Lefschetz fixed point formula

Term paper MATH 272a

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Abstract

In this paper, we will present the *Lefschetz fixed point formula* and some of its applications. We first introduce the concept of simplicial approximation and then use it to give a proof of the *Lefschetz formula*. This proof is due to Hopf. In a second part, applications of this theorem to maps on projective spaces and Euclidean neighborhood retracts are considered. We will conclude by an application of the theorem to prove a simplified version of the Hopf index theorem for vector fields on compact manifolds.

1 Introduction

The importance of fixed point theorems cannot be overemphasized. Those theorems always have proved to find applications both in mathematics and in fields that apply mathematics, such as economics (see [1]) or control theory (see [2]) and *fixed point theory* is becoming an indepedent and highly multidisciplinarial field of study on its own.

Fixed point theorems can roughly be classified into two categories: those that come from analysis and those that come from topology. In this paper, we will focus on a topological result : the Lefschetz fixed point theorem. This theorem can be viewed as a considerable generalization of the celebrated *Brouwer's fixed point theorem*, which states that any map from an $n-$ball to itself has a fixed point.

In a first part, we will present the proof of the theorem; it is worth noting that the approach followed is due to Hopf. The reader can find in [6] or [3] an alternative proof in the case of a map on a manifold based on duality and the Thom isomorphism.

We will mainly follow [4] for the proof and yield the details which are omitted or left as exercise, which are principally gathered in Lemmas 1 and 2.

For a thourough treatment of the subject, including a discussion of the converse of the theorem, the reader is referred to the excellent book of R.F. Brown [5].

In a second part, we will consider maps on the real and projective spaces, and see that for instance any map $f : \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ admits a fixed point. $\mathbb{R}\mathrm{P}^{2n}$ is said to have the *fixed point property*. We will also see that the structure of the cohomology ring yields in some cases very simple sufficient conditions on a map for it to have a fixed point.

Finally, we'll use the *tubular neighborhood theorem* to derive a result concerning tangent vector fields on compact manifolds.

2 Definitions

In this section, we will give a few definitions necessary to the understanding of the results. The first relevant definition concerns simplicial maps:

Definition 1 (Simplicial map)**.** *A map $f : K \to L$, K and L simplicial complexes, is **simplicial** if it maps each simplex of K to a simplex of L by a* linear *map taking vertices to vertices*

A simplicial map is thus determined only by its value on the vertices, since any linear map from a simplex to another simplex is determined by its value on the vertices of the simplices.

We now introduce another definition which will become handy in the proof of the main theorem.

Definition 2 (Star of a simplex)**.** *Let K be a simplicial complex and σ a simplex in K. The **star** of σ, denoted by $\mathbf{St}\sigma$ is the subcomplex of K made of all the simplices that contain σ.*

Similarly, the **open star** of a simplex σ, denoted by $\mathbf{st}(\sigma)$ is the union of the *interior* of the simplices that contain σ.

The *Lefschetz fixed point* formula gives a *sufficient* condition for a map $f : X \to X$, for X a space satisfying some hypothesis, to have a fixed point. This condition is expressed in terms of the *trace* of the induced map on homology groups, $f_* : H_n(X) \to H_n(X)$. In general, for a homomorphism $\phi : \mathbb{Z}^n \to \mathbb{Z}^n$, with matrix (a_{ij}), $\mathrm{tr}\ \phi = \sum_{i=1}^{n} a_{ii}$. In the case of finitely generated abelian groups, we give the following definition

Definition 3. *Let A be a finitely generated abelian group and $\phi : A \to A$ and homomorphism. The* trace *of ϕ, denoted $\mathrm{tr}\ \phi$, is the trace of the induced homomorphism $\phi' : A/torsion \to A/torsion$.*

We now can give the definition of the *Lefschetz number*

Definition 4 (Lefschetz number)**.** *Let $f : X \to X$ be a map of a space with finitely generated homology groups which moreover vanish in large dimension, the Lefschetz number of f, denoted $\tau(f)$ is given by*

$$\tau(f) = \sum_n (-1)^n \mathrm{tr}(f_* : H_n(X) \to H_n(X))$$

3 Preliminary results

In this section, we will state results which are necessary to the proof of the *Lefschetz formula*. Some results have an interest of their own though, mainly the simplicial approximation theorem, which we state below.

