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How about efficient provision through negotiated favor-trading? How does that depend on network structure? Characterize efficient frontier as well as Lindahl outcomes (with strategic foundations) in terms of eigenvalues and eigenvectors of a matrix of marginal payoff relationships. Conceptually: market outcomes ↔ network centrality measures.
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Outline

1. Setup
2. Efficiency
3. Lindahl Outcomes and Network Centrality
4. Conclusions
Players: \( N = \{1, 2, \ldots, n\} \);
The Model

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The Model

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  Think of 0 as status quo outcome.

- **costly actions:** \( \frac{\partial u_i}{\partial a_i} < 0; \)

- **positive externalities:** \( \frac{\partial u_i}{\partial a_j} \geq 0 \) if \( i \neq j \).
The Environment: An Example

prevailing wind

river flow

Town X

Town Y

Town Z
**B**: The (Marginal) Benefits Matrix

**Definition**

\[ B_{ij} = \begin{cases} 
\frac{\partial u_i}{\partial a_j} - \frac{\partial u_i}{\partial a_i} & \text{if } i \neq j \\
0 & \text{otherwise}
\end{cases} \]

How much \( i \)'s values help, measured in units of own effort.

We assume \( B(a) \) is irreducible for all \( a \).
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The Benefits Matrix

We can think of $B(a)$ as a network.
Outline

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4. Conclusions
Example: Is a Pareto Improvement Possible?

\[ B(0) = \begin{bmatrix} 0 & 8 \\ 0.2 & 0 \end{bmatrix} \]
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Example: Is a Pareto Improvement Possible?

\[ B(0) = \begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix} \]

Result

A Pareto improvement on the status quo \( a = 0 \) exists if and only if \( B_{12} \cdot B_{21} > 1 \).
A More Complicated Example
Pareto Frontier Characterization

Definition

The spectral radius \( r(M) \) is the maximum magnitude of any eigenvalue of \( M \).
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A Pareto improvement on the status quo $a = 0$ exists if and only if $r(B(0)) > 1$. 
Pareto Frontier Characterization

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**Proposition**

An interior action profile $a$ is Pareto efficient if and only if $r(B(a)) = 1$. 
Proof Sketch: $a^*$ Pareto-efficient $\Rightarrow r(B(a^*)) = 1$

Take PE $a^*$, assume $\frac{\partial u_i}{\partial a_i}(a^*) = -1$. 
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$a^*$ solves Pareto problem: max. $\sum_i \theta_i u_i(a)$. 

$\theta_B(a^*)$ is non-negative, irreducible and square. $\theta$ is non-negative. Perron-Frobenius: an eigenvalue $\lambda$ of $B$ has a nonnegative left (right) eigenvector if and only if $\lambda = r(B)$. Moreover, $B$ has an eigenvalue $\lambda \in \mathbb{R}$ equal to $r(B)$. 
Proof Sketch: \( a^* \) Pareto-efficient \( \Rightarrow r(B(a^*)) = 1 \)

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FOC: \( \forall j \sum_{i \neq j} \theta_i \frac{\partial u_i}{\partial a_j} - \theta_j = 0 \)

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**Perron-Frobenius**: an eigenvalue \( \lambda \) of \( B \) has a nonnegative left (right) eigenvector if and only if \( \lambda = r(B) \).
Proof Sketch: $a^\ast$ Pareto-efficient $\Rightarrow r(B(a^\ast)) = 1$

Take PE $a^\ast$, assume $\frac{\partial u_i}{\partial a_i}(a^\ast) = -1$.

$a^\ast$ solves Pareto problem: max. $\sum_i \theta_i u_i(a)$.

$$\sum_{i \neq j} \theta_i \frac{\partial u_i}{\partial a_j} - \theta_j = 0$$

$\theta B(a^\ast) = \theta$

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Interpretation of Spectral Radius

Vague Statement

The spectral radius measures the number/intensity of *cycles* in the benefits matrix.
Spectral Radius in Terms of Cycles

\[ B(0) = \begin{bmatrix}
0 & 0 & 7 & 0.5 \\
5 & 0 & 6 & 0.5 \\
0 & 0 & 0 & 0.5 \\
0.5 & 0.5 & 0.5 & 0
\end{bmatrix} \]

