Expectations, Networks, and Conventions

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Iterated Average Expectations: What, Why?

- Iterated average expectations:
  - what is each agent’s expectation of random variable? what is his expectation of the average expectation? etc.
  - each agent takes the average with respect to a network of weights on other players.

- Relevant in:
  - coordination games of incomplete information with linear best reply;
  - pricing in speculative over-the-counter financial markets.

- Special cases:
  - beauty contests with heterogeneous priors;
  - complete-information network games.

- This paper:
  - a Markov formalism for studying iterated average expectations;
  - application to games (and financial markets).
Network Game with Asymmetric Information: Timing

1. Nature draws $\theta$ (“external state”).

2. Agents $i = 1, \ldots, n$ receive private signals $(t^i)_{i=1}^n$ whose joint distribution depends on $\theta$
   - arbitrarily distributed—induce higher-order beliefs.

3. Agents form beliefs about $\theta$ and about others’ signals, based on their signals $t^i$ and priors.

4. Agents choose actions, seeking to coordinate with $y(\theta)$ and with each other.

5. The external state $\theta$ is revealed and payoffs are enjoyed.

Static!
Network Game with Asymmetric Information: Payoffs

Fix $y : \Theta \rightarrow \mathbb{R}$, i.e., $y \in \mathbb{R}^\Theta$. Agents choose $a^i \in [y_{\text{min}}, y_{\text{max}}]$; ex post payoffs:

$$u^i = -(1 - \beta)(a^i - y(\theta))^2 - \beta \sum_j \gamma^{ij}(a^i - a^j)^2.$$

$$a_{BR}^i(t_i) = (1 - \beta)E^i[y \mid t^i] + \beta \sum_j \gamma^{ij}E^i[a^j \mid t^i].$$

(best response $= \text{weighted average of conditional expectation of } y$ and conditional expectation of neighbors’ average action)

(Leads to iterated average expectations.)
Contributions:

- Substantive results—apply to consensus expectations, $\beta \uparrow 1$:
  1. common priors: play common \textit{ex ante} expectation (Shin and Williamson 1996; Samet 1998);
  2. heterogeneous priors and complete information: $E_i^y$ weighted by $i$'s network centrality (Ballester, Calvo-Armengol, Zenou 2006);
  3. contagion of (second-order) \textbf{optimism}:
     - when others view others as (a little) overoptimistic, asset values driven to maximum.
  4. tyranny of the (relatively) \textbf{ignorant}:
     - prior of least informed agent is most influential.

- Methodological (applies to iterated average expectations generally):
  1. unified treatment of network structure and incomplete information;
  2. new use of Markov chains to understand incomplete-information game.
(Interim) Environment

agents

states of the world, types \((finite sets)\)

belief functions

(subjective) expectation operators

network weights

\(i, j \in N = \{1, 2, \ldots, I\}\)

\(\theta \in \Theta, \quad t^i \in T^i\)

\(T = T^1 \times T^2 \times \cdots \times T^I\)

\(\pi^i(\cdot \mid t^i) \in \Delta(\Theta \times T^{-i})\)

\(E^i[\cdot \mid t^i]\)

e.g. for \(y \in \mathbb{R}^{\Theta}, \ E^i y \in \mathbb{R}^{T^i}\)

\(\Gamma = (\gamma^{ij})_{i,j}, \text{ nonnegative}\)

\(I\)-by-\(I\), row-stochastic
Network Game with Linear Best Response: More Explicit

Fix $y \in \mathbb{R}^\Theta$. Agents choose $a^i \in [y_{\min}, y_{\max}]$; ex post payoffs:

$$u^i = -(1 - \beta)(a^i - y(\theta))^2 - \beta \sum_j \gamma^{ij}(a^i - a^j)^2.$$ 

$$a^i_{BR}(t_i) = (1 - \beta)E^i[y | t^i] + \beta \sum_j \gamma^{ij}E^i[a^j | t^i].$$

**Result 0:** As $\beta \uparrow 1$, under connectedness conditions (sufficient: complete network, full support), in equilibrium everyone plays same action, independent of identity and type. Call that action $c$, the **consensus expectation**. For all results, assume $c$ is well-defined.
Simple Special Cases

**Result 1:** if all agents’ beliefs $\pi^i$ are compatible with a common prior, then consensus expectation is the common prior expectation of $y$.