Theorem 1 (Simplicial approximation theorem)**.** *Let K be a finite simplicial complex and L an arbitrary simplicial complex, then* any *map $f : K \to L$ is homotopic to a map $g : K' \to L$ that is simplicial with respect to some iterated barycentric subdivision K' of K. Furthermore, g can be chosen such that for all simplices $\sigma \in K$, $f(\sigma) \subset \mathbf{st}\ g(\sigma)$*

We omit the proof of this theorem, which is mainly technical (not tedious though) and does not bring much to the understand of what will follow. The theorem states that, from a topological point of view, the study of maps between simplicial complexes can be carried out by studying maps that are simplicial, which is,as we saw, a much stronger condition. The second statement of the theorem is not usually found in the literature but is of importance in the proof of the Lefschetz formula.

Now, we turn to more algebraic lemmas.

Lemma 1. *Given a short exact sequence of finitely generated abelian groups $0 \to A \to B \to C \to 0$ and chain maps $\alpha : A \to A$, $\beta : B \to B$ and $\gamma : C \to C$, we have $\mathrm{tr}\beta = \mathrm{tr}\alpha + \mathrm{tr}\gamma$*

Proof. Let us first assume A, B and C are free.

Consider the following commutative diagram :

$$\begin{array}{ccccccccc} 0 & \to & A & \stackrel{i}{\to} & B & \stackrel{j}{\to} & C & \to & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \to & A & \stackrel{i}{\to} & B & \stackrel{j}{\to} & C & \to & 0 \end{array}$$

Write $\Lambda(\delta)$ the set of eigenvalues of δ. It is an easy consequence of exactness and commutativity that $\Lambda(\beta) = \Lambda(\alpha) \cup \Lambda(\gamma)$. Explicitely, let $\mathrm{b} \in B$ and λ be an eigenvalue of β, $\beta\mathrm{b} = \lambda\mathrm{b}$; we have $B = kerj \oplus (kerj)^{\perp}$ and by exactness of the rows, $\ker j = \mathrm{im}\ \ i$. So b is such that $\mathrm{b} = i\mathrm{a}$ for some $\mathrm{a} \in A$ or $j\mathrm{b} = \mathrm{c}$ for some $\mathrm{c} \in C, \mathrm{c} \neq 0$.

In the first case, we have by naturality

$$i\ \alpha\mathrm{a} = \beta i\ \mathrm{a} = \beta\mathrm{b} = \lambda\mathrm{b} = \lambda i\ a = i\ \lambda a \tag{1}$$

so λ is an eigenvalue of α, since $i : A \to \mathrm{im}\, i$ is a bijection.

For the second case, we have similarly

$$\lambda\mathrm{c} = j\ \lambda\mathrm{b} = j\ \beta\mathrm{b} = \gamma j\ \mathrm{b} = \gamma\mathrm{c} \tag{2}$$

and λ is an eigenvalue of γ.

The general case can easily be reduced to the case A, B and C free. First, we factor out the torsion in B and then, by exactness, also in A . Those transfomations clearly leave the traces of α and β unchanged. Now if C has some remaining torsion, write c_k the generators in C of finite order. By surjectivity of j, there exists $b_l \in B$ with $j(b_l) = c_k$ for some k, l. Replace A by $A' = A \cup a_l$ where a_l are new elements of infinite order and set $i(a_l) = b_l$, $\alpha(a_l) = 0$. Now, we can factor out the torsion in C and the sequence will remain exact. The traces of α, β and γ are clearly unchanged. ∎

Lemma 2. *Given a cellular chain complex C_n of finitely generated abelian groups and a chain maps $g_n : C_n \to C_n$,*

$$\sum_{i=1}^{n}(-1)^n \mathrm{tr}(g_n : C_n \to C_n) = \sum_{i=1}^{n}(-1)^n \mathrm{tr}(g_n : H_n \to H_n)$$

where H_n is the n^{th} homology group of C_n.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \to & C_n & \xrightarrow{d_n} & C_{n-1} & \to & \cdots & \to & C_1 & \xrightarrow{d_1} & C_0 & \to & 0 \\ & & \downarrow g_n & & \downarrow g_{n-1} & & & & \downarrow g_1 & & \downarrow g_0 & & \\ 0 & \to & C_n & \xrightarrow{d_n} & C_{n-1} & \to & \cdots & \to & C_1 & \xrightarrow{d_1} & C_0 & \to & 0 \end{array}$$

