number of signals. If there exists a chain of players $m_1, m_2, \ldots, m_k$ such that $m_i \in B(m_{i-1})$ for each $i \geq 2$ and $m_1 \in B(n)$, we say that m_k is in player n's *personal subnetwork* $\hat{B}(n)$. A necessary condition for a network to aggregate or diffuse information is that the size of $\hat{B}(n)$ should grow without bound as n becomes large.

Definition 2. The network $\mathbb{Q}$ features **expanding subnetworks** if for any integer K, we have

$$\lim_{n \to \infty} \mathbb{P}\left(|\hat{B}(n)| < K\right) = 0.$$

Proposition 1. If $\mathbb{Q}$ diffuses information, then $\mathbb{Q}$ features expanding subnetworks.

To see why this condition is necessary, suppose $|\hat{B}(n)| < K$ for some player n. We can bound this player's expected utility by what she could attain with access to K independent signals. Since this need not reach the level u^*, we cannot guarantee diffusion with infinitely many such players.[2]

With a basic necessary condition for diffusion in hand, we organize further analysis of the SSLM according to our two key principles. It will become clear that each learning metric corresponds to a particular learning principle: diffusion to the improvement principle and aggregation to the large-sample principle. Though we focus on utility-based metrics, we comment first on behavioral outcomes and belief evolution.

2.2 Herding and Cascades

Historically, the SSLM literature focuses on long-run patterns of behavior and belief dynamics: The central phenomena are herding and informational cascades. Herding means that all players conform in their behavior after some time, while an informational cascade occurs if all players ignore their private signals after some time. Formally, herding occurs if there is a random variable x supported on $\{0, 1\}$ such that the sequence of actions $\{x_n\}_{n \in \mathbb{N}}$ converges almost surely to x. Defining a cascade requires additional notation. For some thresholds $\underline{\alpha}$ and $\overline{\alpha}$, player n optimally ignores her signal s_n whenever $q_n \in C \equiv [0, \underline{\alpha}] \cup [\overline{\alpha}, 1]$. We call C the *cascade set* of beliefs, and if $q_n \in C$ we say that player n cascades. An informational cascade occurs if, with probability 1, all players with sufficiently high indices cascade. In a complete network, a cascade implies herding. If the social beliefs $\{q_n\}_{n \in \mathbb{N}}$ ever reach the cascade set, no new information is revealed, and q_n remains constant thereafter.

Though herding and informational cascades are clearly related, Smith and Sørensen (2000) make clear these are distinct notions.[3] Players can herd even while it remains possible

[2] See Lobel and Sadler (2015, Theorem 1) for more detail.

[3] Herrera and Hörner (2012) give a precise characterization of when cascades occur.

for some signal realization to change their behavior, because, with some positive probability, such signals may never occur. Indeed, if all players observe the entire history, herd behavior is guaranteed. This is because the sequence of social beliefs q_n is a martingale, and the martingale convergence theorem implies that it converges almost surely. At the same time, the "overturning principle" of Smith and Sørensen (2000) says that whenever a sufficiently late-moving player takes a different action from her predecessors, the social belief must change substantially. These facts combined show that herding must occur, whether or not there is a cascade, and whether or not aggregation occurs. Now, when *is* there a cascade? If private beliefs are unbounded, then clearly there is never a cascade, because signals can be arbitrarily strong. However, Smith and Sorensen note that the martingale $\{q_n\}$ must converge to a random variable supported on the cascade set, and then it is said that a *limit cascade* occurs. In this limiting sense, both herding and cascades always emerge in the complete network, regardless of the signal structure.[4]

In a general network, we need not find such regular patterns of long-run behavior or belief evolution. In a version of the SSLM with $B(n) = \{n-1\}$ for all n, Celen and Kariv (2004) demonstrate that, even though herd-like behavior appears for long stretches, true herds or cascades may be absent. Lobel and Sadler (2015) show that social beliefs along a subsequence of players could converge almost surely to a random variable that puts positive probability strictly outside the cascade set. If these beliefs converge to a point outside the cascade set, actions along the subsequence will follow an i.i.d. sequence of random variables. Nevertheless, networks that diffuse information display outcomes that are closely related to informational cascades. In a network that diffuses information, players' *ex ante* utility is at least as high as if they were in a cascade, even if social beliefs do not converge to the cascade set.

2.3 The Improvement Principle

We now return to welfare in the SSLM, focusing first on the improvement principle and information diffusion.

2.3.1 Two Lemmas on the Improvement Principle

There are two basic steps in any application of the improvement principle: choosing whom to imitate and determining if improvement is possible. We express the selection component through *neighbor choice functions*.[5] A neighbor choice function $\gamma_n : 2^{\{1,\dots,n\}} \to \mathbb{N}$ for player n

[4] Herding also occurs under alternative sampling rules. For instance, Banerjee and Fudenberg (2005) study a sequential decision model in a continuum of players, in which each decision-maker samples several predecessors uniformly at random; in their model, herding is a robust outcome.

[5] The concept of a neighbor choice function is implicit in the work of Acemoglu et al. (2011), which builds improvements on the neighbor with the highest index. The formalization used here was introduced by Lobel and Sadler (2015).

selects a neighbor from any realization of $B(n)$. We require that $\gamma_n(S) \in S$ if S is nonempty; otherwise, we take $\gamma_n(S) = 0$. In what follows, we use γ_n to refer to both this function and its (random) value. Given our original network $\mathbb{Q}$ and a sequence of neighbor choice functions $\{\gamma_n\}_{n\in\mathbb{N}}$, we implicitly define a new random network $\mathbb{Q}_\gamma$. In the network $\mathbb{Q}_\gamma$, we include only those links in $\mathbb{Q}$ that are selected by the neighbor choice functions. In any realization of this network, each player has at most one neighbor.

We say that an improvement principle holds if, for some sequence of neighbor choice functions, the following heuristic procedure leads to diffusion. Each player n discards all observations of neighbors' decisions except the observation of player γ_n's decision. Player n then chooses an action to maximize expected utility given this single observation and her private signal. Proving that an improvement principle holds entails showing that player n can earn strictly higher utility than her chosen neighbor m whenever $\mathbb{E}_\sigma[u(x_m, \theta)] < u^*$.

Lemma 1 (Improvement Principle)**.** Suppose there exists a sequence of neighbor choice functions $\{\gamma_n\}_{n\in\mathbb{N}}$ and a continuous, increasing function $\mathcal{Z}$ such that:

(a) The network $\mathbb{Q}_\gamma$ features expanding subnetworks.

(b) For all $u < u^*$, we have $\mathcal{Z}(u) > u$.

(c) For any $\epsilon > 0$, there exists N_ϵ such that for any $n \geq N_\epsilon$, with probability at least $1-\epsilon$,

$$\mathbb{E}_\sigma\left[u(x_n, \theta) \mid \gamma_n\right] > \mathcal{Z}\left(\mathbb{E}_\sigma\left[u(x_{\gamma_n}, \theta)\right]\right) - \epsilon. \tag{1}$$

Then the network $\mathbb{Q}$ diffuses information.

This result extends Lemma 4 of Lobel and Sadler (2014), and its proof is essentially identical. Condition (c) expresses the key intuition: For all neighbors except some that γ_n selects with negligible probability, player n can make an improvement. To apply Lemma 1, we must construct a suitable improvement function $\mathcal{Z}$.

Lemma 2. There exists a continuous, increasing function $\mathcal{Z}$, with $\mathcal{Z}(u) > u$ for all $u < u^*$, such that

$$\mathbb{E}_\sigma\left[u(x_n, \theta) \mid \gamma_n = m\right] > \mathcal{Z}\left(\mathbb{E}_\sigma\left[u(x_m, \theta) \mid \gamma_n = m\right]\right). \tag{2}$$

Proof Sketch: We describe a heuristic procedure that a player could follow to obtain the desired improvement; since the players are Bayesian, the actual improvement is at least as high. Suppose player n copies the action of m unless she receives a signal inducing a private belief p_n that is very close to an extreme point $\underline{\beta}$ or $\overline{\beta}$. In the latter case, player n chooses the action the signal suggests. Conditional on following her signal, it is as though player n receives very nearly the expert signal $\tilde{s}$, so her expected utility is an average of her neighbor m's utility and something arbitrarily close to u^*. Thus, improvements can accumulate up to the level u^* in the long run.

This argument highlights the significance of the expert signal and why we cannot count on further improvements. Improving on imitation requires following at least the extreme values of the signal, and the expert signal represents an upper bound on what a player can obtain when following an extreme signal.

2.3.2 Sufficient Conditions for Diffusion

There is one more step to connect Lemmas 1 and 2 into a general result, namely bounding the difference between $\mathbb{E}_\sigma[u(x_m,\theta)]$ and $\mathbb{E}_\sigma[u(x_m,\theta)\,|\,\gamma_n = m]$. Player n can imitate player m only if player m is contained in $B(n)$. Therefore, player n's expected utility conditional on imitating player m is not the same as player m's expected utility: Imitation earns m's expected utility *conditional on* n *choosing to imitate player* m—that is, conditional on $\gamma_n = m$. If $\mathbb{E}_\sigma[u(x_m,\theta)]$ and $\mathbb{E}_\sigma[u(x_m,\theta)\,|\,\gamma_n = m]$ are approximately equal for large n, then Lemmas 1 and 2 immediately imply information diffusion.

Proposition 2 (Diffusion)**.** Suppose there exists a sequence of neighbor choice functions $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\mathbb{Q}_\gamma$ features expanding subnetworks, and for any $\epsilon > 0$, there exists N_ϵ such that for any $n \geq N_\epsilon$, with probability at least $1-\epsilon$,

$$\mathbb{E}_\sigma\left[u(x_{\gamma_n},\theta)\,|\,\gamma_n\right] > \mathbb{E}_\sigma\left[u(x_{\gamma_n},\theta)\right] - \epsilon.$$

Then diffusion occurs.

A number of conditions on the network can ensure that

$$\mathbb{E}_\sigma\left[u(x_m,\theta)\,|\,\gamma_n = m\right] = \mathbb{E}_\sigma\left[u(x_m,\theta)\right],$$

and any of these immediately implies information diffusion.

Corollary 1. The network $\mathbb{Q}$ diffuses information if any of the following conditions holds:

(a) The neighborhoods $\{B(n)\}_{n\in\mathbb{N}}$ are mutually independent, and $\mathbb{Q}$ features expanding subnetworks.