- Proof:

\[
a^i(t^i) = (1 - \beta)E^i[y | t^i] + \beta \sum_{j \in N} \gamma^{ij} E^i[a^j | t^i].
\]

\[
EA^i(t^i) = (1 - \beta)E\left[E^i[y | t^i]\right] + \beta \sum_{j \in N} \gamma^{ij} \sum_{t^j \in T^j} E\left[E^i[a^j | t^i]\right]
\]

**Result 2:** if agents have heterogeneous priors about $y$ but there is complete information about their beliefs, then letting $e$ be eigenvector centrality of $\Gamma$ (unique $e \in \Delta(N)$ s.t. $e\Gamma = e$), we have:

\[
c = \sum_i e^i E^i y.
\]

- Reason: boils down to network game from Ballester, Calvo-Armengol, Zenou (2006):

\[
a^i = (1 - \beta)E^i y + \beta \sum_{j \in N} \gamma^{ij} a^j.
\]
Result 3: Contagion of Optimism

- Simple version:
  - Suppose each \( i \) is **certain** that all counterparties have first-order expectations of \( y \) at least \( \delta > 0 \) greater than his own...
  - except those types that already have nearly the most optimistic expectations (above \( f \)): they are certain that all counterparties have **weakly** greater expectations.
  - Then consensus expectation is very high: \( c \geq \bar{f} \).

- More interesting version:
  - Suppose each type of each \( i \) **expects** that average counterparty has first-order expectations of \( y \) at least \( \delta > 0 \) greater than his own...
  - except those types that already have nearly the most optimistic expectations (above \( f \)): their average expectation of counterparties’ first-order expectations are allowed to be less by up to \( \varepsilon \).
  - Then consensus expectation is very high:
    \[
    c \geq \frac{\bar{f}}{1 + \varepsilon / \delta}
    \]
Result 4: Tyranny of (Relatively) Uninformed

- **Simple version:**
  - Suppose $j$ has no information about the state.
  - while everyone else has very precise (but imperfect) information.
  - Then consensus expectation is $j$’s prior expectation.

- **More interesting version:**
  - Suppose $j$ has very precise (but imperfect) information about the state.
  - while everyone else has even more precise (but imperfect) information.
  - Then consensus expectation is $j$’s prior expectation.
**Key Tool for Everything**

- Interaction structure: a weighted graph. Nodes: $S = \bigcup_i T^i$. Edge weights $B$ shown; Markov!

- Treats network weights ($\gamma^{ij}$) and beliefs ($\pi^i$) symmetrically.


- Key fact:

$$c = \sum_{t^i \in S} p(t^i) E^i[y \mid t^i],$$

where $p$ is stationary distribution of $B$. 

![Diagram](image-url)
Iterated Average Expectations

- Fix a random variable $y \in \mathbb{R}^{\Theta}$ (measurable w.r.t. state of the world).
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- Define $x^i_{t^i}(1) = E^i[y \mid t^i]$;
Iterated Average Expectations

- Fix a random variable $y \in \mathbb{R}^\Theta$ (measurable w.r.t. state of the world).

- Define $x^i_{ti}(1) = E^i[y \mid t^i]$;
  - *first-order* expectations of $y$;

- Define
  $$x^i_{ti}(2) = \sum_j \gamma^{ij} E^i[x^j(1) \mid t^i].$$
Iterated Average Expectations

- Fix a random variable $y \in \mathbb{R}^\Theta$ (measurable w.r.t. state of the world).

- Define $x^i_t (1) = E^i [y | t^i]$;
  - first-order expectations of $y$; a random variable measurable with respect to $i$’s information.

- Define
  $$ x^i_{t^i} (n + 1) = \sum_j \gamma^{ij} E^i [x^j (n) | t^i]. $$
  - This is an expectation of an average (taken across population).
  - It is $(n + 1)^{\text{st}}$-order: an expectation over $n^{\text{th}}$-order expectations.

- In any rationalizable strategy profile, actions played are
  $$ a^i (t^i) = \sum_{n=0}^{\infty} (1 - \beta) \beta^n x^i_{t^i} (n + 1) $$
  The limit as $\beta \uparrow 1$ always exists and is equal to $\lim_{n \to \infty} x^i_{t^i} (n)$ if that limit exists.
Professional investment may be likened to those newspaper competitions [in which] each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view ... We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees.