We write $Z_n = \ker d_n$, the space of n-cycles, $B_n = \mathrm{im}\, d_{n+1}$ the space of n-boundaries and $H_n = Z_n/B_n$. We have the following commutative diagrams of short exact sequences where by abuse of notation we write g_n for the induced maps on H_n, B_n and Z_n

$$\begin{array}{ccccccccc} 0 & \to & Z_n & \xrightarrow{i} & C_n & \xrightarrow{j} & B_{n-1} & \to & 0 \\ & & \downarrow g_n & & \downarrow g_n & & \downarrow g_{n-1} & & \\ 0 & \to & Z_n & \xrightarrow{i} & C_n & \xrightarrow{j} & B_{n-1} & \to & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \to & B_n & \xrightarrow{k} & Z_n & \xrightarrow{l} & H_n & \to & 0 \\ & & \downarrow g_n & & \downarrow g_n & & \downarrow g_n & & \\ 0 & \to & B_n & \xrightarrow{k} & Z_n & \xrightarrow{l} & H_n & \to & 0 \end{array}$$

where $i : Z_n \to C_n$ and $k : B_n \to Z_n$ are the inclusion maps and $j : C_n \to B_{n-1}$ and $l : Z_n \to H_n$ are the obvious projections. The maps $g_n : B_n \to B_n$, $g_n : Z_n \to Z_n$ and $g_n : H_n \to H_n$ are well defined because the g_n are chain maps.

Now, applying the previous lemma to those diagrams we get the relations :

$$\operatorname{tr}(g_n : C_n \to C_n) = \operatorname{tr}(g_n : Z_n \to Z_n) + \operatorname{tr}(g_n : B_{n-1} \to B_{n-1})$$

$$\operatorname{tr}(g_n : Z_n \to Z_n) = \operatorname{tr}(g_n : B_n \to B_n) + \operatorname{tr}(g_n : H_n \to H_n)$$

Substituting the second equation into the first and then multiplying by $(-1)^n$ and summing over n, gives us the result. ■

4 Lefschetz fixed point formula

We are now ready to give the main result of this paper, the celebrated *Lefschetz fixed point formula*.

Theorem 2 (Lefschetz fixed point formula)**.** *Let X be a finite simplicial complex and $f : X \to X$ a map. If $\tau(f) \neq 0$, then f has a fixed point*

Proof. We will prove the contraposition. Take X as in the hypothesis and $f : X \to X$ without fixed point. Our goal is to use the simplicial approximation theorem to build subdivisions of X, say Y and Z and a map $g : Y \to Z$, such that

1. Y is a further subdivision of Z
2. g is a simplicial map

3. g is homotopic to f

4. for σ is a simplex in Y, $g(\sigma) \cap \sigma = \phi$

First, we pick up a metric d on X such that d restricts to the standard euclidean metric on each simplex of X. This can be done for instance by viewing X, which is finite, as a subcomplex of a simplex Δ^N and restrict the euclidean metric on Δ^N to X.

A finite simplicial complex being compact and f having no fixed points tells us that $\exists\, \varepsilon > 0$ such that $d(x, f(x)) > \varepsilon$, $\forall x \in X$.

Let Z be a subdivision of X such that $\mathbf{st}(\sigma) < \varepsilon/2$. Such a Z can be obtained by applying successive barycentric subdivisions. The simplicial approximation theorem applied to $f : Z \to Z$ gives us Y, a subdivision of Z, and a *simplicial* map $g : Y \to Z$ homotopic to f with the property $f(\sigma) \subset \mathbf{st}\ g(\sigma)$. We may assume moreover that Y is such that all its simplices have diameter less than $\varepsilon/2$.

Now, let x be a point a simplex $\sigma \in Y$. We have $d(x, \sigma) < \varepsilon/2$ and, by the above construction, $d(f(x), g(\sigma) < \varepsilon/2$. On the other hand, $d(x, f(x)) > \varepsilon$; thus $g(\sigma) \cap \sigma = \oslash$.