(b) There exists a sequence of neighbor choice functions $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\mathbb{Q}_\gamma$ is deterministic and features expanding subnetworks.

(c) There exists a sequence of neighbor choice functions $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\mathbb{Q}_\gamma$ features expanding subnetworks and the random vector $\{B(i)\}_{i=1}^m$ is independent of the event $\gamma_n = m$ for all $n > m$.

Proposition 2 unifies and extends earlier results from Acemoglu et al. (2011) and Lobel and Sadler (2015), applying broadly whenever we can bound the difference

$$\mathbb{E}_\sigma\left[u(x_m,\theta)\,|\,\gamma_n = m\right] - \mathbb{E}_\sigma\left[u(x_m,\theta)\right]$$

using properties of the network.[6] Though the improvement principle is generally robust to features of the network $\mathbb{Q}$, we can construct examples, using highly correlated neighborhoods, in which information fails to diffuse. Lobel and Sadler (2015) highlight through several examples that asymmetric information about the overall network can disrupt the improvement principle even if connectivity is not an issue.

[6] Lobel and Sadler (2015) define a measure of network distortion to bound this difference. Assuming that $\{B(n)\}_{n\in\mathbb{N}}$ are conditionally independent given the state of an underlying Markov chain with finitely many states, they further generalize part (a) of the corollary by applying an improvement principle to the minimum utility across all states of the Markov chain.

2.3.3 Failure to Aggregate

Although the improvement principle can ensure only information diffusion, it is worth asking whether we can do better and aggregate information in many of these networks, particularly if players have multiple neighbors. This question is incompletely answered at present, but results in the literature suggest that aggregation generally fails unless some players have large neighborhoods.

Proposition 3 (Failure to Aggregate)**.** The random network $\mathbb{Q}$ fails to aggregate information if $\mathbb{Q}$ satisfies any of the following conditions:

(a) $B(n) = \{1, 2, \ldots, n-1\}$ for all n.

(b) $|B(n)| \leq 1$ for all n.

(c) $|B(n)| \leq M$ for all n and some constant M, the neighborhoods $\{B(n)\}_{n \in \mathbb{N}}$ are mutually independent, and

$$\lim_{n \to \infty} \max_{m \in B(n)} m = \infty \quad \text{almost surely.}$$

This result is Theorem 3 of Acemoglu et al. (2011). The complete network—and any network in which players have at most one neighbor—will fail to aggregate information. Recall that this means there is *some* signal structure for which, with positive probability, late movers' actions do not approach optimality given all of society's information. In this case, aggregation fails for any signal structure leading to bounded private beliefs. Part (c) says that, in general, aggregation fails if there is a bound on neighborhood size and players' observations are independent. We cannot dispense with the condition of independence, because we can construct example networks, with correlated neighborhoods and $|B(n)| \leq 2$ for all n, that aggregate information. The degree to which we can relax independence is an open question, and a more detailed understanding of the boundary between diffusion and aggregation would constitute a valuable contribution to this literature.

2.4 The Large-Sample Principle

Part (a) of Proposition 3 already demonstrates that large samples alone are insufficient to ensure aggregation, and indeed classical papers (Banerjee 1992; Bikhchandani et al. 1992; Smith and Sørensen 2000) that study sequential learning in a complete network focus on this failure and associated behavioral patterns. To aggregate information from a large sample, infinitely many observations must contain at least some new information, which means that infinitely many players must respond to their private signals. However, if all players observe a large sample, social information will overwhelm the private signals. In some sense the complete network is a knife-edge case in which the large-sample principle fails because no one is forced to rely on a private signal. If we disrupt some of the connections in the network, creating a subsequence of "sacrifical lambs" with no social information, this group can provide enough information for the rest of the network to learn the true state.

Proposition 4 (Aggregation)**.** Suppose there exists a sequence of players $\{m_i\}_{i\in\mathbb{N}}$ such that $\{B(m_i)\}_{i\in\mathbb{N}}$ are mutually independent,

$$\sum_{i\in\mathbb{N}} \mathbb{P}\left(B(m_i) = \emptyset\right) = \infty,$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(m_i \in B(n)\right) = 1$$

for each i. Then the network $\mathbb{Q}$ aggregates information.

Proposition 4 follows from standard martingale convergence arguments, examples of which are found in several papers on social learning.[7] The sequence of players $\{m_i\}_{i\in\mathbb{N}}$ provides enough information to fully reveal the state asymptotically, and all players observe those in this sequence with probability approaching 1. The sequence may consist of an arbitrarily small portion of the network, highlighting an important discontinuity in learning outcomes when we compare a complete network with an almost-complete network.

The basic insight of this proposition generalizes in several directions. Depending on the signal distribution, the sacrificial lambs could have nonempty neighborhoods as long as some signal realizations still dominate the available social information. We also need not have everyone in the network observe the sacrificial lambs: As long as some players observe the large sample and aggregate information, others can learn through imitation. Theorem 4 in Acemoglu et al. (2011) explores both possibilities, giving a more general result. However, the bounds of applicability for the large-sample principle are imprecisely known. Given an infinite set of players following their private signals, there are potentially many ways a network can collect, aggregate, and disperse this information. The literature is still missing a general characterization of networks that aggregate information.

2.5 The SSLM with Heterogeneous Preferences

The improvement principle and the large-sample principle respond differently when we introduce preference heterogeneity. On an intuitive level, the improvement principle should suffer because imitation no longer guarantees the same payoff that a neighbor obtains. If a neighbor's preferences are sufficiently different, copying could result in a relatively lower payoff; long-run learning requires not only improvement but also compensation for this gap. However, heterogeneity also raises the prospect that more players will choose to follow private information, which suggests that the large-sample principle has more room to operate as long as neighborhoods are sufficiently large.

[7] See for instance, Goeree et al. (2006), Acemoglu et al. (2011), and Lobel and Sadler (2014).

We extend the model of the previous section to allow preferences with common and private components.[8] Each player n privately observes a type $t_n \in (0,1)$, and a player of type t will earn utility

$$u(x,\theta,t) = \begin{cases} 1-\theta+t & \text{if } x=0 \\ \theta+1-t & \text{if } x=1. \end{cases}$$

A player's type t neatly parameterizes her trade off between error in state 0 and error in state 1. Action 1 is chosen only if the player believes that $\theta = 1$ with probability at least t. Hence, to choose action 1, players with high types require more convincing information.

2.5.1 No Improvement Principle

Using an example, we illustrate how heterogeneous preferences disrupt the improvement principle. Suppose the signal structure is such that $\mathbb{G}_0(r) = 2r - r^2$ and $\mathbb{G}_1(r) = r^2$, and consider the network topology $\mathbb{Q}$ in which each agent observes her immediate predecessor with probability 1. Suppose player 1 has type $t_1 = \frac{1}{5}$, and all other players n have type $t_n = 1 - t_{n-1}$ with probability 1. Even though the network satisfies our connectivity condition, information diffusion fails.

An inductive argument will show that all players with odd indices err in state 0 with probability at least $\frac{1}{4}$, and likewise players with even indices err in state 1 with probability at least $\frac{1}{4}$. For the first player, observe that $\mathbb{G}_0\left(\frac{1}{5}\right) = \frac{9}{25} < \frac{3}{4}$, so the base case holds. Now suppose the claim holds for all players of index less than n, and n is odd. The social belief q_n is minimized if $x_{n-1} = 0$, taking the value

$$\frac{\mathbb{P}_\sigma(x_{n-1} = 0 \,|\, \theta = 1)}{\mathbb{P}_\sigma(x_{n-1} = 0 \,|\, \theta = 1) + \mathbb{P}_\sigma(x_{n-1} = 0 \,|\, \theta = 0)} \geq \frac{\dfrac{1}{4}}{\dfrac{1}{4} + 1} = \frac{1}{5}.$$

It follows that player n will choose action 1 whenever $p_n > \frac{1}{2}$. We obtain the bound

$$\mathbb{P}_\sigma(x_n = 1 \,|\, \theta = 0) \geq 1 - \mathbb{G}_0\left(\frac{1}{2}\right) = \frac{1}{4}.$$

An analogous calculation proves the inductive step for players with even indices. Hence, all players err with probability bounded away from 0, and since private beliefs are unbounded, diffusion fails.

In the example, each player has a single neighbor whose preferences are substantially different from her own. The difference in preferences means that the choice of this neighbor provides less useful information. Suppose a player and her neighbor are diners choosing between an Italian and a Japanese restaurant. If the neighbor prefers Japanese food, then

[8] Smith and Sørensen (2000) consider a case in which players may have completely opposed preferences, leading to an outcome they call "confounded learning."

observing this neighbor choose the Japanese restaurant is a weak signal of quality, but if the neighbor chooses the Italian restaurant, the choice provides strong information. If our player prefers Italian food, she would benefit more if the signal qualities were reversed.

Lobel and Sadler (2014) offer more general results on this model. If types are i.i.d. random variables and there is a uniform bound on neighborhood size, we can always find a type distribution with support $(0, 1)$ such that diffusion fails. Whether an improvement principle holds depends on the relative frequency of strong signals versus strong preferences. For an improvement principle to hold, the signal distribution must have a thicker tail than the type distribution, meaning that strong signals should be far more common than strong preferences.

2.5.2 Robust Aggregation

Preference heterogeneity has a very different impact on the large-sample principle. If players have large neighborhoods, then we can eliminate the need for sacrificial lambs because, with rich enough support in the type distribution, there is always a chance that preferences roughly balance against available social information, making the private signal relevant to a player's decision. Goeree et al. (2006) first noted this effect in a complete network, but the argument can apply much more broadly. Analogous to Proposition 4 of the previous section, we have the following result.[9]

Proposition 5. Suppose preference types are i.i.d. with full support on $(0, 1)$, and there exists a sequence of players $\{m_i\}_{i\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(m_i \in B(n)\right) = 1$$

for each i. Then information aggregates.

2.6 Remarks

Asymptotic outcomes of sequential observational learning are now well understood, but important challenges remain. A significant gap in our knowledge concerns short-run dynamics and rates of learning in these models. Lobel et al. (2009) study learning rates in special cases with $|B(n)| = 1$ for each player n, linking the rate of learning to the tail of the private belief distribution. However, a more general characterization of learning rates and short-run outcomes is absent outside of particular examples. The complexity of Bayesian updating in a network makes this difficult, but even limited results would offer a valuable contribution to the literature.

