Gibbons's explanation of Rubinstein's infinite-horizon bilateral bargaining model

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May 20, 2005

1 Introduction

Gibbons provides a solution to Rubinstein's infinite-horizon bilateral bargaining model (pp. 70-71) that makes use of stationarity in a nice, intuitive way. Since it took me a while to understand what he was saying, I'll try to explain it here more explicitly than he does.

2 Gibbons's informal argument

Two players are engaged in a sequential bargaining game to divide up a 1-unit sum into shares s and $1-s$. Player 1 makes the first offer, s_1. If Player 2 accepts, the game ends with payoffs $(s_1, 1-s_1)$; if he rejects he makes a counteroffer s_2, which produces payoffs $(s_2, 1-s_2)$. The game continues until an offer is accepted.

There is no backwards induction solution to an infinite horizon game because the game, by definition, has no end from which to work backwards. But assume you could pick a round (we'll say round τ) and somehow collapse all future outcomes of the game into a pair of continuation payoffs for the two players, $(s^*, 1 - s^*)$. These continuation payoffs are the expected value of continuing the game beyond round τ for each of the two players.

Now that we've collapsed the infinite future down to a single payoff vector, we can move back two rounds to $\tau - 2$ and find a backwards induction solution at that point. Player 1 (who we assume has the right to make the offer at $\tau - 2$) will take $1 - \delta(1 - \delta s^*)$ for himself and offer the remainder $(\delta(1 - \delta s^*))$ to player 2.[1]

The key move Gibbons makes in this informal proof is to say that the value of s^* cannot vary from round to round: collapsing the game (as we did before) to $\tau - 2$ or $\tau + 2$ cannot produce a different continuation value s^* from the one we found at τ, since the game would look the same as you peer into the infinite abyss regardless of where you're standing. Therefore, the payoffs that the players receive at $\tau - 2$ through backwards induction using the continuation values at τ must be the same as the continuation values at τ themselves:

$$\begin{aligned} 1 - \delta(1 - \delta s^*) &= s^* && (1) \\ \delta(1 - \delta s^*) &= 1 - s^* && (2) \end{aligned}$$

This is the idea of stationarity: the continuation value of an infinite game must be the same at all periods of the game. This doesn't mean that the discounted payoffs are the same regardless of when they are received: a key feature of any bargaining game is the discount rate that reduces the size of the pie over time. The fact that the

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[1] As Gibbons shows (pg. 70), this is true because player 1 will reject any offer in $\tau - 1$ that provides him a payoff of less than δs^*, so player 2's offer in $\tau - 1$ will be $1 - \delta s^*$; player 1 therefore must offer player 2 at least $\delta(1 - \delta s^*)$ in $\tau - 2$, leaving himself $1 - \delta(1 - \delta s^*)$.

continuation value doesn't change from round to round but the discount rate keeps eating away the real magnitude of that value means that every simple repeated (whether infinitely or not) bargaining game will have a solution in which the first-round offer is accepted. In short, the only change from round to round in these simple games is the ticking of the clock; since the pickings only get slimmer, it's not worth waiting around for something good to happen.

Using either of the above equations, a bit of algebra reveals that the only value of s^* for which this is true is

$$s^* = \frac{1}{1+\delta} \qquad . \tag{3}$$

This is the only division of the pie that could make each round of the game look like the rounds that come $n \in 2, 4, 6, etc.$ rounds earlier or later.[2]

3 Another look at it

Here is an even more intuitive way of approaching the problem that I believe does not make explicit use of stationarity.

The equilibrium bargain in the first round of the infinite-horizon bilateral bargaining game gives the offerer the largest share of the prize that player 2 will tolerate. (If the proposed share were higher, player 2 would reject; if it were lower, player 1 could do strictly better by raising his share.) Let's say that share is p^*, which leaves $1 - p^*$ for the offerer. Our goal is to learn more about what the value of p^* could be.

Imagine that Player 1 fails to propose p^* in the first round of the bilateral bar-

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[2] In other words, this is the only division of the pay that would make each time a given player had the right to propose look just like every other time.

gaining game and instead proposes something *larger*, e.g. $p^* + k$. By construction, the proposal will be rejected. Player 2 will then offer p^*, which will be accepted. Since time has passed to get to the second round, the payoffs of this outcome of the game would be $(\delta(1-p^*), \delta p^*)$.

This is not a good outcome for player 1; if he had made the correct bid in the first round (p^*) his payoff would have been strictly higher (this is true because $\delta > 1$ and $p^* \geq .5$).

What should he do instead? We know that he should propose to take p^* in the first round, but let's look further into the necessary properties of p^*. Most importantly, p^* must make player 2 indifferent between his implied round 1 outcome $(1 - p^*)$ and the payoff he would get by rejecting the first round offer and proposing p^* in the second (δp^*). Now, a tiny bit of algebra tells us what p^* needs to be:

$$\begin{aligned} 1 - p^* &= \delta p^* \\ p^* &= \frac{1}{1+\delta} \end{aligned} \tag{4}$$

This is the same result as above.

Fundamentally, this approach makes use of just two facts about the game:

- The opponents will have identical optimal strategies, and
- The optimal strategy will always be to offer your opponent a payoff equal to the discounted payoff that opponent could receive in the next period.

I'm not sure enough of the definition of stationarity to know if my argument uses it at all.

If we continue the logic above and write out the payoffs that would result from the optimal strategy over many rounds (assuming, contrary to usual logic, that the game would progress past the first round because of repeated player errors), we see more clearly this pattern that the proposer (whose payoffs are in boldface below) consistently makes a proposal that equates the current payoff his opponent will receive with the (discounted) payoff the opponent would receive in the next period, given that he makes the optimal proposal:

$$\text{[Game] 1} \quad \left\{\begin{array}{ll} \boldsymbol{\frac{1}{1+\delta}} & \frac{\delta}{1+\delta} \\[4pt] \frac{\delta^2}{1+\delta} & \boldsymbol{\frac{\delta}{1+\delta}} \end{array}\right.$$

$$\text{[Game] 2} \quad \left\{\begin{array}{ll} \boldsymbol{\frac{\delta^2}{1+\delta}} & \frac{\delta^3}{1+\delta} \\[4pt] \frac{\delta^4}{1+\delta} & \boldsymbol{\frac{\delta^3}{1+\delta}} \end{array}\right.$$

$$\text{[Game] 3} \quad \left\{\begin{array}{ll} \boldsymbol{\frac{\delta^4}{1+\delta}} & \frac{\delta^5}{1+\delta} \\[4pt] \frac{\delta^6}{1+\delta} & \boldsymbol{\frac{\delta^5}{1+\delta}} \end{array}\right.$$

etc.