It is sufficient to study $\tau(g)$, since we clearly have $\tau(f) = \tau(g)$. Write Y^n for the $n-$skeleton of Y. This map g induces a map on the *cellular chain complex* $H_n(Y^n, Y^{n-1})$, $g_* : H_n(Y^n, Y^{n-1}) \to H_n(Y^n, Y^{n-1})$. To see this, it suffices to notice that, because g is simplicial, $g(Y^n) \subset Z^n$, the n-skeleton of Z and since Y is a subdivision of Z, $Z^n \subset Y^n$.

By lemma 2,

$$\tau(g) = \sum_{i=1}^{n} (-1)^n \mathrm{tr}\left(g_n : H_n(Y^n, Y^{n-1}) \to H_n(Y^n, Y^{n-1})\right)$$

But, since $g(\sigma) \cap \sigma = \oslash\ \forall \sigma$, $g_* : H_n(Y^n, Y^{n-1}) \to H_n(Y^n, Y^{n-1}) = 0, \forall n$, $\tau(g) = 0$.

■

This theorem can be generalized to spaces that are homeomorphic to a retract of an open set in $\mathbb{R}^n$. Such spaces are called ENR for *euclidean* neigborhood retract. In particular, every topological manifold is an ENR. More information on ENR can be found in Bredon appendix E and a complete discussion is in Brown Chapter 3.

Corollary 1. *If X is a compact ENR and $f : X \to X$ is such that $\tau(f) \neq 0$ then f has a fixed point.*

Proof. First, we notice that if X is a compact ENR, X is also a retract of a finite simplicial complex. This fact follows from X being a retract of $V \subset \mathbb{R}^n$, V open, and we can, by iterated barycentric subdivision, triangulate $\mathbb{R}^n$ with a simplicial complex such that every simplex touching X is included in V. Let us call the union (which is finite) of those simplices L and the retract from L to X is the restriction to L of the retract from V to X.

The converse is also true, i.e. a retract of a simplicial complex is an ENR. The proof requires more technical details and can be found in [3] Appendix E.

By the above discussion, we can consider the case of a retract of a finite simplicial complex. Suppose $r : K \to X$ is such a retraction, and $f : X \to X$ a map, then $fr : K \to X \subset K$ has the same fixed points as f. We know that $H_n(K) = H_n(X) \oplus H_n(K, X)$ where $r_* : H_n(K) \to H_n(X)$ is the projection map, thus $\mathrm{tr}(f_* r_* : H_n(K) \to H_n(K)) = \mathrm{tr}(f_* : H_n(X) \to H_n(X))$ and Theorem 2 yields the result. ■

5 Applications

We will now consider some applications of this theorem.

The theorem yields the following generalization of the *Brouwer fixed point theorem*.

Corollary 2. *Every map $f : X \to X$ where X is a retract of a contractible finite simplicial complex has a fixed point.*

Proof. If X is contractible, the only non trivial homology group of X is $H_0(X) = \mathbb{Z}$, so $\tau(f) = 0$.

■

Corollary 3. *Every map $f : \mathbb{R}\mathrm{P}^{2n} \to \mathbb{R}\mathrm{P}^{2n}$ has a fixed point.*

Proof. Since $\mathbb{R}\mathrm{P}^{2n}$ has, modulo torsion, the same homology groups as a point, $\tau(f) \neq 0$. ■

The preceding corollary can be proved without the help of *Lefschetz fixed point* theorem, by considering S^{2n} the double couver of $\mathbb{R}\mathrm{P}^{2n}$. A function without fixed point would lift to a function $g : S^{2n} \to S^{2n}$ such that $g(x) \neq x, g(x) \neq -x,\ \forall x$ and such a function can easily be shown to be homotopic to both the identity and the antipodal map, which have different degrees on S^{2n}, hence a contradiction.