[9] Note that players need not know precisely who is responding to a private signal. Knowing that some players have a positive probability of following a private signal is enough to statistically identify the state.

There is also little consideration in this literature of where information comes from. Mueller-Frank and Pai (2014) and Ali (2014) are exceptions. Each studies a version of the SSLM with a complete network in which players, rather than being endowed with a signal, must pay a cost to learn about the payoffs of the available actions. If these costs are arbitrarily low for some players, then the players learn the true state asymptotically; essentially, we can trade an assumption of strong exogenous signals for an assumption of sufficiently low costs to acquire strong signals.

Another challenge arises from the strong assumptions in the SSLM. Bayesian rationality demands much from players in these models, raising questions about how realistic such a representation of behavior is. Even in the sequential framework, general networks necessitate extremely complex reasoning. Though this is certainly cause for concern, the proofs of our results suggest these models can still provide a useful benchmark. Analysis centers on relatively simple heuristics, and we show that Bayesian players must perform at least as well. Selecting a neighbor to imitate and following the most popular choice among a large group are intuitive procedures that do not require perfect rationality to succeed. Thus, implicit in these results about Bayesian agents is a broader class of results on a whole range of heuristics. The absence of repeated decisions or strategic interactions in the SSLM is a more fundamental limitation, which forces a departure from this framework to address certain questions.

3. Repeated Linear Updating (DeGroot) Models

The sequential social learning models that we discussed in the previous section are so tractable because each agent makes only one decision, even though different agents make their decisions at different times. Models of this sort provide many insights, but they also constrain influence to flow in only one direction. In contrast, many of the most fundamental substantive questions about social learning are inherently about the *dynamics* of individual opinions and choices in a world where individuals make many decisions and can repeatedly influence one another. For example: Can network structure explain lasting disagreements in a segregated society, in which members of two groups repeatedly observe each other's views but persist in holding different opinions? When everyone's initial opinion can influence everyone else, whose opinions end up being particularly influential?

One fruitful approach to these questions, which we focus on in this section, models the dynamics of repeated updating without giving up tractability by having equations of motion that are linear in agents' estimates and stationary over time.

3.1 Framework

3.1.1 The Basic DeGroot Model

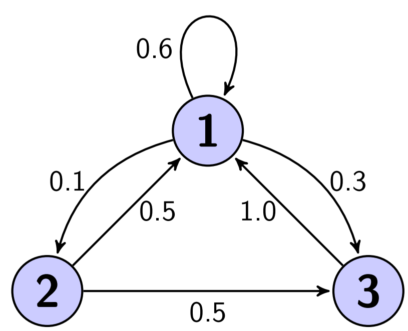
The basic idea behind *DeGroot repeated linear updating models* is that agents start out with some initial estimates, and then all agents update those estimates simultaneously at discrete times $t = 1, 2, 3, \ldots$ Someone's estimate in a given period is obtained by taking a weighted average of some others' estimates. More formally, let X be a convex subset of a vector space,[10] and let $N = \{1, 2, \ldots, n\}$ be a set of agents. The *estimate* or *opinion* of agent i at time t is written $x_i(t)$, and the updating rule for estimates is

$$x_i(t) = \sum_j W_{ij} x_j(t-1) \tag{3}$$

for all positive integers t; the initial values $x_i(0)$ are exogenous. Here $\mathbf{W}$ is an n-by-n matrix with nonnegative entries, with the property that every row sums to 1: for each i, we have $\sum_j W_{ij} = 1$. A typical entry, W_{ij}, is called the *weight* agent i places on agent j. One interpretation is that $\mathbf{W}$ represents a social network: each agent has immediate access, not to everyone's estimates, but only to those of a subset of agents. Those agents are the only j's for which W_{ij} is positive, and the weights represent how an agent averages the opinions she can observe. The simplest case is that she places equal weight on the previous opinion of everyone she observes, so that $W_{ij} = 1/d_i$ whenever W_{ij} is nonzero, where d_i is the number of agents whose estimates i can observe.

$$\mathbf{W} = \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.5 & 0 & 0.5 \\ 1.0 & 0 & 0 \end{bmatrix}$$

(a)



(b)

Figure 2: A weight matrix (a) and the corresponding weighted directed graph (b).

We can also go in the opposite direction, from an updating matrix to a network. Given any $\mathbf{W}$, we can view it as a graph with N as the set of nodes. There is a directed link, or arrow, from i to j if i places nonzero weight on j according to $\mathbf{W}$; we label that arrow by

[10] The simplest example is $X = \mathbb{R}$, and for simplicity, we will sketch many proofs only for this special case. But one of the virtues of DeGroot's formulation of this process is that we can also think of X as consisting, for example, of a set of probability distributions.

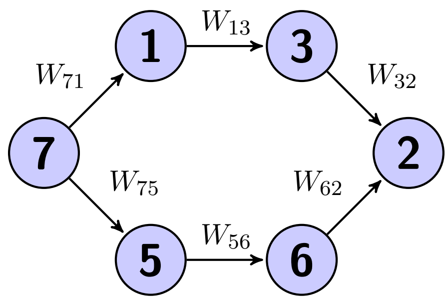


Figure 3: The sequences $s = (7, 1, 3, 2)$ and $s' = (7, 5, 6, 2)$ are both paths of length 3 from agent $i = 7$ to agent $j = 2$, and are therefore both members of the set N^4_{72}. If these are the only such paths, $(\mathbf{W}^3)_{ij} = w(\mathbf{W}, s) + w(\mathbf{W}, s') = W_{71}W_{13}W_{32} + W_{75}W_{56}W_{62}$.

the corresponding weight W_{ij}. (See Figure 2 for an example.) Thus, abusing terminology slightly, we sometimes identify a matrix with the corresponding network (weighted directed graph) and speak of a link in $\mathbf{W}$ from i to j.

3.1.2 Variations

Some variations on this basic model immediately come to mind. In (3), we can allow time-dependent weights, resulting in an updating rule $\mathbf{x}(t) = \mathbf{W}(t)\mathbf{x}(t-1)$ (see, for example, (Chatterjee and Seneta 1977)). The weights can also be stochastic, with a distribution that is either fixed or changing over time—a case we will discuss in detail later. Another variation permits each agent to hold some persistent "original" or "private" estimate $y_i \in X$, on which she always puts some weight, resulting in the updating rule of Friedkin and Johnsen (1999):

$$x_i(t) = \alpha_i \sum_j W_{ij}x_j(t-1) + (1 - \alpha_i)y_i.$$

Finally, there are related models with a discrete set of possible opinions, sometimes called "voter" models (see Mossel and Tamuz 2014 for details).

3.1.3 Matrix Powers and the Connection with Markov Chains

We can write the updating rule (3) in matrix notation as $\mathbf{x}(t) = \mathbf{W}\mathbf{x}(t-1)$, where $\mathbf{x}(t) = (x_i(t))_{i \in N} \in X^n$ is a vector stacking everyone's time-t estimates. Iterating this shows that

$$\mathbf{x}(t) = \mathbf{W}^t\mathbf{x}(0). \tag{4}$$

Thus, the evolution of the estimate vector $\mathbf{x}(t)$ essentially reduces to the dynamics of the matrix powers $\mathbf{W}^t$. Recall that each row of $\mathbf{W}$ sums to 1, and that the entries of this matrix are nonnegative. In other words, $\mathbf{W}$ is a row-stochastic (or Markov) matrix. The iterates $\mathbf{W}^t$ of Markov matrices have been extensively studied, because they capture the t-step transition probabilities of a Markov chain in which the probability of a transition from i to j is W_{ij}. (For introductions, see Seneta 2006 and Meyer 2000, Chapter 8.) Indeed, readers who know Markov chains can quickly absorb the main facts we present about the DeGroot model by reducing them to familiar facts about Markov chains.

There are two useful ways to think of the entry $(\mathbf{W}^t)_{ij}$ in addition to its definition as an entry of a matrix power. First, equation (4) shows that $(\mathbf{W}^t)_{ij} = \frac{\partial x_i(t)}{\partial x_j(0)}$; in words, $(\mathbf{W}^t)_{ij}$ is the derivative of i's time-t estimate with respect to j's initial estimate. In this sense, $(\mathbf{W}^t)_{ij}$ measures how much j influences i in t steps. The second way of thinking of $\mathbf{W}^t$ is as a sum over of various paths of indirect influence. Let N_{ij}^{t+1} be the set of all sequences of $t+1$ agents (i.e., elements of N) starting at i and ending at j. Any such sequence s is called a *walk* of t steps from i to j. We can associate this sequence with a product of t elements of $\mathbf{W}$: the weights in the network $\mathbf{W}$ we meet as we walk from i to j along the sequence s. For example, the sequence $s = (7, 1, 3, 2)$ in the set N_{72}^4 corresponds to the product $w(\mathbf{W}, s) := W_{71}W_{13}W_{32}$, which we call the *weight of s in* $\mathbf{W}$. We can prove inductively that

$$(\mathbf{W}^t)_{ij} = \sum_{s \in N_{ij}^{t+1}} w(\mathbf{W}, s). \tag{5}$$

In other words, $(\mathbf{W}^t)_{ij}$ adds up all the weights of t-step ($(t+1)$-agent) sequences that take us from i to j. These sequences are the conduits of j's influence on i in t steps of updating: i pays attention to somebody, who pays attention to somebody, ..., who pays attention to j. Each such sequence contributes to j's influence on i according to its weight in $\mathbf{W}$. A simple example of how all this looks in a concrete case is depicted in Figure 3.

A theme of the rest of this section is to study the powers of $\mathbf{W}$—when they converge to a limit, how fast, what this limit looks like—to derive substantive conclusions about social learning.

3.1.4 Foundations

In contrast to the sequential learning literature, the mechanics of DeGroot models came first and efforts at microeconomic foundations came later. Here we present two economic rationales for DeGroot-type rules. We defer a discussion of the history—and of some other rationales for the DeGroot model—to Section 3.5.1.