Player 4 is essential.
Spectral Radius in Terms of Cycles

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Value of cycle \( c = (1, 2, 4) \):

\[ v(c; B) = B_{21}B_{42}B_{14} \]

\[ = 5 \cdot \frac{1}{2} \cdot \frac{1}{2} \]

\[ r(B) > 1 \iff \lim_{\ell \to \infty} \sum_{c \text{ cycle of length } \leq \ell} v(c; B) > 1 \]

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If large multilateral negotiation is costly, when can most of the benefits be achieved in smaller groups?
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Formalization: a

- Arbitrary “target” Pareto-efficient $a^*$; two groups, $M, M^c$. 
Efficient Separation

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- $(m_i)_{i \in N}$ deters deviations from $a^*$ if the restriction of $a^*$ to $M$ is Pareto efficient given new payoffs (resp. $M^c$).

- Cost of separation $c_M(a^*)$ defined as the infimum of $\sum_{i \in N} m_i(a^*)$, taken over deviation-deterring transfers.
Proposition

\[ c_M(a^*) \leq \sum \frac{\theta_i}{\theta_j} B_{ij}(a^*) a_j^*, \]

where the summation is taken over all ordered pairs \((i, j)\) such that one element is in \(M\) and the other is in \(M^c\).
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A minimum cut in a graph with suitable weights \(W\).
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A minimum cut in a graph with suitable weights \(W\).

- RHS can be small even when groups provide large benefits to each other.
- Small when spectral gap of \(W\) is small.
Takeaways

- Largest eigenvalue of benefits matrix diagnoses inefficiency:
  - At 0: is it greater than 1?
  - Interior: is it different from 1?
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- A player is essential to achieving any Pareto improvement on 0 iff his removal changes $r(B(0))$ from $>1$ to $<1$.
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- Additional results: spectral radius as a measure of inefficiency.
  - \( r(B(a)) - 1 \) is the rate at which effort would have to be taxed to make the outcome \( a \) Pareto efficient.
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- Largest eigenvalue of benefits matrix diagnoses inefficiency:
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  - \( r(B(a)) - 1 \) is the rate at which effort would have to be taxed to make the outcome \( a \) Pareto efficient.
  - Measures the returns on the best egalitarian improvement.
Outline

1. Setup
2. Efficiency
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4. Conclusions
Multiple Pareto Efficient, Individually Rational Outcomes

\[ u_1 = a_2 - \frac{1}{2} a_1^2 \]

\[ u_2 = a_1 - a_2^2 \]
Multiple Pareto Efficient, Individually Rational Outcomes

\[ u_1 = a_2 - \frac{1}{2} a_1^2 \]
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\[ B(a) = \begin{bmatrix} 0 & \frac{1}{a_1} \\ \frac{1}{2a_2} & 0 \end{bmatrix} \]
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Multiple Pareto Efficient, Individually Rational Outcomes

From now on, assume set of IR points is bounded.
Conceptually: complete the missing markets for externalities to achieve efficient provision.
**Lindahl Outcome**

**Conceptually**: complete the missing markets for externalities to achieve efficient provision.

**Definition**

A *Lindahl outcome* is an $\alpha^*$ such that there is a schedule of prices \( \{P_{ij} : i \neq j\} \) satisfying, for each $i$,

\[
\alpha^* \in \arg\max \text{ weak budget balance } u_i(\alpha)
\]
**Conceptually**: complete the missing markets for externalities to achieve efficient provision.

**Definition**

A *Lindahl outcome* is an $\mathbf{a}^*$ such that there is a schedule of prices \( \{P_{ij} : i \neq j\} \) satisfying, for each $i$,

\[
\mathbf{a}^* \in \text{argmax } u_i(\mathbf{a})
\]

subject to weak budget balance

\[
\sum_{j:j \neq i} P_{ij} a_j \leq a_i \sum_{j:j \neq i} P_{ji}.
\]
Conceptually: complete the missing markets for externalities to achieve efficient provision.