— Keynes, *The General Theory*... (1936)
Convergence to a Consensus Expectation: Theorem

Suppose beliefs and interactions are jointly connected (definition to follow). Then for any $y \in \mathbb{R}^\Theta$

- The limit

$$c = \lim_{\beta \to 1} a^i(t^i)$$

$$c = \lim_{\beta \to 1} (1 - \beta) \sum_{n=0}^{\infty} \beta^n x^i_{t^i}(n + 1)$$

exists for every agent $i$ and every type $t^i \in T^i$.

- This limit does not depend on $i$ or on $t^i$.

- This consensus expectation is a weighted average of various types’ first-order beliefs:

$$c = \sum_i \sum_{t^i \in T^i} p(t^i) E^i[y \mid t^i],$$

where $\sum_i \sum_{t^i \in T^i} p(t^i) = 1$ and $p$ depends only on $\pi^i(t^j \mid t^i)$’s and $\Gamma$. 
Convergence Result: Proof Idea

Define $S = \bigcup_{i \in \mathbb{N}} T_i$, union of type spaces.

- $x^i(n)$ is a vector in $\mathbb{R}^{T^i}$—one entry per type of $i$, recording that type’s conditional expectation.
- Can stack these in a vector $x(n) \in \mathbb{R}^S$.
- Formula defining $x(n+1)$ is linear—in fact, “Markovian”—in $x(n)$.
- So we can write the iteration conveniently via an interaction structure matrix, $B$:

$$x(n+1) = Bx(n).$$

Then analysis comes down to powers of this $B$ matrix – and these are well-understood.

- Alternate proof: reduction of incomplete-information to just another network game.
The Interaction Structure: The Matrix $B$

$$
x_{t_i}^i(n+1) = \sum_j \gamma^{ij} E_{t_i}^i x^j(n) = \sum_{j \in N} \gamma^{ij} \sum_{i \in T^j} \pi_{t_i}^i(t^j) x_{t_i}^j(n)
$$

$$
x(n+1) = B x(n).
$$
The Interaction Structure: The Matrix $B$

$$x^i_t(n+1) = \sum_j \gamma^{ij} E^i_t x^j(n) = \sum_{j \in N} \gamma^{ij} \sum_{i \in T^j} \pi^i_t(t^j)x^j_t(n)$$

$$x(n+1) = Bx(n).$$

$$\Rightarrow x(\infty) = B^\infty x(1)$$
The Interaction Structure: The Matrix $B$

$$x_{ti}^i(n + 1) = \sum_j \gamma_{ij}^i E_{ti}^i x^j(n) = \sum_{j \in N} \gamma_{ij}^i \sum_{t^j \in T^j} \pi_{t^i}^i(t^j) x_{t^j}^j(n)$$

$$x(n + 1) = Bx(n).$$

$$\Rightarrow x(\infty) = B^\infty x(1) = p' x(1) 1,$$

where $p'$ is stationary distribution of $B$. 

\[ \gamma_{ij}^i \pi_{t^i}(t^j | t^i) \]
The Interaction Structure: The Matrix $B$

\[ x_{ti}^i(n + 1) = \sum_j \gamma^{ij} E_{ti} E_{ij} x^j(n) = \sum_j \gamma^{ij} \sum_{t_i \in T_i} \pi_{ti}^i(t^j) x_{ti}^j(n) \]

\[ x(n + 1) = B x(n). \]

\[ \Rightarrow x(\infty) = B^\infty x(1) = p' x(1) 1, \text{ so } c = \sum_i \sum_{t_i \in T_i} p^i(t^i) x(1) \]
Beliefs and Interactions are Jointly Connected

We say beliefs and interactions are jointly connected if for any nonempty proper subset of types $R \subseteq T^1 \cup T^2 \cup \cdots \cup T^I$ there is some $t^i \in R$ and $t^j \notin R$ so that $\gamma^{ij} \pi^i(t^j | t^i) > 0$. 
Joint Connection Subtleties

- **Sufficient Conditions:**
  - network is complete and there are full support beliefs
  - network is complete and there are no non-trivial common knowledge events
  - network is connected and there are full support beliefs

- **Not Sufficient Conditions:**
  - network is connected and no non-trivial common knowledge events
Result 2: Complete Information

- Define $e$ to be the eigenvector centrality of the network $\Gamma$: unique vector summing to 1 so that

$$
e^i = \sum_j \gamma^{ji} e^j \quad \forall i$$

- If there is no incomplete information, type spaces are singletons and $B = \Gamma$. So $p = e$.