Persuasion Bias

To our knowledge, DeMarzo et al. (2003) were the first to discuss in detail how the DeGroot updating rule might arise in a quasi-Bayesian way from an imperfect optimization within a standard microeconomic model. (However, see Lehrer and Wagner 1981, who give some axiomatic foundations for iterated weighted averaging schemes.) Suppose agent i starts out with a private signal $\mu + \epsilon_i$, where μ is a normally distributed state that the agents are trying to estimate, and the ϵ_i are mean-zero, normally distributed errors that are independent of the state and of each other. Consider a case in which the prior distribution the agents hold about μ is diffuse, so that they have very imprecise prior information—indeed, consider an (improper) prior over μ that is uniform over the real line. Then the posterior expectations of μ conditional on the private signals alone are $x_i(0) = \mu + \epsilon_i$. Now suppose that before forming a time-t estimate, each agent observes the previous estimates, $x_j(t-1)$, of some subset of the others. Then Bayesian updating at $t = 1$ corresponds to (3) with suitable weights (Gelman et al. 2013, Sections 2.5 and 3.5; DeGroot 2005, Section 9.9): An agent optimally pools others' estimates with her own past estimate by taking a linear combination of all those estimates, with coefficients accounting for the different precisions of different agents' information.

In future periods, there is still something left to learn: If an agent is not connected to everyone, then her contacts' revised estimates convey information about what is known elsewhere in the network. It turns out that linear averaging is also the Bayesian updating rule in future periods, but the optimal weights change over time.[11] DeMarzo et al. (2003) motivate the DeGroot process with unchanging weights as a behavioral heuristic: It is complicated to revise the weights optimally, so agents stick with the weights of the first period. DeMarzo et al. (2003) also suggest that this captures a *persuasion bias* or *echo chamber* effect, much studied by social psychologists, in which people tend to be unduly swayed by things they hear repeatedly.

Myopic Best-Reply Dynamics

Another microfoundation imagines agents learning to play a game. Suppose agents take actions $x_i \in X$ and have payoffs making reaction functions linear in the actions of the others. For instance, suppose $\mathbf{W}$ is a matrix each of whose rows sums to 1; $W_{ii} = 0$; X is a normed space, and agents have the payoffs

$$u_i(x_1, x_2, \ldots, x_n) = -\sum_{j \neq i} W_{ij} \|x_i - x_j\|^2.$$

Then agent i's best response to a profile $\mathbf{x}_{-i} = (x_j)_{j \neq i}$ is $\sum_j W_{ij} x_j$. This is a simple coordination game: It is costly to make a choice (e.g., of a technology or a language) that differs from that of one's neighbors. Any action profile in which all agents take the same action is a Nash equilibrium of such a game. What equilibrium will be reached, and how long will it take? One approach to answering these questions is to consider myopic best-reply dynamics: In each period, players select best responses to last-period actions. This is a simple way for them to incorporate information from their surroundings and attempt to coordinate with their neighbors (i.e., to learn how to play the game). Such a rule gives rise to exactly the dynamics of the basic DeGroot model, described by (3); see Golub and

[11] For instance, if i heard j's initial information and j is not connected to anyone except i, then there is no reason for i to put any weight on j in the future. More generally, Bayesian agents adjust the weights they place on others in order to optimally incorporate new information, while trying not to overweight the old. See Mossel and Tamuz (2010) for details.

Jackson (2012).[12]

Other Foundations and a Critique

The microfoundations discussed for DeGroot updating are not fully satisfying. There is a lot of myopia and limited cognition—not to mention quite particular functional form assumptions—inherent in both the persuasion bias and the best-response foundations. Whether actual people behave as if these assumptions hold is an important empirical question about which the evidence is only beginning to come in (Corazzini et al. 2012; Chandrasekhar et al. 2015).

The DeGroot model is a continuing focus of study, despite these important caveats, because it allows us to understand the evolution of beliefs rather completely—both via intuitions and via formal results—as a function of the underlying network. A reason for this tractability is the connection with matrix powers and Markov chains described in Section 3.1.3, and the rest of this section explores some of what that connection opens up. Looking beyond this analysis, the hope is that it provides a useful benchmark for the study of other processes of repeated learning in networks.

3.2 The Long-Run Limit: Consensus Estimates and Network Centrality

Let us start with the basic DeGroot model of Section 3.1.1 with a fixed matrix $\mathbf{W}$ of updating weights. There are three basic questions concerning the long-run behavior of individuals' beliefs that we answer in this section:

(i) Does each individual's estimate settle down to a long-run limit? That is, does $\lim_{t\to\infty} x_i(t)$ exist for every i?

(ii) When there is convergence, are the limiting estimates the same across agents? In other words, *is there consensus in the long run* for all possible vectors of initial estimates?[13]

(iii) When there is a consensus in the long run, what is this consensus? How does it depend on the matrix $\mathbf{W}$ and the initial estimates?

It is question (iii) that offers the richest connection between outcomes and network structure; the answers to the first two questions set the stage for characterizing that connection.

[12] In some ways it is more appealing to think of each node i as a continuum of players, who all make the same observations and move simultaneously. What is observable about a node is the average action of its members. In this case, there is no need for the restriction $W_{ii} = 0$, because individuals can care about coordinating with those in their own node. Also, the assumption of myopic actions is more reasonable when each agent is negligible; see Section 4.1.

[13] Berger (1981) discusses a necessary and sufficient condition for a weighting matrix to generate consensus *for a single* $\mathbf{x}(0)$; this condition is in terms of both $\mathbf{W}$ and $\mathbf{x}(0)$. Our discussion here characterizes the $\mathbf{W}$ under which consensus is reached for all $\mathbf{x}(0)$, and thus this condition depends only on $\mathbf{W}$.

3.2.1 The Strongly Connected Case: Convergence to a Consensus

We first answer the prior questions in an important special case—that of *strongly connected* networks; the general case reduces to this one. The network $\mathbf{W}$ is *strongly connected* (or the matrix is *irreducible*) if any agent i has a directed path in the network $\mathbf{W}$ to any agent j.[14] Equivalently, it is impossible to partition the agents into two nonempty sets so that at least one of the sets has no directed link to the other in $\mathbf{W}$.

In strongly connected networks, the answer to (i) and (ii) is affirmative once we rule out a "small" set of network structures. To give an idea of the sort of sufficient condition we need, suppose some agent k has a positive self-weight, so that $W_{kk} > 0$. We can prove via (5) in Section 3.1.3 that strong connectedness of the network and this positive self-weight assumption imply that, *for some* q, *we have* $(\mathbf{W}^q)_{ij} > 0$ *for every* i *and* j. Any strongly connected matrix $\mathbf{W}$ having this property is called *primitive* (Seneta 2006, Definition 1.1).[15] It is a standard fact that $\mathbf{W}$ is primitive if and only if $\lim_{t\to\infty} x_i(t)$ exists for each i and the value of the limit is independent of i.[16]

3.2.2 The Strongly Connected Case: Influence on the Consensus

Answering (iii) highlights the most interesting connection between the DeGroot updating process and the structure of the network in which agents communicate (i.e., $\mathbf{W}$): that limiting consensus estimates are a linear combination of various agents' initial estimates, weighted by those agents' *network centralities*.

Heuristically, from (4) we can write the following equation for the (constant) vector of limiting estimates: $\mathbf{x}(\infty) = \mathbf{W}^\infty \mathbf{x}(0)$. We can make this rigorous: There is a limit $\mathbf{W}^\infty$ of the sequence $(\mathbf{W}^t)_t$ that satisfies this equation for any $\mathbf{x}(0)$. Moreover, since we have already seen that, for any starting estimates, all agents converge to some consensus, it follows that all rows of $\mathbf{W}^\infty$ must be equal to the *same* row vector, which we will call $\boldsymbol{\pi}^\mathsf{T}$. Thus, a typical entry of $\mathbf{x}(\infty)$ is equal to

$$x(\infty) = \sum_i \pi_i x_i(0). \tag{6}$$

In words, the consensus estimate is a *linear* combination of initial estimates, and the coefficients π_i do not depend on the initial estimates $\mathbf{x}(0)$, but only on the network. The coefficient

[14] Recall the interpretation of $\mathbf{W}$ as a network from Section 3.1.1. For basic definitions of graph-theoretic notions, see Jackson (2010) or other textbooks that discuss directed graphs.

[15] The general necessary and sufficient condition for primitivity of a strongly connected $\mathbf{W}$ is aperiodicity: The greatest common divisor of all cycles (walks that return to their starting point) is 1.

[16] To show that primitivity implies convergence, first note that since agents form their estimates by taking convex combinations, the sequences $\max_i x_i(t)$ and $\min_i x_i(t)$ are each monotone in t. Thus the maximum and minimum must converge to the same point as long as the distance between them gets arbitrarily small. Let $W_{\min} > 0$ be the minimum of the entries in $\mathbf{W}^q$. As the two most extreme agents put weight at least $W_{\min}$ on each other in q steps of updating, the difference between the maximum and minimum entries in $\mathbf{x}$ decreases by at least a factor of $1 - W_{\min} < 1$ every q steps. For more discussion, as well as a proof of the converse, see statements 8.3.10 and 8.3.16 in Meyer (2000).

π_i measures how much i's initial estimate affects the consensus.

We explore in some detail how these *influence coefficients* π_i relate to the underlying network $\mathbf{W}$. The vector $\boldsymbol{\pi}$ satisfies the following equation[17] with $\lambda = 1$:

$$\boldsymbol{\pi}^\mathsf{T}\mathbf{W} = \lambda\boldsymbol{\pi}^\mathsf{T}. \tag{7}$$

Indeed, $\boldsymbol{\pi}^\mathsf{T}$ is the *unique* nonnegative, nonzero vector that sums to 1 and satisfies this equation (for any value of λ). Such a vector is called a left-hand *eigenvector centrality* of $\mathbf{W}$, and its entries are called agents' (left-hand) eigenvector centralities. A typical row of the system of equations (7), with $\lambda = 1$, reads $\pi_i = \sum_j W_{ji}\pi_j$. In words, i's influence is a weighted sum of the influences of those who put weight on i, with π_j weighted by how much weight j puts on i.

In short, one of the nicest features of the DeGroot model is that we can express the influence of each agent's initial estimate on the final consensus in terms of agents' centralities according to a natural and much-studied measure.[18] Recalling the connection with Markov chains, we also observe that $\boldsymbol{\pi}$ is the unique stationary distribution of the Markov chain described by $\mathbf{W}$.

We now summarize our findings on both convergence and the form of the limit.

Proposition 6. The following hold for all values of $\mathbf{x}(0)$ if $\mathbf{W}$ is strongly connected and primitive:

(i) Each $x_i(t)$ converges to a limit as $t \to \infty$.

(ii) They all converge to the *same* limit.

(iii) This limit is equal to $\sum_i \pi_i x_i(0)$, where π_i is i's left-hand eigenvector centrality in $\mathbf{W}$.