**Definition**

A *Lindahl outcome* is an \( a^* \) such that there is a schedule of prices \( \{P_{ij} : i \neq j\} \) satisfying, for each \( i \),

\[
\mathbf{a}^* \in \text{argmax } u_i(\mathbf{a})
\]

weak budget balance

\( a \) satisfies **weak budget balance** for prices \( P \) if

\[
\sum_{j:j \neq i} P_{ij}a_j \leq a_i \sum_{j:j \neq i} P_{ji}.
\]

Main theorem: characterization in terms of network centrality.
Lindahl Outcome Graphically

Pareto frontier

\[ u_2 = 0 \]

\[ u_1 = 0 \]
Lindahl Outcome Graphically

Pareto frontier

\[ u_1 = c_1 \]

\[ u_2 = 0 \]

\[ u_1 = 0 \]

\[ u_2 = c_2 \]
Lindahl Outcome Graphically

Pareto frontier

$u_1 = c_1$

$u_2 = 0$

$u_2 = c_2$

$u_1 = 0$

Lindahl outcome
Centrality Property

**Definition**

\( a \in \mathbb{R}_+^n \) has the centrality property (or is a centrality action profile) if \( a \neq 0 \) and

\[
a = B(a; u) a.
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Centrality Property

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\( \mathbf{a} \in \mathbb{R}_+^n \) has the centrality property (or is a centrality action profile) if \( \mathbf{a} \neq \mathbf{0} \) and

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\mathbf{a} = \mathbf{B}(\mathbf{a}; \mathbf{u}) \mathbf{a}.
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**Fixed-point definition of actions.**

Agents taking high actions are those who benefit a lot (at the margin) from others who are taking high actions.
Definition

\( \mathbf{a} \in \mathbb{R}_+^n \) has the *centrality property* (or is a *centrality action profile*) if \( \mathbf{a} \neq \mathbf{0} \) and

\[
\mathbf{a} = \mathbf{B}(\mathbf{a}; \mathbf{u}) \mathbf{a}.
\]

\[
\mathbf{a}_i = \sum_{j \neq i} B_{ij}(\mathbf{a}) \cdot \mathbf{a}_j
\]

- Fixed-point definition of actions.

Agents taking high actions are those who benefit a lot (at the margin) from others who are taking high actions.
The Main Theorem

Definition

\( a \in \mathbb{R}^n_+ \) has the centrality property if \( a \neq 0 \) and

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The Main Theorem

Definition

\[ a \in \mathbb{R}^n_+ \text{ has the centrality property if } a \neq 0 \text{ and } a = B(a; u) a. \]

Theorem

A nonzero \( a \) is a Lindahl outcome if and only if it has the centrality profile.
Four questions:

1. How is it proved?
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2. What is eigenvector centrality?
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Rest of the Talk

■ Four questions:
  1. How is it proved?
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■ Rest of talk:
  2. Background on eigenvector centrality.
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Rest of talk:

2. Background on eigenvector centrality.
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4. Conclusions
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Centrality Property $\iff$ Lindahl Outcome

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D´avila, Eeckhout, and Martinelli (JPET 09), Penta (JME 11); see also Yildiz (Games 03).

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Theorem

If $0$ is inefficient and utilities are strictly concave, then: in any efficient perfect equilibrium, a Lindahl outcome is played.
Selecting an Outcome: A Bargaining Game

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Implementing Theory Rationale

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  To avoid equilibrium selection fight, Lindahl mechanism is the best bet.
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Walk Interpretation of Eigenvector Centrality

Vague Statement

A node’s centrality measures the number/intensity of **walks** in the benefits matrix that end at that node.
Walks and their Values

\[ B(0) = \begin{bmatrix}
0 & 0 & 7 & 0.5 \\
5 & 0 & 6 & 0.5 \\
0 & 0 & 0 & 0.5 \\
0.5 & 0.5 & 0.5 & 0
\end{bmatrix} \]

Value of walk \( w = (3, 1, 2) \):

\[ v(w; B) = B_{13}B_{21} = 7 \cdot 5 \]
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Walks can repeat nodes: e.g., 
\((3, 1, 2, 4, 3, 2)\).
Centrality in Terms of Walks

Define

\[ V^↓_i(\ell; B) = \sum_{w \text{ a walk ending at } i \text{ of length } \ell} v(w; B). \]
Centrality in Terms of Walks

Define

$$V_i^\downarrow(\ell; B) = \sum_{\text{w a walk ending at } i \text{ of length } \ell} v(w; B).$$

Fact

Assume $B(a)$ is aperiodic. $a$ has the centrality property if and only if

$$\frac{a_i}{a_j} = \lim_{\ell \to \infty} \frac{V_i^\downarrow(\ell; B)}{V_j^\downarrow(\ell; B)}.$$

Each agent’s effort proportional to the total value of long walks he terminates (“total incoming benefits”).
Contributions

\[ \text{PE} \iff \theta = \theta B(a) \iff r(B(a)) = 1 \]
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\[ \text{Lindahl} \iff P_{ij} = \theta_i B_{ij} \iff a = B(a)a \]
Looking at the benefits network sheds light on public goods problem.
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Summary