Corollary 4. *Every map* $f : \mathbb{R}\mathrm{P}^{2n+1} \to \mathbb{R}\mathrm{P}^{2n+1}$ *such that* $f_* : H_{2n+1} \to H_{2n+1}$ *is multiplication by* $m \neq 1$ *has a fixed point.*

Proof. We have that, modulo torsion, the non trivial homology groups of $\mathbb{R}\mathrm{P}^{2n+1}$ are $H_0(\mathbb{R}\mathrm{P}^{2n+1}) = \mathbb{Z}$ and $H_{2n+1}(\mathbb{R}\mathrm{P}^{2n+1}) = \mathbb{Z}$. Hence $\tau(g) = 1 + (-1)^{2n+1}m = 1 - m$ ■

Those two corollaries can also be examined in the light of linear algebra. Any *linear* map $A : \mathbb{R}^n \to \mathbb{R}^n$ induces a map on projective space $\bar{A} : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$, and eigenvectors of A correspond to fixed points of $\bar{A}$. If n is odd, the characteristic polynomial of A is odd therefore has a real eigenvalue and a real eigenvector, which implies that $\bar{A}$ has a fixed point. On the other hand, for n even, the rotation $A(x_1, x_2, \ldots, x_{2k}) = (x_2, -x_1, \ldots, x_{2k}, -x_{2k-1})$ has no fixed points. This map being a rotation, thus orientation preserving, we have $m = 1$.

Corollary 5. *Every map* $f : \mathbb{C}\mathrm{P}^{2n} \to \mathbb{C}\mathrm{P}^{2n}$ *has a fixed point.*

Proof. To deduce this fact, homology is not sufficient and we need the additional structure of cohomology, namely the cup product. The universal coefficient theorem yields that the cohomology groups are dual to the homology groups and thus $\mathrm{tr}(f_i^*) = \mathrm{tr}(f_*^i)$

The cohomology ring of $\mathbb{C}\mathrm{P}^l$ is given by $\mathbb{Z}[x]/x^{l+1}$ where x is a generator of $H_2(\mathbb{C}\mathrm{P}^l)$. Let $f_2^* : H^2(\mathbb{C}\mathrm{P}^l) \to H^2(\mathbb{C}\mathrm{P}^l)$ be multiplication by m. By naturality of the cup product, $f_k^* : H^k(\mathbb{C}\mathrm{P}^l) \to H^k(\mathbb{C}\mathrm{P}^l)$ is multiplication by m^k. The Lefschetz number is thus given by

$$\begin{aligned} \tau(f) &= \sum_k (-1)^k \mathrm{tr}\, f_*^k & (3) \\ &= 1 + m + m^2 + \cdots + m^{2n} & (4) \\ &= \begin{cases} \frac{1-m^{2n+1}}{1-m} & if\ m \neq 1 \\ n+1 & if\ m = 1. \end{cases} & (5) \end{aligned}$$

We see that $\tau(f)$ is always non zero. ■

Corollary 6. *Every map* $f : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ *such that* $f_2^* : H^2(\mathbb{C}\mathrm{P}^l) \to H^2(\mathbb{C}\mathrm{P}^l)$ *is multiplication by* $m \neq -1$ *has a fixed point.*

Proof. From equation 5, we see that the Lefschetz number of such a map is non zero. ■

The case of the torus is particularly interesting. We denote by T^n the $n-$torus $\mathbb{R}^n/\mathbb{Z}^n$.

Corollary 7. *If $f : T^n \to T^n$ is a map such that $f^* : H^1(T^n; \mathbb{C}) \to H^1(T^n; \mathbb{C})$ does not have $+1$ as an eigenvalue, then f has a fixed point.*

Proof. Since $\mathbb{C}$ is a field, we know by the universal coefficient theorem that cohomology is dual to homology. We have $H^*(T^n) = E(\mathbb{C}, x_1, x_2, \dots, x_n)$, the exterior algebra on n generators. Because $\mathbb{C}$ is algebraically closed, we know there is a basis $y_1, y_2, \dots, y_n$ such that $f^* : H^1(T^n) \to H^1(T^n)$ is represented by an upper triangular matrix, say (a_{ij}).

Let us denote by $z_1, z_2, \dots, z_n$ the dual basis. With respect to this basis, $f_*^1 : H_1(T^n) \to H_1(T^n)$ is given by $(a_{ij})^T$ and thus $\operatorname{tr} f_*^1 = \sum a_{ii}$.