See DeGroot (1974) or Statement 8.3.10 in Meyer (2000) for a formal proof. There is a case in which influence can be calculated very explicitly—that of *reversible weights*: Suppose we are given some connected weighted graph $\mathbf{G}$ in the form of a symmetric adjacency matrix—which may describe, say, how much time various pairs spend interacting bilaterally. Assume $W_{ij} = G_{ij}/\sum_i G_{ij}$, so that the influence of j on i is equal to the share of i's time that is spent with j. In that case, one can check that $\pi_i = \sum_j W_{ij}/\sum_{j,k} W_{jk}$.[19] In other words, an agent's centrality is proportional to the amount of her interaction as a fraction of the total interaction in the network.[20]

[17]That it should satisfy this equation is intuitive if we believe (at least heuristically) that $\mathbf{W}^\infty\mathbf{W} = \mathbf{W}^\infty$, and then recall that a typical row of $\mathbf{W}^\infty$ is $\boldsymbol{\pi}^\mathsf{T}$.

[18]Eigenvector centrality is defined rather abstractly, as a vector $\boldsymbol{\pi}$ that satisfies (7), possibly with a proportionality constant. See Bonacich (1987) and Jackson (2010, Chapter 2) for some classic motivations for this kind of definition. A lot is known about the structure of such $\boldsymbol{\pi}$. For some insight on how agents' centralities relate to simple properties of the network (and also an application of network centrality ideas to macroeconomics), see Acemoglu et al. (2012). For the general comparative statics of $\boldsymbol{\pi}$ in $\mathbf{W}$, see Schweitzer (1968) and Conlisk (1985).

[19]One can calculate directly that $\boldsymbol{\pi}^\mathsf{T}\mathbf{W} = \boldsymbol{\pi}^\mathsf{T}$ holds, and then use the fact that a strongly connected $\mathbf{W}$ can have only one eigenvector (up to scale) corresponding to the eigenvalue 1.

[20]The reason for the name *reversible weights* is that, in Markov chain language, this corresponds to the case of $\mathbf{W}$ being a reversible chain. See Levin et al. (2009, Section 9.1).

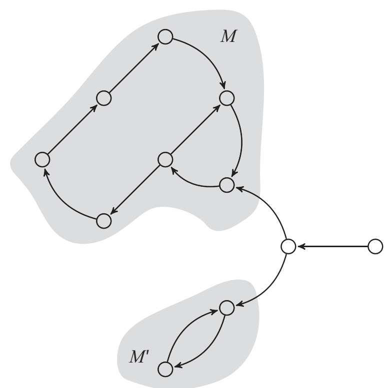


Figure 4: An illustration of a network that is not strongly connected. Here M and M' are closed communicating classes (inside the class, one can follow a directed path from any node to any other), and the remaining nodes are in no closed communicating class.

3.2.3 Beyond Strong Connectedness

If a network is not strongly connected, we can reduce the study of its steady state to the study of convergence to consensus in suitable strongly connected subgraphs. In particular, we can view any directed graph as a disjoint union of strongly connected subgraphs that have no links exiting them (called *closed communicating classes*) and some remaining nodes. (See Figure 4 for an example.) A closed communicating class, by definition, cannot be influenced by anything that goes on outside of it, so the analysis of how its agents' estimates converge reduces to what we have studied above, restricted to that class.

Moreover, we can see that, for any i and large enough q, the weight $(\mathbf{W}^q)_{ij}$ is positive only if j is in a closed communicating class.[21] Thus, the estimates of agents outside all the closed communicating classes are eventually convex combinations of the consensus beliefs of the agents in various closed communicating classes. The details of how this works are given in Meyer (2000, p. 698) and Golub and Jackson (2010, Theorem 2).

From this it follows that if there are two or more closed communicating classes, long-run consensus (the issue contemplated by question (ii)) no longer obtains. If there is only one closed communicating class and $\mathbf{W}$ restricted to that class is primitive, consensus does obtain. Convergence of estimates within individual closed communicating classes to consensus requires only primitivity when $\mathbf{W}$ is restricted to those classes.

An important take-away is that ignoring anyone outside itself—being a closed communicating class—gives a group a lot of power under the DeGroot model. Such a group can certainly sustain its own views in the sense that its long-run consensus depends only on its own initial opinions. Moreover, if that group receives attention from outside, it also gains a decisive influence on the more malleable agents in its society. On the one hand, this feature

[21]In view of the discussion of Markov chains in Section 3.1.3, this corresponds to the statement that if a Markov chain starts outside the closed communicating classes and proceeds with transition probabilities given by $\mathbf{W}$, eventually it will be found, with arbitrarily high probability, inside some closed communicating class. Proving this is a good exercise.

highlights a quirk of the DeGroot model: Very small differences in updating weights (e.g., whether an inward-looking group gets a tiny amount of attention or no attention from the outside world) can make a huge—indeed, a discontinuous—difference in the model's asymptotic predictions.[22] On the other hand, this feature cleanly captures the fact that stubborn, inward-looking groups have a particularly durable internal inertia and, as long as they are not totally ignored by those outside them, substantial external influence. This observation seems consistent with some observations of political and academic persuasion (DeMarzo et al. 2003).

3.2.4 The Large-Population Limit: When is Consensus Correct?

Let us assume networks are strongly connected and primitive (so that there is consensus), and consider again the setting introduced in the discussion of persuasion bias (Section 3.1.4). There is a true state μ, and we assume agents start out with noisy estimates of it and are interested in learning its true value. (Normality is not important for this application.) Are large societies of DeGroot updaters able to aggregate information so well that the consensus becomes concentrated tightly around the truth? More formally, take an infinite sequence of networks $(\mathbf{W}^{(n)})_{n=1}^{\infty}$, with $\mathbf{W}^{(n)}$ having n agents making up a set $N^{(n)}$. Suppose initial estimates $x_i^{(n)}(0)$ in network n are noisy estimates of the true state of the world, according to the stochastic specification given in the discussion of persuasion bias in Section 3.1.4, and let $x^{(n)}(\infty)$—now a random variable—be the consensus reached in network n. Can we say that the $x^{(n)}(\infty)$ converge in probability to μ as n grows? If we can, then in a certain asymptotic sense the agents are as good at learning as someone who had access to everyone's initial information and aggregated it optimally.

Let us assume that the variances of the noise terms ϵ_i are bounded both above and below. Then we have the following result.

Proposition 7 (Golub and Jackson 2010)**.** Under the model just described with random initial beliefs, the $x^{(n)}(\infty)$ converge in probability to μ if and only if $\lim_{n\to\infty} \max_i \pi_i^{(n)} = 0$.

To prove the "if" direction, we use the expression for the limit consensus estimate given by (6) to say that $\mathrm{Var}[x^{(n)}(\infty) - \mu] = \sum_i (\pi_i^{(n)})^2 \,\mathrm{Var}[\epsilon_i]$. This converges to 0 if and only if $\lim_{n\to\infty} \max_i \pi_i^{(n)} = 0$ (here we are using our assumption about the variances of the ϵ_i); then, using Chebyshev's inequality, we conclude that $x^{(n)}(\infty)$ converges in probability to its mean, μ. The converse uses the same variance calculation, and is left as an easy exercise.

To summarize the key point, large societies achieve asymptotically exact estimates of the truth if and only if the influence of the most influential agent decays to 0 as society grows large. Without this, the idiosyncratic noise in someone's initial belief plays a nontrivial role in everyone's asymptotic belief, and asymptotic correctness is not achieved.

To make the condition for failure of good aggregation more concrete, we give a corollary.

[22] This is perhaps a reason to focus more on predictions of the DeGroot model for large but fixed values of t—which are nicely continuous in $\mathbf{W}$–than on the literal $t = \infty$ predictions, whose behavior is analytically cleaner but in many ways more peculiar.

Corollary 2. Suppose we can find an m and $\epsilon > 0$ so that, for each n, there is a group $L^{(n)} \subseteq N^{(n)}$ of m or fewer "opinion leaders," with each individual $i \in N^{(n)} \setminus L^{(n)}$ giving some leader $\ell \in L^{(n)}$ at least ϵ weight ($W_{i\ell}^{(n)} \geq \epsilon$). Then individuals' estimates do not converge in probability to μ.

The proof works by manipulating (7) to show that $\lim_{n\to\infty} \max_i \pi_i^{(n)} = 0$ does not hold, and then using Proposition 7. Thus, societies with a small group that influences everyone cannot achieve asymptotic correctness of beliefs. These issues are explored in more detail in Golub and Jackson (2010).

3.3 Speed of Convergence to the Long-Run Limit: Segregation and Polarization

When estimates converge to a consensus (or to some other steady state we can characterize), it is important to know how fast this happens. For practical purposes, consensus is often irrelevant unless it is reached reasonably quickly. If it takes thousands of "rounds" of updating to reach a consensus, the model's limit predictions are unlikely to be useful. To say it another way, even if a network satisfies conditions (e.g., strong connectedness) that ensure convergence in the long run, we may still empirically observe disagreement. In this case, it is the medium-run (as opposed to $t = \infty$) behavior of the system that is practically relevant, and we would like to theoretically understand how network structure relates to medium-run disagreement. Some obvious questions arise:

(i) How long does it take for differences in estimates to become "small"?

(ii) What do agents' estimates look like as they are converging to consensus?

(iii) What network properties correspond to fast or slow convergence?

As a preliminary, note that how fast consensus is reached depends on both the network and the starting estimates $\mathbf{x}(0)$. In the trivial case where the initial estimates are all identical, consensus is reached instantly, regardless of the network. In the general case, a full analysis of the time it takes estimates to converge would deal with the dependence of this outcome on $\mathbf{W}$ and on $\mathbf{x}(0)$ jointly. We focus on understanding how properties of the *network* affect convergence time, and for that reason we will often think about worst-case convergence time: roughly, how many rounds of updating it takes for differences of opinion to be guaranteed to be "small" for *any* $\mathbf{x}(0)$ we might start with.

3.3.1 A Spectral Decomposition of the Updating Matrix

We saw earlier, in Section 3.1.3, that powers of $\mathbf{W}$ are important. We now build on that with a convenient decomposition. For simplicity, we restrict attention to strongly connected, primitive updating matrices $\mathbf{W}$, as in Section 3.2.1.