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  - Price equilibrium \( \Leftrightarrow \) more central agents (ones at ends of high-value walks) contribute more.
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Characterization of market outcome in terms of centrality:
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- Conceptual punchline: can think of market outcomes using network centrality!
- Encouraging metaphor, but need to address “markets you can take literally”.
Outline

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Further Results

- Analogous characterization with transferable numeraire.
  ▶ Details

- Explicit formulas for centrality action profiles in parameterized economies. (New microfoundations for network centrality measures).
  ▶ Details

- Next step: analogous exercise for Walrasian outcomes in other settings to examine key nodes, robustness of market to removing nodes, etc.
We imagine the designer of a mechanism.

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Foundations for Lindahl: The Design Problem

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Satisfies desiderata?
An Example of a Mechanism

- Mechanism definition:
  - strategy set $\Sigma_i$ for each agent (let $\Sigma = \prod_i \Sigma_i$);
  - an outcome function $g : \Sigma \rightarrow \mathbb{R}_+^n$ to prescribe actions.

- Example:
  - $\Sigma_1 = \Sigma_2 = \mathbb{R}_+^2$;
  - $g(a^{(1)}, a^{(2)}) = \min\{a^{(1)}, a^{(2)}\}$.

- Satisfies desiderata?
  No. Has many inefficient equilibria.
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1. If $H$ is reliable, then $L$ is a sub-correspondence of $\Sigma^*_H$. That is, every Lindahl outcome is an equilibrium outcome of $H$. 

*Proof of theorem*

Explicit condition for uniqueness

Details
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2. There is a reliable mechanism \( H \) such that \( \Sigma^*_H = L \).
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Hurwicz Foundations for Lindahl

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Mechanism \( H \) satisfies \textbf{payoff-uniqueness} under \( u \) if all elements of \( \Sigma^*_H(u) \) are payoff-equivalent (no selection conflict).

Payoff-uniqueness is achievable exactly for those \( u \) such that all Lindahl outcomes under \( u \) are payoff-equivalent. 

Proof of theorem

Explicit condition for uniqueness

Details
- **Public goods.**
  - Classical theory: Wicksell (1896); Lindahl (1919); Samuelson (1954); Coase (1960); Foley (1970); Roberts (1973, 1974).
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Technical: network (eigenvector) centrality.

- Concepts: Markov (1906); Leontief (1928); Katz (1953); Bonacich (1987).
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- Recent applications: Brin and Page (1998); Ballester, Calvó-Armengol, and Zenou (2006); Acemoglu et al. (2012).
Intuition for Choice of Prices

\[ P_{ij} = \theta_i B_{ij}(\alpha) \]
Suppose agent is maximizing $u_i(x_1, x_2, \ldots, x_n)$ subject to $
abla_j p_j x_j \leq m$.
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Proof of Cycles Formula for Spectral Radius

**Proposition**

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r(B) = \lim_{\ell \to \infty} \left[ \sum_{c \text{ a cycle of length } \leq \ell} v(c; B) \right]^{1/\ell}
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Note

\[ \text{trace}(B^\ell) = \sum_i (B^\ell)_{ii} = \sum_{c \text{ a cycle of length } \ell} v(c; B). \]
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- Let \( d \) be such that \( \lambda^d \in \mathbb{R}^+_n \) for every eigenvalue \( \lambda \) of \( B \) with \( |\lambda| = r(B) \). (Exists by Wielandt, 1950.)
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Original economy (separable case):

\[ u_i(a) = b_i(a_{-i}) - c_i(a_i). \]
The Spectral Radius as a Measure of Inefficiency: Frictions

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The interior action profile \( a \) is a Pareto efficient outcome under \( u_i^{(\tau)}(a) \) if and only if \( \tau = r(B(a)) \).

Write \( \tau = 1 + t \) (where \( t \) is a tax). A tax of \( t = r(B(a)) - 1 \) on contributions would be necessary to dissuade a social planner from increasing contributions.
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$$b_i(a, d) = \frac{i's \; marginal \; benefit}{i's \; marginal \; cost}$$

A direction $d \in \Delta$ is egalitarian at $a$ if every entry of $b(a, d)$ is the same.