- Now consensus expectation is eigenvector weighted complete information expectation

$$c = \sum_i e^i E^i y.$$  

- Ballester, Calvó-Armengol, and Zenou (06) on network games.

- Generalization: if there are common priors over signals, same formula holds (next two slides).
Some General Structure: Type Weights Sum to Agent Centralities

Proposition

Total weight on $i$’s types $= i$’s network centrality

$$\sum_{t^i \in T^i} p(t^i) = e^i$$

Therefore, can write

$$p(t^i) = e^i r(t^i)$$

where

$$\sum_{t^i \in T^i} r^i(t^i) = 1.$$  

$r^i(t^i)$, the type weight on $t^i$, can be thought of as a pseudoprior on type $t^i$ of $i$.

$$c = \sum_{i \in N} e^i \sum_{t^i} r^i_{\pi, \Gamma(t^i)} E_{t^i, y}$$

In general, $r^i_{\pi, \Gamma(t^i)}$ depends on information structure $\pi$ and the network $\Gamma$. 
An important structural property: Separating Effects of Network and Beliefs

**Definition**

Beliefs $\pi = (\pi^i)_{i \in N}$ have **compatible marginals** if there is a profile $(\tilde{\pi}^i \in \Delta(T^i))_{i \in N}$ such that for any $i$, any $t^i \in T^i$, and any $j \in N$

$$\tilde{\pi}^i(t^i) = \sum_{t^j \in T^j} \tilde{\pi}^j(t^j)\pi^j_{t^j}(t^i).$$

Weaker than assuming beliefs arise from a common prior over $T$ and satisfied for free if only two players

**Proposition**

The following are equivalent:

1. Beliefs $\pi$ have compatible marginals.
2. For all irreducible $\Gamma$, type weights $(r_{\pi,\Gamma}^i)_{i \in N}$ are the same, $(\bar{\pi}^i)_{i \in N}$.

If either condition holds, then $\tilde{\pi}^i = \bar{\pi}^i$ for each $i$. 
Implication of Compatible Marginals

So compatible marginals implies:

\[ c = \sum_{i \in N} e^i \sum_{t^i \in T^i} \bar{r}_\pi^i (t^i) E_{t^i}^i y = \sum_{i \in N} e^i \cdot \text{"\(i\)'s prior expectation of \(y\)."} \]
Result 3: Second-Order Optimism

Suppose there is a real number $\bar{f}$ and $\delta, \varepsilon > 0$ such that there is second-order optimism:

- Every type whose first-order expectation of $y$ is below $\bar{f}$ expects the first-order expectation, averaged across his counterparties, to be at least $\delta$ above his own:

$$\sum_j \gamma^{ij} (E^i E^j y)(t_i) \geq (E^i y)(t_i) + \delta.$$ 

- Every type whose first-order expectation of $y$ is at least $\bar{f}$ expects the first-order expectation, averaged across his counterparties, to be almost as large as his own, with a shortfall of at most $\varepsilon$:

$$\sum_j \gamma^{ij} (E^i E^j y)(t_i) \geq (E^i y)(t_i) - \varepsilon.$$ 

Then the consensus expectation of $y$ satisfies

$$c \geq \frac{\bar{f}}{1 + \varepsilon / \delta}.$$
Let $B$ be an ergodic Markov chain and suppose that there are $\delta, \varepsilon > 0$ and a real number $\bar{f}$ such that:

- For every $s$ such that $f(s) < \bar{f}$, we have $\mathbb{E}_{W_0=s}[f(W_1)] \geq f(s) + \delta$.
- For every $s$ such that $f(s) \geq \bar{f}$, we have $\mathbb{E}_{W_0=s}[f(W_1)] \geq f(s) - \varepsilon$.