Lemma 3. For generic[23] $\mathbf{W}$, we may write

$$\mathbf{W}^t = \sum_{\ell=1}^{n} \lambda_\ell^t \mathbf{P}_\ell, \tag{8}$$

where the following properties are satisfied:

(a) The numbers $\lambda_1 = 1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the n distinct eigenvalues of $\mathbf{W}$, ordered from greatest to least according to modulus.

(b) The matrix $\mathbf{P}_\ell$ is a projection operator corresponding to the nontrivial, one-dimensional eigenspace of λ_ℓ.

(c) $\mathbf{P}_1 = \mathbf{W}^\infty$ and $\mathbf{P}_1\mathbf{x}(0) = \mathbf{x}(\infty)$;

(d) $\mathbf{P}_\ell \mathbf{1} = \mathbf{0}$ for all $\ell > 1$, where $\mathbf{1}$ is the vector of all 1's.

The import of (c) is that we have encountered $\mathbf{P}_1$ before, in Section 3.2.1. It is equal to $\mathbf{W}^\infty$ and corresponds to the eigenvalue $\lambda_1 = 1$ (which, as (a) states, is always an eigenvalue of any row-stochastic matrix $\mathbf{W}$). All other eigenvalues are strictly smaller. In other words, the leading term of (8) corresponds to the steady state we studied in Section 3.2.1, and the other terms are deviations from that asymptotic steady-state weighting matrix.

3.3.2 Speed of Convergence to Consensus

How fast the steady-state summand $\mathbf{P}_1$ comes to dominate depends mainly on $|\lambda_2|$, the magnitude of the second-largest eigenvalue of $\mathbf{W}$. When this number is not too large (say, $|\lambda_2| = 0.6$), all the terms after the first term in (8) become negligible as soon as t is at all large (e.g., $t \geq 10$). Thus, the quantity $1 - |\lambda_2|$, called the *absolute spectral gap*, is an important measure of this system's tendency to equilibrate (Levin et al. 2009, Section 12.2). Systems with a small spectral gap (large second eigenvalue) exhibit very slow decay of the nonstationary part in the worst case. The following is a simple formal version of this statement:

Proposition 8. Consider the DeGroot updating process given by (3). For generic $\mathbf{W}$,

$$\frac{1}{2}|\lambda_2|^t - (n-2)|\lambda_3|^t \leq \sup_{\mathbf{x}(0)\in[0,1]^n} \|\mathbf{x}(t) - \mathbf{x}(\infty)\|_\infty \leq (n-1)|\lambda_2|^t.$$

[23] In this lemma, drawing a $\mathbf{W}$ from some measure absolutely continuous with respect to Lebesgue measure buys us a lot: for instance, the luxury of not having to deal with repeated eigenvalues, which—as will become apparent—would require substantial extra bookkeeping. This is mostly a convenience for exposition; Proposition 8, for example, holds with only minor adjustments without assuming any genericity (see Debreu and Herstein 1953, Theorem V and Meyer 2000, Section 7.9 for a flavor of the arguments). However, some of the results in DeMarzo et al. (2003) that we will discuss do break down under highly symmetric network structures. It is a good exercise to replicate the ensuing discussion, replacing uses of this lemma with the corresponding facts about the Jordan canonical form, to see what survives and what requires major adjustment.

Here $\|\cdot\|_\infty$ is the supremum norm, so that $\|\mathbf{x}(t) - \mathbf{x}(\infty)\|_\infty$ is the largest deviation from consensus experienced by any agent. The proof of Proposition 8 is a good exercise in matrix analysis.[24] The result provides a succinct answer to question (i): $|\lambda_2|^t$ is a precise estimate of how much deviation from consensus this network permits at time t. This basic insight has been applied in a variety of ways; for example, in an elaboration of the DeGroot model, Acemoglu et al. (2010) use it to characterize the long-run influence of "forceful" agents (who sway others much more than they themselves are swayed).

We will now explore the λ_2 term of (8) in more detail, to better understand both its magnitude and its structure in terms of the underlying network.

3.3.3 One-Dimensionality of Deviations from Consensus: A Left–Right Spectrum

Just as the $\ell = 1$ term in (8) describes the steady-state component to which estimates are converging, the $\ell = 2$ term of (8) describes the dominant term in the *deviation* from consensus (i.e., in what is left over after we subtract the steady-state component). This $\ell = 2$ term can be seen as corresponding to a *metastable* or *medium-run* state in which most disagreement is gone but a critical persistent part remains.[25]

In view of this, let us now dig deeper into the structure of $\mathbf{P}_2$. Since it is an operator that projects onto a one-dimensional space, we may write $\mathbf{P}_2 = \boldsymbol{\sigma}\boldsymbol{\rho}^\mathsf{T}$, where $\boldsymbol{\rho}^\mathsf{T}$ is a left-hand eigenvector of $\mathbf{W}$ corresponding to eigenvalue λ_2 and $\boldsymbol{\sigma}$ is a right-hand eigenvector of $\mathbf{W}$ corresponding to eigenvalue λ_2.[26]

As a consequence, once t is large enough that $|\lambda_3|^t$ is small relative to $|\lambda_2|^t$, the difference $\mathbf{x}(t) - \mathbf{x}(\infty)$ is essentially $\lambda_2^t\boldsymbol{\sigma}(\boldsymbol{\rho}^\mathsf{T}\mathbf{x}(0))$. Thus, if λ_2 is a positive real number, *individual i's deviation from consensus is proportional to σ_i, irrespective of* $\mathbf{x}(0)$.[27] DeMarzo et al. (2003) note a striking interpretation of this: Across many different issues, the ordering of agents' medium-run views is determined by a single, network-based number—a position σ_i on a left-right spectrum. For example, if estimates are real numbers (so $X = \mathbb{R}$), then agents' deviations from consensus on a given issue are either ordered the same as σ_i or in the opposite order (depending on whether $\boldsymbol{\rho}^\mathsf{T}\mathbf{x}(0)$ is positive or negative). More generally, $\boldsymbol{\rho}^\mathsf{T}\mathbf{x}(0) \in X$ determines the "axis" of disagreement if initial estimates are given by $\mathbf{x}(0)$. If $X = \mathbb{R}$, this is a scalar, but in general $\boldsymbol{\rho}^\mathsf{T}\mathbf{x}(0)$ is a vector; in the medium run, all deviations from consensus are proportional to it.

[24] The key equation is $\mathbf{x}(t) - \mathbf{x}(\infty) = \sum_{\ell=2}^{n} \lambda_\ell^t \mathbf{P}_\ell \mathbf{x}(0)$, which follows from (8) and Lemma 3. The upper bound follows by applying the triangle inequality to this equation. The lower bound is obtained by using the same equation for a suitable choice of $\mathbf{x}(0) \in [0,1]^n$: Take a nonzero vector $\mathbf{z} \in \text{span}(\mathbf{P}_2)$, and assume (by scaling it) that its largest entry is equal to exactly $1/2$; let $\mathbf{x}(0) = \frac{1}{2}\mathbf{1} + \mathbf{z}$, and use Lemma 3 along with standard norm inequalities.

[25] We are in this regime after enough time has passed for the $\ell \geq 3$ terms in (8) to die away, but not the $\ell = 2$ term.

[26] See Meyer (2000, statements 7.2.9 and 7.2.12) for details. Since $\mathbf{P}_2\mathbf{1} = \mathbf{0}$ by Lemma 3(d), we know that $\boldsymbol{\rho}^\mathsf{T}\mathbf{1} = \mathbf{0}$ (i.e., that the entries in $\boldsymbol{\rho}$ add up to 0).

[27] A correction of DeMarzo et al. (2003) that is due to Taubinsky (2011).

3.3.4 How Does the Deviation from Consensus Depend on Network Structure?

The decomposition presented in Section 3.3.1 opens the door to many neat mathematical results. It yields a rich set of statistics we can use to think about polarization and deviation from consensus, and we can compute these statistics efficiently. On the other hand, what we have seen so far leaves something to be desired. We would like more concrete, hands-on insights about how the "visible," geometric structure of the social graph relates to the persistence of disagreement. Fortunately, there is a large field of applied mathematics, particularly probability theory, devoted to studying this.[28] The basic take-away is that what makes networks slow to converge is being *segregated*. There are many different ways to capture this. We mention two of them informally, and refer the reader to relevant studies for the technical details.

The *bottleneck ratio*, *Cheeger constant*, or *conductance* is defined as

$$\Phi_\star(\mathbf{W}) = \min_{\substack{M \subseteq N \\ \pi(M) \leq \frac{1}{2}}} \frac{\sum_{i \in M, j \notin M} \pi_i W_{ij}}{\sum_{i \in M} \pi_i}.$$

The bottleneck ratio is small if there is some group (having at most half the influence in society) that pays a small amount of attention outside itself, relative to its influence. The attention or weight summed in the numerator is weighted by influence. The situation where the bottleneck ratio is small corresponds to the existence of a bottleneck. In the case of reversible communication (recall the end of Section 3.2.2), the second-largest eigenvalue of $\mathbf{W}$ can be bounded on both sides in terms of this bottleneck ratio (Levin et al. 2009, Theorem 13.14). That, in turn, yields bounds on the decay of disagreement (recall Proposition 8). For a sophisticated use of these sorts of bounds in a paper on the DeGroot model, see Acemoglu et al. (2009).

Another approach is to think of segregation probabilistically. Imagine, for example, that there are two groups of equal size (say, boys and girls). Friendships happen within a group with probability p_s and between groups with probability p_d. Given these probabilities, friendships are independent across pairs. The matrix $\mathbf{W}$ is then formed based on the friendship graph as described at the end of Section 3.2.2. In that case, it turns out that we can characterize the rate of convergence quite precisely. In particular, λ_2 converges in probability to $\frac{p_s}{p_d} - 1$ as the random network grows large. By Proposition 8, we can convert this into a bound on the worst-case disagreement at any particular time. This gives a clean way of saying that in a simple model of social segregation, it is segregation—and not network density, or anything else—that makes all the difference for the speed of convergence (Golub and Jackson 2012; Chung and Radcliffe 2011).

One more fact worth noting: The metastable structure of disagreement discussed in Section 3.3.3 is related to network structure. For example, in the two-group random graph just

[28] In probability theory and statistics, this question comes up in seeking to characterize the *mixing time*, a measure of how quickly Markov chains equilibrate and reach their stationary distributions—which is important in Markov Chain Monte Carlo statistical methods. See Levin et al. (2009) for details.

discussed, agents within each group converge to something resembling an internal consensus (with essentially all agents in one group being on the same side of the eventual consensus). Then we can approximate the situation with just two nodes (corresponding to the two groups) communicating bilaterally until the whole network reaches a consensus. For more on this, see Golub and Jackson (2012).