Proposition

At any $a$, there is a unique egalitarian direction $d_{eg}(a)$. Every entry of $b(a, d_{eg}(a))$ is equal to the spectral radius of $B(a)$. 
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- At any \( a \), the matrix \( B(a) \) is nonnegative and irreducible.
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In other words, for each $i$,

$$\rho = \frac{\sum_i B_{ij} d_j}{d_i} = \frac{\sum_j \frac{\partial u_i}{\partial a_j} d_j}{-\frac{\partial u_i}{\partial a_i} d_i}.$$
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- By uniqueness of the Perron vector, there is no other egalitarian direction.
$B(0) = \begin{bmatrix} 0 & 0 & 7 \\ 5 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix}.$

Geometric mean of weights along a cycle is always a lower bound on $r(B(0))$. Cycles also provide an upper bound. If no cycles, then $r(B(0)) = 0$. 
Cycles Interpretation

\[ B(0) = \begin{bmatrix} 0 & 0 & 7 \\ 5 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix}. \]

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\[ r(B(0)) > 1 \]

(lots of cycles)
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\[ r(B(0)) \geq (5 \cdot \frac{1}{2} \cdot \frac{1}{2})^{1/3} > 1 \]
Gross Substitutes

Assumption (Gross Substitutes)

Let $p_j > 0$ be the price of $j$’s effort and $1$ be $i$’s wage. Let

$$a^*(p) = \arg\max_a u_i(a) \text{ subject to } \sum_{j \neq i} p_j a_j \leq a_i.$$
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If only \( p_j \) increases, then for \( k \neq i, j \), the demand \( a_k^* \) does not strictly decrease (in the strong set order); \( a_i^* \) does not strictly increase.
The Proof that $L \subseteq \Sigma^*_H$ (Hurwicz, Maskin, Postlewaite)

Consider a Lindahl outcome $\alpha$ under preferences $u$. 
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The Proof that \( L \subseteq \mathbf{\Sigma}^*_H \) (Hurwicz, Maskin, Postlewaite)

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![Diagram showing the linearization of preferences](image-url)
The Proof that $L \subseteq \Sigma^*_H$ (Hurwicz, Maskin, Postlewaite)

Note that each agent’s “better-than-$a$” set is strictly larger under $\hat{u}$ than under $u$.

By Maskin’s theorem, whatever $\Sigma^*_H$ implements under $\hat{u}$ must also be implemented under $u$. 

![Diagram](image-url)
The Proof that $L \subseteq \Sigma^*_H$ (Hurwicz, Maskin, Postlewaite)

Construct preferences increasingly “near” $\hat{u}$ so that IR and PE alone force outcome of $\Sigma^*_H$ to be near $a$.

By continuity, $a$ must be one of the outcomes implemented under $\hat{u}$. 
Suppose now preferences of the form $u_i(a, m_i)$, where $m$ is the net transfer of “money” $i$ receives.
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Assume for this slide that $\frac{\partial u_i}{\partial a_i} = -1$. 

Define $\theta_i(a, m_i) = \left[ \frac{\partial u_i}{\partial m_i}(a, m_i) \right] - 1$: inverse marginal utility of income.

Proposition

The action profile $a$ is a Lindahl outcome if and only if $\theta_i = \theta_B$ where $m_i = \theta_i(-a_i + \sum_j B_{ij} a_j)$. 
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\( a \) has the centrality property if and only if \( a = (I - G)^{-1}h \).
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Say \( h = 1 \). Then \( a_i = \left( \text{total value of walks in } G \text{ ending at } i \right) \).
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- The current active player proposes a direction \( d \in \Delta \) and an upper bound \( s_i \).

Result: as \( \min_i \delta_i \to 1 \), the MPE payoffs converge to Lindahl payoffs. Does not depend on ratios \( (1 - \delta_i) / (1 - \delta_j) \).

Citations:

Yildiz (Games '03), D’avila and Eeckhout (JET '08), D’avila, Eeckhout, and Martinelli (J Pub Econ Th '09), Penta (J Math Econ '11).
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- If anyone says “no”, then the next proposer is active. Otherwise, play $a = d \min_j s_j$.

Result: as $\min_i \delta_i \to 1$, the MPE payoffs converge to Lindahl payoffs.

- Does not depend on ratios $(1 - \delta_i)/(1 - \delta_j)$.

Citations:

- Yildiz (Games ’03), Dávila and Eeckhout (JET ’08), Dávila, Eeckhout, and Martinelli (J Pub Econ Th ’09), Penta (J Math Econ ’11).