Then, letting $p$ denote the ergodic distribution of $B$, we have

$$p(s : f(s) \geq \bar{f}) \geq \frac{1}{1 + \varepsilon/\delta}.$$ 

Proved by using the identity $\mathbb{E}_{W_0 \sim p}[f(W_1) - f(W_0)] = 0$ and then expanding the l.h.s. using the conditions above.
Result 4: Tyranny of the Uninformed

- Suppose that there is common knowledge that one agent knows (or believes that he knows) nothing while other agents all know (or believe that they know) they know everything.

- Consensus expectation is the ex ante expectation of the least informed agent.

- Can (and have) approximated.
Tyranny of the Uninformed

- Work with explicit ex ante stage.
- Suppose agents $i$ receive signals of what the state is, equal to the true state with probability $1 - \varepsilon^i$, and erroneous with probability $\varepsilon^i$.
- Common interpretation of signals:
  - agree about distribution of signals given the state;
  - but may disagree about prior probabilities of states.
- If all $\varepsilon^i \downarrow 0$, but one of them ($j$) much slower than the others, then only $j$’s priors over states will matter.
  - How much slower it has to be depends on the beliefs in a subtle way.
  - Study ergodic distribution of $B$ when some edges are near 0.
Consider $B(\zeta)$ where $\varepsilon^i$ is a function of $\zeta$.

Consensus depends on ergodic distribution of $B(\zeta)$.

$B(0)$ is disconnected.
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Skeleton of “leading edges” will determine stationary distribution in the low-$\zeta$ limit.

Argument is via minimum mean first passage times (Cho and Meyer 2000).
Conclusion

- Consensus expectations exist and have economically interesting properties, interpretations and applications.

- By studying “interaction structure” $B$ that treats network and beliefs symmetrically (à la Morris 2000, “Contagion”), can generalize both classical beauty contest results and complete-information network results.

- Key connections between machinery and applications:
  - Contagion of optimism via functions of Markov chains.
  - Complete information limits as nearly reducible Markov chains.
  - DeGroot interpretation.

- Can see when incomplete information and beliefs “project nicely” from the full analysis (compatibility).
Other Related Literature


- de Martí and Zenou (2014) “Network Games with Incomplete Information.”

- Bergemann, Heumann, and Morris (JET 2015), “Information and Volatility.”


An Over-the-Counter Market

- $I$ populations of agents: continuum of each population, with each individual in population $i$ having the same type or signal $t^i$.
- All agents are risk-neutral and there is no discounting.
- They are trading an asset $Y$.
- Suppose that an agent in population $i$ holds the asset.
  - With probability $1 - \beta$, state is realized and the agent consumes the realization of the asset.
  - If not, then with (subjective) probability $\gamma^{ij}$ agent $i$ must sell the asset in a market where he faces population $j$.
  - Competitive market, so the price is equal to population $j$’s subjective valuation.
- As $\beta \uparrow 1$, valuation of agent $i$ tends (in any symmetric, Markov, subgame-perfect trading equilibrium) to
  $$\lim_{\beta \uparrow 1} (1 - \beta) \sum_{n=0}^{\infty} \beta^n x_i(n + 1).$$
Separability of Network Structure and Type Weights: Proof

1. $p \in \Delta(S)$ defined by $pB = p$.

2. $p(t^i) = e^i r^i(t^i)$ by the proposition.

3. Plug (2) into (1) to reduce characterization of $p$ to finding weights $r^i(t^i) \in \Delta(T^i)$ such that

$$e^i r^i(t^i) = \sum_{j \in N} \gamma^{ji} e^j \sum_{t^j \in T^j} r^j(t^j) \pi^j_{ti}(t^i).$$

(*)

4. Assume $\pi$ have consistent marginals. Set $r^i = \bar{r}^i \in \Delta(T^i)$ to be the marginals in the definition of type-consistency. Then (*) boils down to $e = \Gamma e$, which holds by definition.

5. Conversely, suppose that $r^i(t) = \bar{r}^i(t)$, independent of $\Gamma$. Then use $e^i = \sum_{j \in N} \gamma^{ji} e^j$ to write

$$\sum_{j \in N} \gamma^{ji} e^j \bar{r}^i(t^i) = \sum_{j \in N} \gamma^{ji} e^j \sum_{t^j \in T^j} \bar{r}^j(t^j) \pi^j_{ti}(t^i).$$

6. Because we can vary $\gamma^{ji} e^j$ freely, this implies type-consistency.