The connection between the iteration of Markov matrices and the structure of the corresponding network has spawned a huge body of literature, surveyed in book-length treatments. For more, see Levin et al. (2009), Montenegro and Tetali (2006), and Aldous and Fill (2002).

3.4 Time-Varying Updating Matrices

The most restrictive and unrealistic feature of the DeGroot model as we have presented it is that updating weights remain fixed over time and are deterministic. Relaxing this assumption is a major concern of the literature on this model, especially in statistics and engineering, going back to Chatterjee and Seneta (1977).

It turns out that it is fairly easy to extend the DeGroot model to a richer one, the *stochastic DeGroot model*, in which the weights agents use are stochastic. For this section, we will assume that X is a compact, convex subset of a normed vector space. Suppose that $\mathbf{x}(t) = \mathbf{W}(t)\mathbf{x}(t-1)$ and the matrices $\mathbf{W}(t)$ are independent and identically distributed random variables. Now the $\mathbf{x}(t)$ are also random variables, but they can be analyzed using what we already know. Indeed, define $\overline{\mathbf{x}}(t) = \mathbb{E}[\mathbf{x}(t)]$ and $\overline{\mathbf{W}} = \mathbb{E}[\mathbf{W}(t)]$. Because expectation commutes with matrix multiplication for independent matrix-valued random variables, we have

$$\overline{\mathbf{x}}(t) = \mathbb{E}[\mathbf{x}(t)] = \mathbb{E}[\mathbf{W}(t)\mathbf{W}(t-1)\cdots\mathbf{W}(1)\mathbf{x}(0)] = \overline{\mathbf{W}}^t\mathbf{x}(0). \tag{9}$$

In words, the *expectation* of the DeGroot process follows the law of motion of a nonrandom DeGroot process with updating matrix $\overline{\mathbf{W}}$. Thus, we know immediately that if the process $\overline{\mathbf{x}}(t)$ does not converge to a consensus vector $\overline{\mathbf{x}}(\infty)$ for all profiles of initial estimates, then neither can $\mathbf{x}(t)$ converge in probability to any random vector of consensus beliefs.[29] Remarkably, the converse also holds: If the $\overline{\mathbf{x}}(t)$ converge to a consensus $\overline{\mathbf{x}}(\infty)$ for all profiles of initial estimates, then all agents' estimates in the random updating process converge almost surely to some (random) consensus.[30] There is even a neat condition characterizing when

[29]That is, the random beliefs cannot converge to a consensus even in the weakest sense of convergence for random variables. This fact relies on the equivalence between L^1 convergence and convergence in probability for random variables taking values in a compact set.

[30] We will give the flavor of a direct argument, assuming $\overline{\mathbf{W}}$ is strongly connected. The essential idea is to focus on the expectation of the difference between the maximum and minimum estimates in $\mathbf{x}(t)$. Since $\overline{\mathbf{W}}$ is primitive (recall from Section 3.2.1 that this is equivalent to $\overline{\mathbf{x}}(t)$ converging), a stochastic version of the argument we gave for convergence in Section 3.2.1 shows that this expected difference decreases over time to 0. Since the difference is a bounded random variable, it must also converge to 0 almost surely (see, for example, Acemoglu et al. 2010, Theorem 1). This is easy to extend to the case in which $\overline{\mathbf{W}}$ is not strongly connected because the assumed condition on $\overline{\mathbf{W}}$ means it can have only one closed communicating class (recall Section 3.2.3).

$\overline{\mathbf{x}}(t)$ converges to a consensus vector for all starting beliefs: that all eigenvalues of $\overline{\mathbf{W}}$ except λ_1 (which is equal to 1) are less than 1 in modulus. To summarize:

Proposition 9. The following are equivalent:

(i) All eigenvalues of $\overline{\mathbf{W}}$ except λ_1 (which is equal to 1) are less than 1 in modulus.

(ii) The process described by (9) converges to a consensus limit for all values of $\overline{\mathbf{x}}(0)$.

(iii) For any value of $\mathbf{x}(0)$, the random variables $\mathbf{x}(t)$ in the stochastic DeGroot model converge almost surely to a (random) consensus $\mathbf{x}(\infty)$.

For detailed discussions of all this, see Tahbaz-Salehi and Jadbabaie (2008). Using (9) and parallelling Section 3.2.2, we can say that $\mathbb{E}[x_i(\infty)] = \sum_i \overline{\pi}_i x_i(0)$, where $\overline{\boldsymbol{\pi}}^{\mathsf{T}}$ is the left-hand eigenvector of $\overline{\mathbf{W}}$ corresponding to the eigenvalue 1 (i.e., an influence vector). This gives us some information about the consensus.

To analyze the medium-run behavior of the random process, one can build on the analysis of Section 3.3, moving between expectations and realized random estimates in a way analogous to the above treatment of the long run.

Independent and identically distributed updating matrices are the simplest way to relax the assumption of constant weights, but there are many other directions that have been explored. DeMarzo et al. (2003) study a version of the DeGroot model in which the weights agents place on themselves change over time, while the relative weights placed on others remain the same. Chatterjee and Seneta (1977) explore some basic issues of convergence in a model where weights are changing over time. As we note in the next section, there are elaborations of the model in which the weights actually depend on others' opinions.

3.5 Remarks

3.5.1 Some History and Related Literatures

To our knowledge, the social psychologist John French (1956) was the first to discuss a special case of the DeGroot model—one with each agent placing equal weights on her contacts—as a way to think about the evolution of opinions over time. He motivated his early formulation using ideas from physics, particularly the balance of "forces" of influence. French's ideas were developed by Harary (1959), who recognized that French's process was related to the mathematics of Markov chains. Harary generalized French's results on convergence by using the theory of directed graphs, but continued working with a model in which all agents place equal weights on their contacts. DeGroot (1974) appears to have been the first to write a fully general version of the process, with arbitrary weights, as a model of opinion updating, and to point out the connection between consensus opinions and the stationary distribution of a corresponding Markov chain (recall Section 3.2.2).

There are a variety of other names for the DeGroot model, corresponding to expositions in other literatures. In philosophy, it is often called the Lehrer–Wagner model, for Lehrer and Wagner (1981) worked on a related model around the same time as DeGroot. They focused on justifying the influence coefficients (recall Section 3.2.2) as a normatively reasonable scheme for aggregating views in a network of peers. As we have mentioned, Friedkin and Johnsen (1999), motivated by sociological theories of disagreement, studied versions of this model in which each agent persistently weights an "initial" or otherwise fixed opinion.

A recent literature studies versions of the DeGroot model in which opinions that are too far from one's own are weighted little or not at all (Hegselmann and Krause 2002; Lorenz 2007). In engineering and control theory, there is a large literature on "gossip" algorithms, which use DeGroot-type rules as a means of pooling information or synchronizing behavior across devices (see Shah 2009 for a survey). This literature extends and generalizes many aspects of the classic models presented above. For example, Moreau (2005) considers nonlinear updating rules, and gives generalizations of the conditions for consensus discussed above.

3.5.2 A Review and a Look Ahead

In some ways, the DeGroot model is an intuitive and reasonable heuristic—as we've seen formally in sections 3.1.4 and 3.2.4—but we do not have a tight characterization of when it is, in some sense, the "right" or "best" feasible heuristic.[31] Our empirical knowledge is also incomplete: The literature has only begun to explore how well the DeGroot rule fits the behavior of real people. Both of these issues present obvious avenues for further study, theoretical and empirical. In this section we have tried to demonstrate the DeGroot model's great tractability and its capacity to produce social learning dynamics with a rich but manageable structure. Because of these features, the DeGroot model has become an important benchmark with which to compare other learning dynamics, and a starting point for more sophisticated imperfectly rational models. For an example of the latter, see Section 4.2.

4. Repeated Bayesian Updating

Like the DeGroot model studied in Section 3, the models we are about to present have agents revising actions and beliefs repeatedly. But they are different from the DeGroot model in seeking, like the sequential models of Section 2, to accommodate potential coarseness of communication (e.g., observations of a binary choice) and more rational learning rules. While

[31] Carroll (2015) is an inspiring example of how nonstandard modeling of agents' optimization problems can rationalize linear rules in quite a different (incentive theory) setting.

the dynamics become more complicated, some of the main substantive conclusions—such as convergence to a common opinion or action—survive.

We adopt a standard notation across each model in this section. Time is discrete, the possible states of the world are $\theta \in \Theta$, and each player n chooses an action $x_n(t)$ in period t. We represent the network as a directed graph $G = (V, E)$, where the set of vertices V is either finite or countably infinite, and we write $B(n) = \{m : nm \in E\}$ for player n's set of neighbors.

4.1 Myopic Updating

One popular approach to modeling repeated updating of beliefs assumes that players ignore future effects of their decisions. Players myopically choose the best action today based on current beliefs without regard for any effects on other players or future information availability. Beliefs are still rational given the information players observe, so we can think of these models as an approximation to fully rational behavior with heavy discounting. Alternatively, we can think of each node in a network as corresponding to a continuum of identically informed agents, with only the average action of the continuum being observed by others. In this case, individuals cannot affect anyone's observation or decision, and again agents simply take the best action given their beliefs.

4.1.1 Continuous Actions: Revealing Beliefs

Geanakoplos and Polemarchakis (1982) and Parikh and Krasucki (1990) introduced the first models of this type. In every period, each player takes an action $x_n(t) \in [0, 1]$ that perfectly reveals her belief at the beginning of that period about the probability of some event E. Players have a common prior and are endowed with different information about the state. A central insight (subsequently developed further in work by Mueller-Frank 2013; 2014) is that each player's beliefs allow her neighbors to infer what that player could have seen last period, and therefore narrow down the set of states. When this process stops, it is common knowledge between any pair of neighbors what their beliefs are. By an extension of Aumann's (1976) reasoning on the impossibility of agreeing to disagree, it cannot be common knowledge that these beliefs are different, so consensus is reached in any connected network. Moreover, when there are only finitely many states in Θ, a player's belief can change at most finitely many times, so the opinions converge in finite time. In the Gaussian environment of Section 3.1.4, in which agents receive normal signals about a normally distributed state, similar reasoning works. There, finiteness of the state space is replaced by finite dimensionality of the unknowns (see Theorem 3 of DeMarzo et al. 2003 and Mossel and Tamuz 2010).