An element of $H_2(T^n)$ will be dual to an element of $H^2(T^n)$, and we know those have the form $y_i \smile y_j$. We write its dual, formally, as $z_i \smile z_j$. Using naturality of the cup product, we have

$$\begin{aligned}
\langle y_i \smile y_j, f_*(z_i \smile z_j)\rangle &= \langle f^*(y_i \smile y_j), z_i \smile z_j\rangle && (6)\\
&= \langle f^* y_i \smile f^* y_j, z_i \smile z_j\rangle && (7)\\
&= \left\langle \sum_l a_{il} y_l \smile \sum_m a_{im} y_m, z_i \smile z_j \right\rangle && (8)\\
&= a_{ii}a_{jj} - a_{ij}a_{ji} && (9)\\
&= a_{ii}a_{jj} && (10)
\end{aligned}$$

For the last equality, one has to remember that (a_{ij}) is upper triangular. Since a basis for $H_2(T^n)$ is given by $\{z_i \smile z_j | i < j\}$,

$$\operatorname{tr} f_*^2 = \sum_{i<j} a_{ii}a_{jj}$$

Similarly, we have for $H_l(T^n)$ that

$$\operatorname{tr} f_*^l = \sum_{i_1 < i_2 < \cdots < i_l} a_{i_1 i_1} a_{i_2 i_2} \dots a_{i_l i_l}$$

Putting those together, we get

$$\tau(f) = 1 - \sum a_{ii} + \sum_{i<j} a_{ii}a_{jj} - \cdots + (-1)^n a_{11}a_{22}\cdots a_{nn} = \Pi(1 - a_{ii})$$

from which we conclude

$$\tau(f) = \det(I - A).$$

■

We now give a last application of this theorem to index theory of vector fields on manifolds. Let M be a smooth $n-$dimensional manifold. We know it can be embedded in $\mathbb{R}^k$ for some k and let d be a distance function in $\mathbb{R}^k$. We define the tubular neighborhood of M as follows :

Definition 5 (Tubular neighborhood)**.** $V_\varepsilon(x) = \left\{x \in \mathbb{R}^k | d(x, M) < \varepsilon\right\}$

This definition can be easily extended to the case of compact submanifold of a manifold, but this generalization is not necessary for our purpose. The name tubular neighborhood comes from the consideration of a 1-dimensional manifold embedded in $\mathbb{R}^3$.

For $x \in R^k$ we define $c(x) \in M$ to be the closest point to x in M.Tubular neighborhoods satisfy the following important result:

Theorem 3 (Tubular neighborhood theorem)**.** *Let M be a smooth compact manifold, for ε small enough, the map $\pi : (V_\varepsilon, \partial V_\varepsilon) \to M : x \to c(x)$ is a smooth surjective map*

We omit the proof of this result, the reader can find it in [3].

Proposition 1. *If M is a compact manifold which admits a continuous non-zero tangent vector field, there exists $f : M \to M$ without fixed point and homotopic to the identity.*

Proof. The proof is a trivial application of the tubular neighborhood theorem. We first embed M in $\mathbb{R}^k$ for some k and then work in the metric space $\mathbb{R}^k$. Let $v : M \to TM$ be the non zero vector field. Since M is compact, $\forall \varepsilon > 0,\ \exists c > 0$ such that $\forall x$ the vector field $c\, v(x)$ has length less than ε.

The function $f_t(x) = \pi(x + t\ c\ v(x))$ has no fixed point at $t = 1$ for if it had one, then $\pi(x + c\, v(x)) = x$ and $v(x)$ is both tangent and normal to M, which is impossible. This family of maps clearly define a homotopy from f to the identity on M. ∎

This fact can now be used to prove this simplified version of the Hopf index theorem:

Theorem 4. *Let M be a compact smooth manifold, if $\chi(M) \neq 0$, then M does not admit a non zero continuous tangent vector field.*

Proof. It suffices to notice that $\chi(M) = \tau(\mathbb{1})$. If M had a non zero continuous tangent vector field, the previous proposition asserts the existence of a function $f : M \to M$ *without* fixed point homotopic to the identity, hence with $\tau(f) = \chi(M) \neq 0$ which is a contradiction. ∎

Corollary 8. *S^n admits a non zero continuous tangent vector field iff n is odd.*

Proof. We know the S^n has a CW structure made of 1 0-cell and 1 n-cell, so $\chi(S^n) = 1+(-1)^n$ so S^{2n} cannot have a non zero continuous tangent vector field by proposition 1

In the case of n odd, embed S^{2n-1} in R^{2n} and define the vector field $v(x_1, x_2, \ldots, x_{2n} = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n+1})$, v is clearly tangent, non zero and continuous

■

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