We can even say something about the correctness of the consensus in each of the settings just discussed. When there are finitely many states, it takes nongeneric priors for different knowledge to lead to the same posterior probability of any event. Therefore, it holds generically that agents' beliefs perfectly reveal their knowledge, and all information is perfectly aggregated as soon as consensus is reached (Mueller-Frank 2014). For somewhat different reasons, perfect aggregation also holds in the Gaussian environment just discussed. There, agents end up holding the beliefs that would be held by someone who observed all the private information initially dispersed in society. (This is another aspect of a result mentioned above, Theorem 3 of DeMarzo et al. 2003).

4.1.2 Discrete Actions

The conclusions of the previous section depend on a very strong assumption about belief revelation, but we can obtain conformity with far coarser communication protocols. For the rest of Section 4, we assume that $x_n(t) \in \{0, 1\}$.[32] Two observations allow us to characterize long-run behavior in this family of models. First, an imitation principle holds: Any player asymptotically earns at least as high a payoff as any neighbor because imitation is an available strategy. Second, beliefs evolve according to a martingale, so each individual's belief must converge almost surely to a (random) limit. Together, these observations imply that long-run behavioral conformity is a robust outcome in any connected network. Barring indifference, all players converge on the same action in finite time. An important question is then whether the limiting action is optimal.

Bala and Goyal (1998) provide a seminal contribution to this literature, studying a model of social experimentation in arbitrary networks.[33] Let Y denote an arbitrary space of outcomes. Conditional on the state θ, each action $x \in \{0, 1\}$ is associated with a distribution $F_{x,\theta}$ over outcomes. Players share a common utility function $u : \{0, 1\} \times Y \to \mathbb{R}$; if player n chooses action $x_n(t)$ in period t, she earns expected utility

$$\int u(x_n(t), y)\mathrm{d}F_{x_n(t),\theta}(y).$$

In every period, a player has beliefs about the underlying state and chooses an action to maximize current-period expected utility.

Each player n observes the outcome of her action in each period as well as the actions and outcomes of all players $m \in B(n)$. Given these observations, player n updates her beliefs about the underlying state and carries these into the next period. This updating is imperfect in that players use information only from neighbors' realized outcomes in Y; they do not infer additional information from the actions neighbors choose. A player learns the true payoff distribution for any action that a neighbor takes infinitely often, which immediately implies an imitation principle, and players will eventually converge on the same action. In

[32] Many of the models we discuss can accommodate more general action spaces, but the binary case captures the key insights in each.

[33] Within the literature on learning in games, there are antecedents considering agents who can observe their neighbors in particular networks; see Ellison and Fudenberg (1993) for an example with agents arranged on a line using a simple rule of thumb based on popularity weighting.

general, players can converge on a suboptimal action, but in large networks, relatively mild conditions will ensure they learn the optimal one.

One sufficient condition is having enough diversity in initial beliefs. Any player with a prior close to the truth will choose the optimal action for several periods, generating information that can persuade others to adopt this action. If some players have priors arbitrarily close to the truth, this guarantees that the optimal action is played for arbitrarily many periods by a single player, and this implies convergence to the optimal action throughout the network.

A condition relating more closely to the network structure is that infinitely many players have locally independent neighborhoods. Two neighborhoods $B(m)$ and $B(n)$ are locally independent if they are disjoint; this means that players m and n have independent information. As long as the distribution of initial beliefs is such that each player has some positive probability of selecting the optimal action in the first period, then infinitely many players with locally independent neighborhoods will sample the optimal action. Some of these players will obtain positive results and continue using the optimal action, gathering more information and ensuring that all players eventually learn the best action.

The key intuition behind both conditions is to make sure that some player samples the optimal action infinitely often. This guarantees that someone will obtain positive results, and this knowledge will spread to the rest of the network via the imitation principle. The second condition highlights the importance of independent information sources. If a small group is observed by everyone,[34] then all players receive highly correlated information, and this can cause convergence to a suboptimal action, even in an infinite network.

Gale and Kariv (2003) respond more directly to the limitations of the SSLM we discussed earlier. The authors study a similar model of observational learning in which players each receive a single informative signal at the beginning of the game, but they eliminate the sequential structure: Each player in the game makes a separate decision in each period. Strategic interactions still pose technical challenges, so they assume that in each period, players choose the myopically optimal action given their current beliefs. Given this behavior, belief updating based on observed neighbors' choices is fully rational. In this context, the imitation principle is directly analogous to the improvement principle in the SSLM: Players can guarantee the same expected utility as a neighbor through imitation, and they may improve based on their other information. Similar results have been established by studying the improvement principle in more general settings. Rosenberg, Solan, and Vielle (2009) relax the assumption that agents observe all their neighbors every period, allowing intermittent observation. Mueller-Frank (2013) generalizes beyond the case of decision rules that maximize expected utility, considering arbitrary choice correspondences; he also permits the decision rules not to be common knowledge.

Results on the optimality of eventual outcomes in the observational learning model are less complete but follow a similar intuition to those of Bala and Goyal (1998). Examples suggest that behavior converges quickly in a dense network, limiting the available information and making suboptimal long-run behavior more likely. In a sparse network, learning takes

[34] Bala and Goyal (1998) refer to this group as the "Royal Family."

longer but leads to better asymptotic outcomes.

4.2 Local Heuristics

An alternative way to simplify repeated updating supposes that players heuristically incorporate information from their neighbors. Jadbabaie et al. (2012) offer a canonical example of this approach that is closely related to the DeGroot model studied in the previous section. Players have priors over a finite set of possible states, and they attempt to learn the true state over time. At the beginning of each discrete period, a player observes an exogenous private signal and the current beliefs of her neighbors. Signals are i.i.d. across time, but they may correlate across players within a single period. A player incorporates her signal into her beliefs via Bayesian updating, and subsequently the player takes a weighted average of this posterior with her neighbors' beliefs to obtain the prior for the following period. The weights are given in a stochastic matrix $\mathbf{W}$ that is fixed over time.

Suppose the state space is $\Theta = \{0, 1\}$, so we can represent the belief of player n at time t as a number $p_n(t) = \mathbb{P}(\theta = 1)$. On receiving the signal $s_n(t)$, player n updates her belief to

$$p'_n(t) \equiv \mathbb{P}\left(\theta = 1 \mid s_n(t)\right) = \frac{\mathbb{P}(s_n(t) \mid \theta = 1)p_n(t)}{\mathbb{P}(s_n(t) \mid \theta = 1)p_n(t) + \mathbb{P}(s_n(t) \mid \theta = 0)\left(1 - p_n(t)\right)}.$$

Players assign weights to each other according to a stochastic matrix $\mathbf{W}$ that is fixed over time. We say m is a neighbor of n if $W_{nm} > 0$, and let G denote the corresponding graph. The player combines her updated belief with the reported beliefs of her neighbors to arrive at

$$p_n(t+1) = W_{nn}p'_n(t) + \sum_m W_{nm}p_m(t).$$

This leads to belief dynamics similar to the DeGroot model, and in fact we may view the DeGroot model as a special case in which signals are always uninformative.

If some signals are informative, then this model leads to the most robust learning outcomes of any we have considered. If $G_{nn} = 1$ for each n, and G is strongly connected, then beliefs converge almost surely to the truth. This occurs regardless of correlations between players' signals and regardless of how influential any player is in the network. Crucial to this finding is the continual flow of new information. As long as someone in the network receives new information in each period, this will spread throughout the network. Jadbabaie et al. (2013) provide an extended study of how the distribution of information across individuals interacts with network structure to determine the speed of learning.

4.3 Rational Expectations

Recently, Mossel et al. (2015) offer a significant contribution to this literature, studying a model of repeated observational learning in a network with fully rational expectations. The state space is $\Theta = \{0, 1\}$, and each player n receives a single informative signal s_n before the first period. Player n observes the choices of each $m \in B(n)$ in every period; let $h_n(t)$ denote the history player n observes by the beginning of period t. Player n earns utility in each period equal to the probability that her action matches the state in that period:

$$u(x_n(t), h_n(t), s_n) = \mathbb{P}(\theta = x_n(t) \mid h_n(t),\, s_n).$$

The information structure is essentially identical to the model of Gale and Kariv (2003), but a crucial difference is that players discount future payoffs at a rate $\lambda \in (0, 1)$, and they play a perfect Bayesian equilibrium.

The authors characterize a general class of networks in which players learn the true state almost surely. We say the graph G is *L-locally-connected* if an edge from n to m implies the existence of a path from m to n with length at most L. If an infinite graph is L-locally-connected and there is a bound d on the number of neighbors any player observes, then all players converge on the optimal action almost surely. The proof builds on familiar imitation and martingale convergence arguments, but it also requires several technical innovations. We can interpret the bounded-degree and L-local-connectedness conditions as a way of ensuring that no one player is too influential. In this sense, the intuition here is similar to earlier models with myopic players.

5. Final Remarks

Research on learning in social networks enjoys a diversity of approaches, providing a rich set of answers to our motivating questions. Long-run consensus is a central finding throughout this literature, occurring for a wide range of information structures and decision rules in large classes of networks. Typically, the main assumption needed is an appropriately defined notion of connectedness.

The consistency of this finding may cause some discomfort because we often observe disagreement empirically, even about matters of fact. Explaining such disagreement is an important task for this literature going forward. Of course, we can get disagreement in these models by assuming that networks are disconnected or that agents' preferences are opposed, but such assumptions are not always appropriate. The DeGroot model, with the detailed predictions it makes about a metastable state with one-dimensional deviations from the consensus, comes closest to providing an account of disagreement in terms of network properties. A theory explaining long-run disagreement, especially one with rational foundations and appropriate sensitivity to network structure, would constitute a valuable contribution.

A central theme throughout this literature is that influential individuals have a negative impact on long-run learning, and the different models offer complementary insights as they elaborate on this point. The tractability of sequential models allows us to separate the improvement principle and the large-sample principle as distinct learning mechanisms. This distinction provides intuition for the extent of learning in different networks and a nuanced understanding of how preference heterogeneity impacts learning. DeGroot models deliver a precise grasp of individuals' influence, as well as global learning rates, allowing more detailed comparisons between networks. The significance of strategic interactions for the correctness of eventual consensus is still poorly understood, presenting an important direction for future work.

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